School of Mathematics University of New South Wales

# MATH3711: Higher Algebra (2007,S1)
## Problem Sheet 1 [1]

1. Given the following equation in a group $x^{-1}yxz^2 = 1$, solve for $y$.

2. In any group $G$, show that $(g^{-1})^{-1} = g$ for any $g \in G$. Show for any $m, n \in \mathbb{Z}$ that $g^m g^n = g^{m+n}$ and $(g^m)^n = g^{mn}$.

3. Prove, disprove or salvage if possible the following statement: Given subgroups $J, H \leq G$, the union $H \cup J$ is a subgroup of $G$.

4. Let $G$ be a group and $H \subseteq G$. Show that $H$ is a subgroup iff it is non-empty and for every $h, j \in H$ we have $hj^{-1} \in H$. This gives an alternative characterisation of subgroups. (There is an analogue of this for subspaces, do you know it?)

5. Let $G$ be a group with group multiplication $\mu : G \times G \longrightarrow G$. We define a new group multiplication by $\nu : G \times G \longrightarrow G : (g, g') \mapsto \mu(g', g)$. We let $G^{\text{op}}$ be the set $G$ equipped with this map. Show that $G^{\text{op}}$ is a group. (It is called the *opposite group* to $G$). Remark: when there are two group structures on a set, then a product expression like $gg'$ can mean two different things depending on which multiplication you use. A simple remedy is to introduce more complicated notation like $g * g' := \nu(g, g'), gg' := \mu(g, g')$. Then the relation between the two group structures is $g * g' = g'g$.

6. Let $GL_n(\mathbb{Z})$ be the set of $n \times n$ matrices $M$ with integer entries such that $M^{-1}$ exists and also has integer entries. Show that $GL_n(\mathbb{Z})$ forms a group when endowed with matrix multiplication.

7. In this question, we identify $1 \times 1$ matrices with their unique entry so that $GL_1(\mathbb{C})$ gets identified with $\mathbb{C}^*$, the non-zero elements in $\mathbb{C}$. Let $\mu$ be the subset of roots of unity of $\mathbb{C}^*$. (Recall that a root of unity is a complex number $\zeta$ such that $\zeta^n = 1$ for some integer $n$). Show that, $\mu$ is a subgroup of $\mathbb{C}^*$. Show that the subset $\mu_n$ of $n$-th (not necessarily primitive) roots of unity is in turn a subgroup of $\mu$.

8. (Mildly non-trivial according to my students in previous years.) Show that any finitely generated subgroup of $\mu$ is cyclic. Show that $\mu$ is not finitely generated and find a non-trivial subgroup of $\mu$ which is not finitely generated.

9. Consider the following permutation in two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

Write the permutation out in cycle notation.

10. Compute $(123)(1354)(123)^{-1}$ (your answer should be in cycle notation).

11. Consider a permutation $\sigma \in S_n$ and a $k$-cycle $(a_1 a_2 \ldots a_k)$. Express the product $\sigma(a_1 \ldots a_k)\sigma^{-1}$ in cycle notation. What is its order?

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12. Write out the multiplication table for $S_3$.

13. For elements $g, h$ in a group, show that $gh$ and $hg$ have the same order.

14. Show that a finite group of odd order has an even number of elements of order 2.

15. In the symmetric group $S_6$, describe all the elements of the subgroup $H$ generated by the 3 generators $(12), (34), (56)$. In particular, what is the order of $H$?

16. Describe explicitly, the subgroup $H$ of $GL_2(\mathbb{C})$ generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

where $\zeta$ is a primitive $n$-th root of unity. This is the binary dihedral group.

17. Determine explicitly the elements of the cyclic group generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

18. Consider the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $Sp_2(\mathbb{R}) := \{A | A \text{ is a real 2 by 2 matrix}, A^t J A = J\}$. Show that $Sp_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$. It is called the symplectic group.

19. Find the order of the permutation $(12)(345)$. More generally, given disjoint cycles $\sigma, \tau$, find the order of $\sigma\tau$.

20. Find the orders of all elements in the dihedral group $D_n$. Find all subgroups of $D_n$.

21. Show that the subset $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ of matrices of determinant 1 is a subgroup.

22. For a $k$-cycle $(a_1 a_2 \ldots a_k)$, when is $(a_1 a_2 \ldots a_k)^2$ a cycle.

23. Let $H < S_5$ be the subgroup of the symmetric group generated by $(23), (34)$. Describe $H$ in such a way that it allows you to compute the order of $H$. What is the order? (Hint: there is a simple reason why $H$ has at most 6 elements.)

24. Let $T$ be a subset of $S$. Show that $\{\sigma \in \text{Perm } S | \sigma(t) = t \text{ for every } t \in T\}$ is a subgroup of Perm $S$. Suppose that $S = S_3$ and $T$ is the subset of 3-cycles. What is the order of this subgroup?

25. Let $S = \mathbb{C} - \{1, 0\}$. Describe the subgroup of Perm $S$ generated by the functions $f : S \longrightarrow S : z \mapsto 1 - z, g : S \longrightarrow S : z \mapsto 1/z$. (In this case, you can describe the group by listing all the elements and writing out the multiplication table).

26. Given subsets $J, K$ of a group $G$, show that if $K \subseteq \langle J \rangle$ and $J \subseteq \langle K \rangle$ then $\langle J \rangle = \langle K \rangle$.

27. Show that if $S$ is a subset of $G$ such that $ss' = s's$ for any $s, s' \in S$ then $\langle S \rangle$ is an abelian group.