THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS

JULY 2004

MATH3710
HIGHER ALGEBRA I

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 7

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(6) If you are ill during this examination and wish to request special consideration, you must inform the invigilator that you are ill, and ensure that the invigilator makes a note of this fact on your examination booklets. These examination booklets will not be marked, and you must apply for special consideration, through the Student Centre, within 3 days of this examination.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (50 marks total) The following questions are worth 5 marks each. Explain your answers briefly.

a) Find the order of $\frac{\mathbb{Z}/mn\mathbb{Z}}{m\mathbb{Z}/mn\mathbb{Z}}$.

b) Let $F \subset \mathbb{R}^2$ be a subset consisting of 2 (distinct) points. Find the group G of isometries T of $\mathbb{R}^2$ such that $T(F) = F$.

c) Let G be a group of order 18 and P a Sylow 3-subgroup. How many left cosets of P are there in G?

d) Let $\sigma \in S_n$ be a permutation and $(a_1 \ldots a_k)$ be a k-cycle in S_n. Write $\sigma(a_1 \ldots a_k)\sigma^{-1}$ in cycle notation.

e) Classify all isomorphism classes of abelian groups of order 42.

f) Show that for $n \geq 4$, the alternating group A_n is generated by 3-cycles.

g) Let G be a group of order 55. Using the Sylow theorems or otherwise, find the composition factors of G.

h) Let G, G' be solvable groups. Is $G \times G'$ solvable too?

i) Let $H, K \leq G$ be groups. An (H, K)-coset in G is a subset of G of the form HgK where $g \in G$ (subset product notation used here). Show that the number of elements in any (H, K)-coset divides $|H||K|$.

j) Let G be a group and $\phi : S \longrightarrow S'$ be an homomorphism of G-sets. For $s \in S$, what is the relationship between $\mathrm{Stab}_G\ s$ and $\mathrm{Stab}_G\ \phi(s)$? What more can be said if ϕ is an isomorphism?

2. (10 marks) Is the exponential map $\exp : \mathbb{R} \longrightarrow \mathbb{R}^*$ a group homomorphism? Is $\mathbb{R}^* \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$?

3. (12 marks) A round cake is divided into 6 equal sectors. In the centre of each sector is placed either an indigo or magenta candle. Use the theorem on counting the number of orbits as in lectures to find the number of essentially different ways the candles may be placed on the cake. You should show working but you need not give a full proof for your answer.

4. (12 marks) Let G be a non-abelian group of order 8. Show that $|Z(G)|$ cannot be 4. Hence or otherwise determine the number of conjugacy classes of G and their sizes i.e. their cardinalities. (Hint: The class equation may be useful.)

Please see over ...

5. (12 marks) Let $G = GL_2(\mathbb{C})$ and G' be the normal subgroup of scalar matrices in G. (You need not show G' is normal). Let $H = SL_2(\mathbb{C})$ and H' be the normal subgroup of scalar matrices in H. Which, if any, of the following groups are isomorphic: $G/G', H/H', G' \times G', G'$ or G? Fully justify your answer.

6. (12 marks) Let G be the group $\langle g, h | g^3 = h^2, h^4 = 1, gh = hg^{-1} \rangle$. Show that G is finite. What is the order of G? Fully justify your answer.

7. (12 marks) An abelian group A is said to be indecomposable if it cannot be written as the direct sum of two non-trivial subgroups. For a prime p, show that $\mathbb{Z}/p^2\mathbb{Z}$ is indecomposable. Write $\mathbb{Q}/\mathbb{Z}$ as the direct sum of indecomposable subgroups.