THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

June 2007

MATH3711
HIGHER ALGEBRA

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (50 marks total) The following are each worth 5 marks. Justify your answers with a brief explanation (but be careful to mention the key points).

a) Are $3, 3i$ associates in $\mathbb{Z}[i]$?

b) What is the order of the rotational symmetry group of a tetrahedron?

c) Is $x^3 + 2x^2 - 2x + 6$ irreducible in $\mathbb{Q}[x]$? Is $\mathbb{Q}[x]/\langle x^3 + 2x^2 - 2x + 6\rangle$ a field?

d) Simplify $\mathbb{C}[x, y]/\langle y - x\rangle$. Is $\langle y - x\rangle \triangleleft \mathbb{C}[x, y]$ prime?

e) Is $\mathbb{Z}[x]$ a UFD? Is it a PID?

f) Is $[\mathbb{Q}(\cos \frac{\pi}{16}) : \mathbb{Q}]$ a power of two?

g) Consider an algebraic field extension K/E and a finite field extension E/F. Is K/F algebraic?

h) Let S be a G-set and $g \in G$. Show $S^g = S^{g^{-1}}$.

i) What are all the ideals of $\mathbb{R}[x]/\langle x^2 - x\rangle$?

j) What is the group of units $(\mathbb{C}[x, y]/\langle xy - 1\rangle)^*$?

2. (10 marks) In this question, we work in the ring $R = \mathbb{Z}[i\sqrt{2}]$. Find the greatest common divisor of $2i\sqrt{2}$ and $2 + i\sqrt{2}$ in R (be sure to show working). What is the ideal $\langle 2i\sqrt{2}, 2 + i\sqrt{2}, 1 + 9i\sqrt{2}\rangle$? (Make sure your answer is in simplest form!)

3. (10 marks) In each question below, make sure you justify your answer fully. Let

$$\alpha := \sqrt[3]{2 + \sqrt{2}}.$$

a) Is α algebraic over $\mathbb{Q}$? If so, find the minimal polynomial of α over $\mathbb{Q}$.

b) What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?

c) What is the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$?

d) Is $\sqrt[4]{2} \in \mathbb{Q}(\alpha)$?

4. (8 marks) Consider the subset $S := \{p(x) \in \mathbb{R}[x] | p(-x) = p(x)\}$ of even polynomials.

a) Show that S is a subring of $\mathbb{R}[x]$.

b) Consider the map $\phi : \mathbb{R}[y] \longrightarrow \mathbb{R}[x] : p(y) \mapsto p(x^2)$. Is ϕ an homomorphism? Justify your answer with a brief explanation.

Please see over ...

c) Show that the image of ϕ is S.

d) Is $\mathbb{R}[y] \simeq S$? Justify fully, your answer.

5. (10 marks) In this question, we let $R = \mathbb{Z}[i]$.

a) Show that $1+i$ is irreducible in R.

b) Show that 2 is reducible in R.

c) Factorise $2x^2 - 18$ into irreducibles in $R[x]$. Make sure you prove that your factors are indeed irreducible.

6. (12 marks) Let G be the group of 3×3-matrices below

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} | a, b, c \in \{1, -1\} \right\}.$$

We let G act on $\mathbb{R}^3$ by matrix multiplication, that is, if $\mathbf{v} \in \mathbb{R}^3$ then $g.\mathbf{v} := g\mathbf{v}$. Consider also the following vectors in $\mathbb{R}^3$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} , \mathbf{v_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} , \mathbf{v_3} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

a) What is the order of G?

b) Find the G-orbit of $\mathbf{v_1}$ and its stabiliser.

c) Prove that $G.\mathbf{v_1} \simeq G.\mathbf{v_2}$ as G-sets.

d) Prove that $G.\mathbf{v_1}$ is not isomorphic to $G.\mathbf{v_3}$ as G-sets.