

# MATH3710: Higher Algebra I,

## Problem Sheet 4

1. Let  $G = AGL_n$ , the group of isometries on  $\mathbb{R}^n$  and  $T$  be the subgroup of translations. Show that  $T \triangleleft G$  and using the third isomorphism theorem show that  $G/T$  is isomorphic to a well known group.
2. Let  $S := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$  be the upper half of the complex plane. Let  $G = SL_2(\mathbb{R})$ . We define a map  $\mu : G \times S \rightarrow S$  as follows. If  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $z \in S$ , let

$$\mu(g, z) = g.z := \frac{az + b}{cz + d}.$$

Show that  $g.z$  is indeed in the codomain  $S$  and that  $\mu$  defines an action of the group  $G$  on  $S$ .

3. Let  $S$  be the set of complex  $2 \times 2$ -matrices and  $G = GL_2(\mathbb{C})$ . We let  $G$  act on  $S$  by conjugation as in lecture 13. Let  $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \in S$  be non-scalar matrices. By using the classification of  $G$ -orbits given in lecture 15, show that  $G.a, G.b$  are isomorphic as  $G$ -sets.
4. Generalise the previous question to the following setup. Let  $S$  be the set of complex  $n \times n$ -matrices and  $G = GL_n(\mathbb{C})$ . We let  $G$  act on  $S$  by conjugation as before. Let  $a, b \in S$  be matrices such that all the eigenvalues for  $a$  are distinct and similarly, all the eigenvalues for  $b$  are distinct. Show that  $G.a, G.b$  are isomorphic  $G$ -sets.
5. Let  $G = S_n$  act on  $\mathbb{R}^n$  by  $\sigma.(x_1, \dots, x_n)^t = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})^t$  as in problem sheet 2 question 18. For  $n = 3$  describe all  $G$ -orbits and determine which  $G$ -orbits are isomorphic and which ones are non-isomorphic.
6. Let  $V = \mathbb{R}^n$  and  $G = GL_n$ . We define an action of  $G$  on  $V$  by  $g.v = g(v)$ . Show that  $V - \mathbf{0}$  is a  $G$ -subset of  $V$  isomorphic to  $G/H$  for some  $H < G$  and compute  $H$  explicitly. Is  $H$  unique?
7. This question gets you to examine the motivating example of lecture 14. Let  $G$  be a group and  $\phi : S \rightarrow T$  an homomorphism of  $G$ -sets. Show

that  $\phi(S^G) \subseteq T^G$ . Now let  $G = \{1, \sigma\}$  and  $S, T = \mathbb{R}^2$ . Let  $G$  act on  $S, T$  by  $\sigma.(x, y) = (-x, y)$  in  $S$  and  $\sigma.(x, y) = (-x, -y)$  in  $T$ . Hence  $S, T$  are the  $G$ -sets of the motivating example in lecture 14. Show that  $S, T$  are non-isomorphic  $G$ -sets by distinguishing their fixed point sets or otherwise.

8. Let  $G \leq GL_2(\mathbb{C})$  be the subgroup of diagonal matrices as in lecture 12. Since  $GL_2(\mathbb{C})$  is also a subgroup of  $\text{Perm } \mathbb{C}^2$ , the natural inclusion  $G \hookrightarrow \text{Perm } \mathbb{C}^2$  defines a permutation representation. Describe all the orbits and stabilisers of the corresponding  $G$ -set.
9. Let  $G$  be a group and  $X$  a  $G$ -set. Show that for any  $J \subset G$ , we have  $X^J = X^{\langle J \rangle}$ . This was the first proposition in lecture 16.
10. Let  $G \leq S_5$  be the subgroup generated by  $(12), (23), (45)$ . The inclusion homomorphism  $G \hookrightarrow S_n$  defines a permutation representation. Find all orbits and stabilisers of the corresponding  $G$ -set.
11. How many different ways can you paint the edges of a regular hexagon if each edge may be painted red, white or pink?
12. Show there are 9 099 ways of colouring the faces of a dodecahedron with 3 different colours.