Daniel Chan

MATH3710: Higher Algebra I, Problem Sheet 6

1. Describe all abelian groups of order 24 up to isomorphism.

2. Let $G = \mathbb{Z}^3$ and H be the subgroup generated by

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}.$$

Write G/H as a product of cyclic groups as in lecture 23.

3. This question completes the proof of the proposition in lecture 24. Let $N \lhd G$. If G is solvable, show that G/N is solvable as follows. Consider a normal chain of subgroups of G

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

with cyclic factors. First observe from the third isomorphism theorem that $G_i N < G$ and that $G_i N/N$ is thus a well-defined group. We wish to show that

$$(*) \qquad 1_{G/N} = N \lhd G_1 N/N \lhd G_2 N/N \lhd \ldots \lhd G_n N/N = G/N$$

is a normal chain of subgroups with cyclic (possibly zero) factors.

(a) Show that $G_i N \lhd G_{i+1} N$ so the above is indeed a normal chain of subgroups. (Use the second isomorphism theorem here if you like).

(b) Using the universal property of quotients or otherwise, show that there is a natural epimorphism $\phi : G_{i+1}/G_i \longrightarrow G_{i+1}N/G_i N$.

(c) Show that the quotient of any cyclic group is cyclic.

(d) Using the isomorphism theorems or otherwise, show that the factors of (*) are indeed cyclic.

4. For a finite group G, define the k-th derived group inductively by, $G^{(k)} := [G^{(k-1)}, G^{(k-1)}]$ and $G^{(1)} = [G, G]$. Show that

$$1 \leq G^{(k)} \leq G^{(k-1)} \leq \ldots \leq [G, G] \leq G$$

is a normal chain of subgroups of G. Show that G is solvable if and only if $G^{(r)} = 1$ for some $r \in \mathbb{N}$.

5. Is the dihedral group solvable?

6. Show that any group of order less than 20 is solvable.

7. Show that any group of order 30 is solvable. Hence or otherwise show that A_5 is simple (i.e. has no non-trivial normal subgroups).

8. Consider the group $G := \langle \sigma, \tau | \sigma^5 = \tau^2 = (\sigma\tau)^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$. Show that G is $\mathbb{Z}/2\mathbb{Z}$.

9. Show that the alternating group A_4 is generated by $g = (12)(34), h = (123)$. Hence write A_4 in terms of generators and relations.

10. The infinite dihedral group is defined to be $D_\infty := \langle g, h | h^2 = 1, hg = g^{-1}h \rangle$. Show that for any dihedral group D_n, there is a group epimorphism $D_\infty \longrightarrow D_n$. Show that there is an action of D_∞ on $\mathbb{Z}$ where, for $n \in \mathbb{Z}$ we have $g.n = n + 1, h.n = -n$. (Hint: look at the proof of the normal form theorem).