School of Mathematics University of New South Wales

MATH3711: Higher Algebra (2006,S1)
Problem Sheet 0, Revision [1]

In this course, we will use a lot of set theory. You should have picked up all of what you need to know in first year maths (MATH1241 or MATH1251). Discrete maths would be very handy but is not necessary. As a quick check, make sure you know the following

Set theory definitions you should know

Domain, codomain, range, invertible function, finite set, one-to-one or injective, onto or surjective, bijection, composition of functions.

If for some reason, you never learnt one of these, don't worry, they are fairly straightforward. Given n sets $S_1, \ldots, S_n$ you should know that the product of them is

$$S_1 \times S_2 \times \ldots \times S_n := \{(s_1, \ldots, s_n) | s_i \in S_i \text{ for every } i\}$$

that is, it is the set of all ordered n-tuples of the form $(s_1, \ldots, s_n)$ where $s_i \in S_i$ for $i = 1, \ldots, n$. There is an obvious generalisation to the case where you have an infinite number of sets S_i (only the notation is trickier).

Set theory facts you should know

There are lots of facts about sets which are intuitively obvious which I expect you to know. You are free to use them in assignments/tests without comment. Below is a sample of set theory facts you should know.

1. If $f : S \longrightarrow T$ is a function and S is finite with $|S|$ elements then the range of f has at most $|S|$ elements.

2. If $f : S \longrightarrow T$ is a one-to-one function and S is finite with $|S|$ elements then the range of f has $|S|$ elements.

3. The composite of injective functions is injective.

4. If S, T are finite sets with $|S|, |T|$ elements respectively then $S \times T$ has $|S||T|$ elements.

5. If f, g are functions such that the composite fg (sometimes denoted $f \circ g$) is the identity, then g is injective and f is surjective.

6. Invertible functions are the same as bijective functions.

Set theory exercises

Computational skills are very important in 1st/2nd year courses. In this course, it is your set-theoretic/logic skills which will be most important. Below are some exercises to test the types of skills you should have.

1. For sets R, S, T, show that there are natural bijections between $(R \times S) \times T$, $R \times (S \times T)$ and $R \times S \times T$. (Consequently, we will often identify these and ignore the difference between them).

[1] by Daniel Chan

2. Find a set S and an injective function $f : S \longrightarrow S$ which is not onto. Find all sets with this property.

3. (You will need to be able to show sets are the same). Let $M_2(\mathbb{R})$ denote the set of 2 by 2 matrices with entries in $\mathbb{R}$. Show that

$$\{A \in M_2(\mathbb{R}) | AA^T = I\} = \left\{ \begin{pmatrix} \cos\theta & \pm\sin\theta \\ -\sin\theta & \pm\cos\theta \end{pmatrix} | \theta \in \mathbb{R} \right\}.$$

4. (You will need to be able to show two functions are the same). Let S be the set $\{\text{Batman}, \text{Robin}\}$ and $f : S \longrightarrow S$ be an injection. Show that $f^2 = \text{id}$ (where $f^2 := f \circ f$ the composite of f with itself).

5. Prove, disprove or salvage if possible the following statements concerning a function $f : S \longrightarrow T$ between sets S, T.

   (a) If $S_1, S_2 \subseteq S$ then $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.

   (b) If $S_1, S_2 \subseteq S$ then $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$.

   (c) If $T_1, T_2 \subseteq T$ then $f^{-1}(T_1 \cap T_2) = f^{-1}(T_1) \cap f^{-1}(T_2)$. (Recall that the pre-image $f^{-1}(T_i)$ of T_i is defined to be $\{s \in S | f(s) \in T_i\}$.)

   (d) If $T_1, T_2 \subseteq T$ then $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$.

6. For sets R, S, T show that $R \subseteq T$ and $R \subseteq S$ if and only if $R \subseteq S \cap T$.

7. For a set S we define the diagonal map to be $\Delta : S \longrightarrow S \times S : s \mapsto (s, s)$ and the projection maps to be $p_1 : S \times S \longrightarrow S : (s, s') \mapsto s, p_2 : S \times S \longrightarrow S : (s, s') \mapsto s'$. For $i = 1, 2$ find $p_i \circ \Delta$ and $\Delta \circ p_i$.

Relations

Let S be a set. A *relation* $\prec$ on S is a subset of $S \times S$ where we write $s \prec s'$ iff (s, s') is in that subset. The relation is said to be *reflexive* if it contains the image of the diagonal map (see exercise 7 above for the definition of this). It is *anti-symmetric* if for every pair $s, s' \in S$ satisfying $s \prec s', s' \prec s$ we have $s = s'$. Finally, it is *transitive* if for any $s, s', s'' \in S$ satisfying $s \prec s', s' \prec s''$ we also have $s \prec s''$. A relation which is anti-symmetric, reflexive and transitive is said to be a *partial order*.

1. Let S be a set and P be the set of all subsets of S (this is called the *power set* of S). Show that $\subseteq$ is a partial order on S.

2. Consider a partial order $\prec$ on a set S defined by $R \subseteq S \times S$. If $T \subset S$, show that $T \times T \cap R$ is a partial order on T. Do you understand what this partial order is? It is called the *induced* partial order.

3. Make up your own examples.

A relation $\prec$ is *symmetric* if for any $s, s' \in S$ with $s \prec s'$ we have $s' \prec s$. A symmetric, reflexive and transitive relation is called an *equivalence relation*. Equivalence relations are usually denoted by more symmetric looking symbols such as $\sim, \equiv$ rather than asymmetric ones like $\prec$. Given an equivalence relation $\equiv$ on a set S, the *equivalence class* of some $s \in S$ is $\{s' \in S | s' \equiv s\}$ and is typically denoted $[s]$. Note reflexivity implies $s \in [s]$.

Proposition 0.1 *Suppose first that $\equiv$ is an equivalence relation on a set S. Then S is the disjoint union of its equivalence classes. Conversely, suppose S is the disjoint union of subsets S_i where i runs through some index set I. Define the relation $\equiv$ on S by $s \equiv s'$ iff s, s' belong to some common subset S_i (for some i). Then $\equiv$ is an equivalence relation whose equivalence classes are precisely the subsets S_i.*

Proof is easy but long and omitted.

The set of equivalence classes is denoted $S/\equiv$.

1. Show that the image of the diagonal map is an equivalence relation and determine its equivalence classes.

2. Define a relation on $\mathbb{Z}$ by $m \equiv n$ iff $2|(m-n)$. Show that $\equiv$ is an equivalence relation and the disjoint union of the above proposition is the partition of $\mathbb{Z}$ into even and odd numbers.

3. Let $S := \mathbb{Z} \times (\mathbb{Z} - 0)$. Define a relation on S by $(m, m') \equiv (n, n')$ iff $mn' = nm'$. Show that the function $f : S/\equiv \longrightarrow \mathbb{Q} : [(m, m')] \mapsto m/m'$ is a well-defined bijection.

Zorn's Lemma

This is equivalent to the axiom of choice in set theory. We will assume it in this course, as do most mathematicians but be warned that some still do not.

Let S be a set partially ordered by $\prec$. If $T \subseteq S$ then an *upper bound* for T is an element $s \in S$ such that $t \prec s$ for all $t \in T$. A *maximal* element of T is an element $t \in T$ such that if $t' \in T$ satisfies $t \prec t'$ then $t = t'$.

A set S with a partial order $\prec$ is *totally ordered* if given any $s, s' \in S$ we have either $s \prec s'$ or $s' \prec s$.

Lemma 0.2 *(Zorn) Suppose S is a partially ordered set such that every totally ordered subset of S has an uppper bound in S. Then S contains a maximal element.*

1. Show that every vector space has a basis.