THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS

JUNE 2003

MATH3710
HIGHER ALGEBRA I

(1) TIME ALLOWED – 3 HOURS

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ATTEMPT ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(6) If you are ill during this examination and wish to request special consideration, you must inform the invigilator that you are ill, and ensure that the invigilator makes a note of this fact on your examination booklets. These examination booklets will not be marked, and you must apply for special consideration, through the Student Centre, within 3 days of this examination.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. (5 marks each part for a total of 50 marks)

True/False. State true or false for the following statements and give a brief justification of your answer.

a) Let G be a group of order 100. Then G contains a subgroup of order 15.

b) The group S_7 is solvable.

c) Any subgroup of SO_3 of order 27 is cyclic.

d) There exists a finitely generated non-cyclic group G with trivial automorphism group $\operatorname{Aut} G$.

e) The group $G = < g, h | g^m = 1, h^n = 1, gh = hg >$ is isomorphic to $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

f) Let G be a group of order n acting on a set X with r elements ($n, r < \infty$). Suppose the group action is non-trivial in the sense that there is some $g \in G$ and $x \in X$ with $g.x \neq x$. If $n > r!$ then G is not simple.

g) For p a prime, the composition factors of a p-group G are all isomorphic.

h) Let Γ be a connected graph with 3 vertices and 4 edges (i.e. 2 pairs of edges). The fundamental group of Γ is trivial.

i) If $H \lhd N \lhd G$ then $H \lhd G$.

j) Let

$$G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$$

and H be the subgroup generated by elements of the form $(a+2^i\mathbb{Z})-(2a+2^{i+1}\mathbb{Z})$ where $a \in \mathbb{Z}, i \geq 1$. Then G/H is isomorphic to the group

$$\mu_{2^\infty} := \{z \in \mathbb{C}^* | z^{2^r} = 1 \text{ for some } r \in \mathbb{N}\}.$$

2. (10 marks) Let $\mu_n < \mathbb{C}^*$ be the subgroup of n-th roots of unity. Show that $\mathbb{C}^* \simeq \mathbb{C}^*/\mu_n$.

3. (10 marks) Let $G = \mathbb{Z}^3$ and H be the subgroup generated by

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}.$$

Write G/H as a product of cyclic groups and find the torsion subgroup of G/H.

Please see over ...

4. (10 marks) Let G be a group of order 65. Use Sylow's theorem to find the number of Sylow 13-subgroups and the number of Sylow 5-subgroups of G. Show that any group of order 65 is cyclic.

5. (10 marks) Determine all finite groups with at most 3 conjugacy classes.

6. (10 marks) Let $G = GL_2(\mathbb{C})$ act on the set S of complex 2×2-matrices by conjugation. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find the stabilisers of A and B. Are the G-orbits of A and B isomorphic as G-sets? Justify your answer fully.