THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS

June 2006

MATH3711
HIGHER ALGEBRA

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 7

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (48 marks total) The following are each worth 4 marks. Explain your answers briefly (but be careful to mention the key points).

a) Is $\mathbb{Z}/9\mathbb{Z}$ a domain?

b) Are $1 - i$ and $1 + i$ associates in $\mathbb{Z}[i]$?

c) What is the order of the rotational symmetry group of a cube?

d) Is $x^4 - 6x^3 - 4x - 2$ irreducible over $\mathbb{Q}$?

e) Is the field $K(\mathbb{C}[x])$ algebraically closed?

f) Simplify $\mathbb{C}[x, y]/\langle x - 2\rangle$ by writing it as a quotient of a polynomial ring with fewer variables. (Only brief explanation required.)

g) Let $\alpha = \cos \frac{\pi}{16}$. Is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ a power of two?

h) Are there fields with exactly 6 elements in them?

i) In $\mathbb{C}[x, y, z]$, are the ideals $\langle x - z, y - z\rangle$ and $\langle x - z, x^2 - xz - y + z\rangle$ equal?

j) Are $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ and $M_2(\mathbb{C})$ isomorphic rings?

k) Is $\mathbb{Z}[x, y]/\langle x^2 + 2\rangle$ a UFD?

l) Is every group of order $n < \infty$ isomorphic to a subgroup of S_{2n}?

2. (6 marks) Find the greatest common divisor of $1 + 5i$ and $3 + i$ in $\mathbb{Z}[i]$. Make sure you show working.

3. (12 marks) Let $\alpha := \sqrt{2} + \sqrt{3}$.

a) What is $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$? Explain fully your reasoning.

b) Show α is algebraic over $\mathbb{Q}$ and find its minimal polynomial.

c) What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?

d) What is the minimal polynomial of α over $\mathbb{Q}(\sqrt{2})$?

4. (6 marks) A cake is divided into 4 equal quadrants. In each is placed a pink, purple or peach coloured candle. How many essentially different ways are there of doing this? Show working.

5. (8 marks)

a) Consider the evaluation homomorphism $\epsilon : \mathbb{C}[x] \longrightarrow \mathbb{C} : p \mapsto p(1)$. Which principal ideal is $\ker \epsilon$?

b) Using the Chinese Remainder Theorem or otherwise, prove that $\mathbb{C}[x]/\langle x^2-1\rangle \simeq \mathbb{C} \times \mathbb{C}$.

6. (12 marks) Let R be the ring $\mathbb{Z}[i]$.

a) Is $1+2i$ irreducible in R? Explain your answer fully.

b) Factorise $2 \in R$ into irreducibles. (Don't forget to show that the factors you give are irreducible.)

c) Is $R/\langle 1+2i\rangle$ a field? If so, what is its characteristic? Which field is it? Make sure you explain fully your reasoning.

7. (8 marks) Let $G = D_3$, the dihedral group with 6 elements. Prove that two G-orbits with the same cardinality are isomorphic as G-sets. Find two non-isomorphic G-sets which have the same number of elements.