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MATH3710: Higher Algebra I, Problem Sheet 3

1. Consider the subgroup $\mathbb{R}$ of $\mathbb{C}$ (you need not show it is a subgroup). Describe geometrically, all the cosets of $\mathbb{R}$ in $\mathbb{C}$. Identify the group $\mathbb{C}/\mathbb{R}$ i.e. show it is isomorphic to a well-known group we have already studied in class.

2. Recall that $\mathbb{R}$ is a group when endowed with addition and $H := \{z \in \mathbb{C} \,||z| = 1\}$ is a subgroup of $\mathbb{C}^*$. Using the exponential function and the first isomorphism theorem, show that H is isomorphic to a quotient group of $\mathbb{R}$. State explicitly what this quotient group is. Show using similar methods or otherwise that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to a subgroup of $\mathbb{C}^*$.

3. Let $\phi : \mathbb{C}^* \longrightarrow \mathbb{C}^* : z \mapsto z^n$ for some positive integer n. Show that ϕ is a group homomorphism. Find $\ker \phi, \text{im } \phi$. What isomorphism does the the first isomorphism theorem give? Verify that the fibres of ϕ are indeed the cosets of $\ker \phi$.

4. Weak version of Chinese remainder theorem. Let m, n be relatively prime positive integers. Consider the homomorphism $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ defined by $\phi(a) = (a + m\mathbb{Z}, a + n\mathbb{Z})$. Find $\ker \phi$. Compare the orders of $\mathbb{Z}/\ker \phi$ and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ to determine the image of ϕ. Use the first isomorphism theorem to find which cyclic group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is isomorphic to.

5. Let T, U be sets and S be their disjoint union. Consider the subset G of $\text{Perm}S$ consisting of permutations σ such that $\sigma(T) = T, \sigma(U) = U$. (Note that G is a subgroup). Use the universal property of products to construct a group isomorphism $G \xrightarrow{\sim} \text{Perm}T \times \text{Perm}U$.

6. Let G be the dihedral group of order $2n$ and N the (unique) cyclic subgroup of order n ($N =< \sigma >$ in the lecture notes). Let H be the group generated by any $\tau \notin N$. Verify the third isomorphism theorem in this case and compute explicitly the isomorphism.

7. Suppose $N \trianglelefteq G, N' \trianglelefteq G'$. Show that $N \times N'$ is naturally a normal subgroup of $G \times G'$ and show $(G \times G')/(N \times N') \simeq (G/N) \times (G'/N')$.

8.