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MATH3710: Higher Algebra I, Problem Sheet 1

1. Given the following equation in a group $x^{-1}yxz^2 = 1$, solve for y.

2. Let $GL_n(\mathbb{Z})$ be the set of $n \times n$ matrices M with integer entries such that M^{-1} exists and also has integer entries. Show that $GL_n(\mathbb{Z})$ forms a group when endowed with matrix multiplication.

3. In this question, we identify 1×1 matrices with their unique entry so that $GL_1(\mathbb{C})$ gets identified with $\mathbb{C}^*$, the non-zero elements in $\mathbb{C}$. Let μ be the subset of roots of unity of $\mathbb{C}^*$. (Recall that a root of unity is a complex number ζ such that $\zeta^n = 1$ for some integer n). Show that, μ is a subgroup of $\mathbb{C}^*$. Show that the subset μ_n of n-th (not necessarily primitive) roots of unity is in turn a subgroup of μ.

4. (Mildly non-trivial according to my students in previous years.) Show that any finitely generated subgroup of μ is cyclic. Show that μ is not finitely generated and find a non-trivial subgroup of μ which is not finitely generated.

5. Consider a permutation $\sigma \in S_n$ and a k-cycle $(a_1 a_2 \dots a_k)$. Express the product $\sigma(a_1 \dots a_k)\sigma^{-1}$ in cycle notation. What is its order?

6. Write out the multiplication table for S_3.

7. In the symmetric group S_6, describe all the elements of the subgroup H generated by the 3 generators $(12), (34), (56)$. In particular, what is the order of H?

8. Describe explicitly, the subgroup H of $GL_2(\mathbb{C})$ generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

where ζ is a primitive n-th root of unity. This is the binary dihedral group.

9. Determine explicitly the elements of the cyclic group generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

10. Consider the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $Sp_2(\mathbb{R}) := \{A | A \text{ is a real 2 by 2 matrix}, A^t J A = J\}$. Show that $Sp_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$. It is called the symplectic group.

11. Find the order of the permutation $(12)(345)$. More generally, given disjoint cycles σ, τ, find the order of $\sigma\tau$.

12. Find the orders of all elements in the dihedral group D_n.

13. Show that the subset $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ of matrices of determinant 1 is a subgroup.

14. For a k-cycle $(a_1 a_2 \dots a_k)$ find $(a_1 a_2 \dots a_k)^n$ for any integer n.

15. Let $H < S_5$ be the subgroup of the symmetric group generated by $(23), (34)$. Describe H in such a way that it allows you to compute the order of H. What is the order? (Hint: there is a simple reason why H has at most 6 elements.)

16. Let T be a subset of S. Show that $\{\sigma \in \text{Perm } S | \sigma(t) = t \text{ for every } t \in T\}$ is a subgroup of Perm S.