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MATH3710: Higher Algebra I, Problem Sheet 2

1. Find the subgroup of $\mathbb{Z}$ generated by 4 and 6.
2. Let G be the symmetric group on 4 symbols S_4 and H be the subset $\{\sigma|\sigma(4) = 4\}$. Show that H is a subgroup. Compute all the left and right cosets of H in G. Verify Lagrange's theorem and the 1-1 correspondence between left and right cosets given in class.
3. Consider $\sigma \in S_6$ defined using 2 line notation by
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}.$$
Write out σ explicitly as a product of transpositions and hence determine whether it is odd or even. Verify your answer by computing $\sigma\Delta$ where Δ is the difference product.
4. Let H, K be subgroups of G of order 3 and 5 respectively. Use Lagrange's theorem to show that $H \cap K = 1$.
5. Let G be a group with prime order. Use Lagrange's theorem to find all subgroups of G. Show that G is cyclic.
6. Using the previous exercise or otherwise, find all subgroups of S_3.
7. Show the associativity of the subset product claimed in lecture 7 i.e. for subsets K_1, K_2, K_3 of a group G we have $(K_1K_2)K_3 = K_1(K_2K_3)$.
8. Let $G = \mathbb{C}^*$ and H be the subset of complex numbers of modulus 1. Show that H is a normal subgroup of G and describe the cosets of H. Show that G/H is isomorphic to a subgroup of $\mathbb{R}^*$.
9. Show that $A_n \leq S_n$ is generated by 3-cycles.
10. Let $G = GL_2$ and let H be the subgroup of elements of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ where $a, c \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Compute all the left and right cosets of H in G. If you know some projective geometry you may wish to show that G/H can be naturally identified with the real projective line.

11. Let G be a group and H be a subgroup of index two. Show that H is normal.

12. Why is $H = A_n$ normal in $G = S_n$? Find a group isomorphic to G/H.

13. Let $z \in \mathbb{C}^*$ and ϕ be multiplication by z. Is ϕ a group homomorphism from a) $\mathbb{C} \longrightarrow \mathbb{C}$, b) $\mathbb{C}^* \longrightarrow \mathbb{C}^*$?

14. Find all isomorphisms $\phi : \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ where p is prime.

15. Isomorphic groups should be identical as far as their group structure is concerned. To illustrate this, consider an isomorphism $\phi : G \longrightarrow G'$. Show

    (a) G is abelian if and only if G' is.

    (b) G, G' have the same order.

    (c) There is a natural bijection between the the subgroups of G and the subgroups of G'. It preserves orders, inclusions and normality.

    (d) If $g \in G$ has order n, so does $\phi(g)$.

16. Show that S_3 and $\mathbb{Z}/6\mathbb{Z}$ both have order 6 (so are isomorphic sets) but are not isomorphic as groups.

17. (Hard?) Let μ be the group of roots of unity introduced in problem sheet 1. Find all isomorphisms $\phi : \mu \xrightarrow{\sim} \mu$.

18. For $\sigma \in S_n$ we let $\Phi(\sigma)$ be the linear transformation $\Phi(\sigma) : (x_1, \ldots, x_n)^t \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^t$. Show that $\Phi : S_n \longrightarrow GL_n$ is a group homomorphism. Determine its image.

19. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group μ_n introduced in problem sheet 1.