Lecture 3: Graphical analysis- Arms race

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Semester 1 2014

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Identify variables of interest Let

$$h = \text{the number of hobbit ICBFs and}$$
$$o = \text{the number of orc ICBFs}$$

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Historical note: China has a "no nuclear first strike policy".

Deterrent hypothesis

Let's posit a more reasonable hypothesis, that the hobbits will use the ICBFs as a deterrent.

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1. Hobbits wish to minimise costs.
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Our A There may or may not be mathematical considerations in determining some answer to this question. We will ignore it by letting h_0 be the number of ICBFs which achieves sufficient damage.

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That depends on the probability p of each hobbit ICBF withstanding an attack from one orc ICBF. Then ph ICBFs remain and we need $h_0 = ph$, that is $h = \frac{h_0}{p}$ and we can plot the point $(\frac{h_0}{p}, \frac{h_0}{p})$.

The relationship between h and o

Let's plot more points

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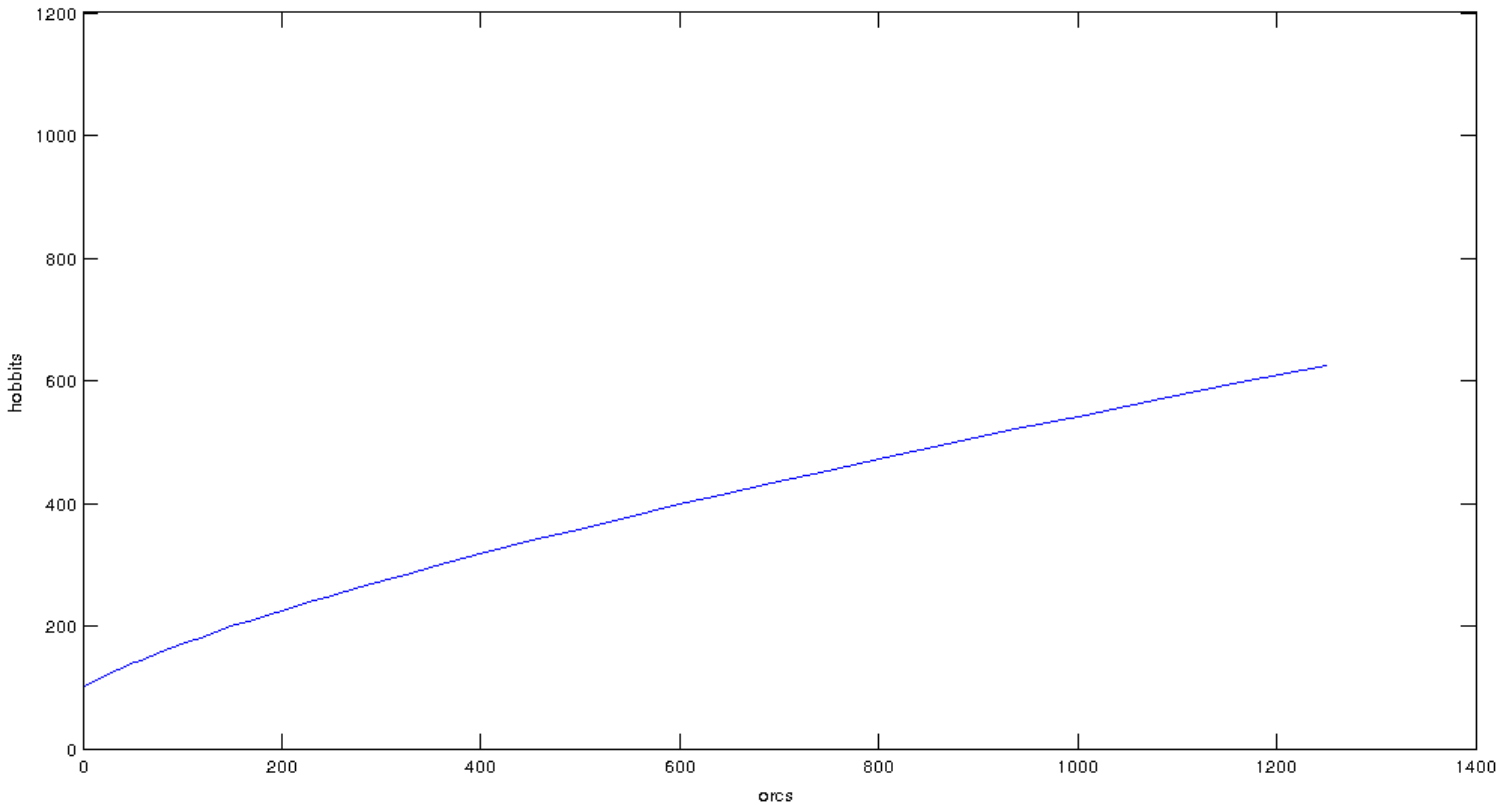
N.B. Here h is not an explicit function of o, but we can plot points of this graph easily enough by looking at various values of n.

Plot

Let's pick $h_0 = 100, p = .4$. Then we can use the computer to plot



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Furthermore, $\frac{do}{dh}$ increases with h, so the o versus h graph is concave up. Thus the h versus o graph is concave down.

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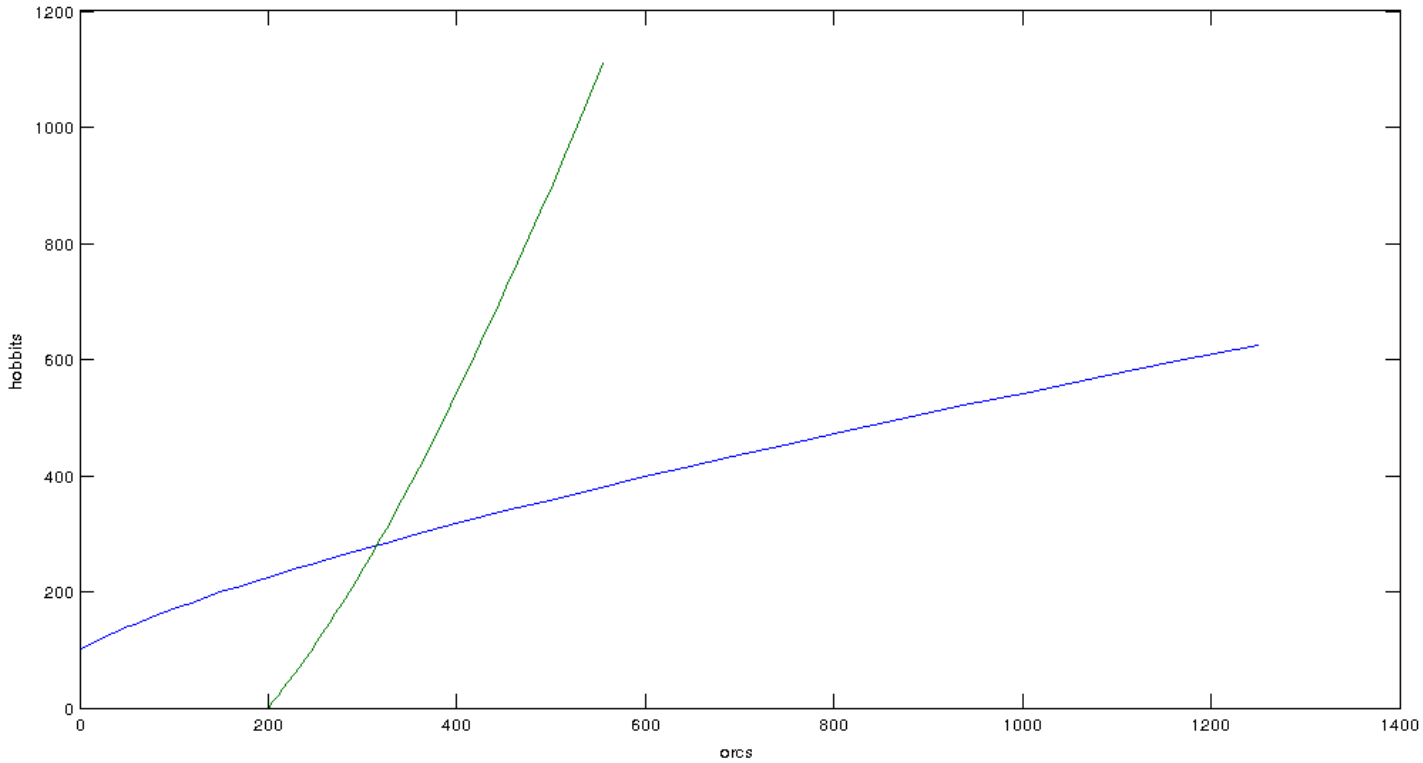
From the orc perspective, we have

Relationship

$$o = \frac{o_0}{q^{h/o}}$$

Combined plot

Picking $o_0 = 200, q = .2$, we have the following combined plot



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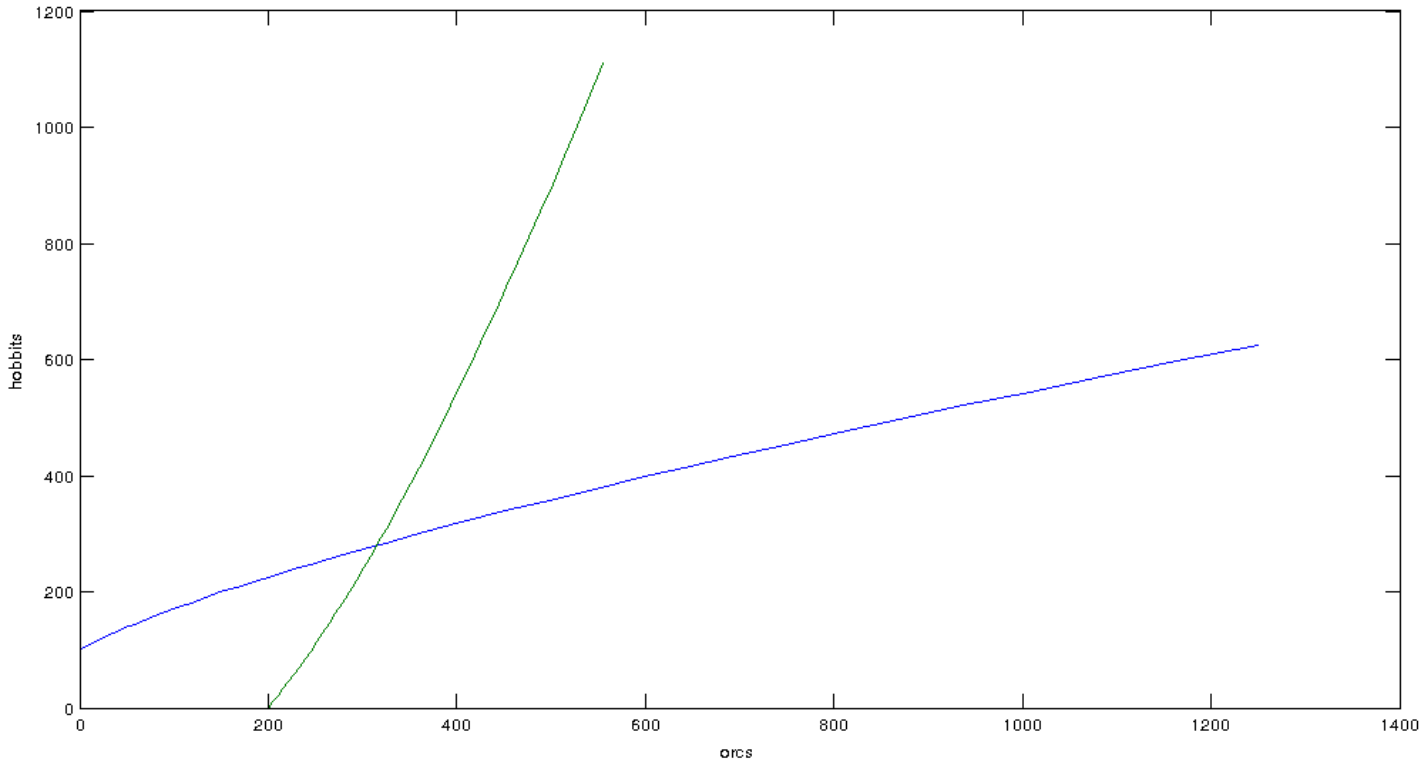

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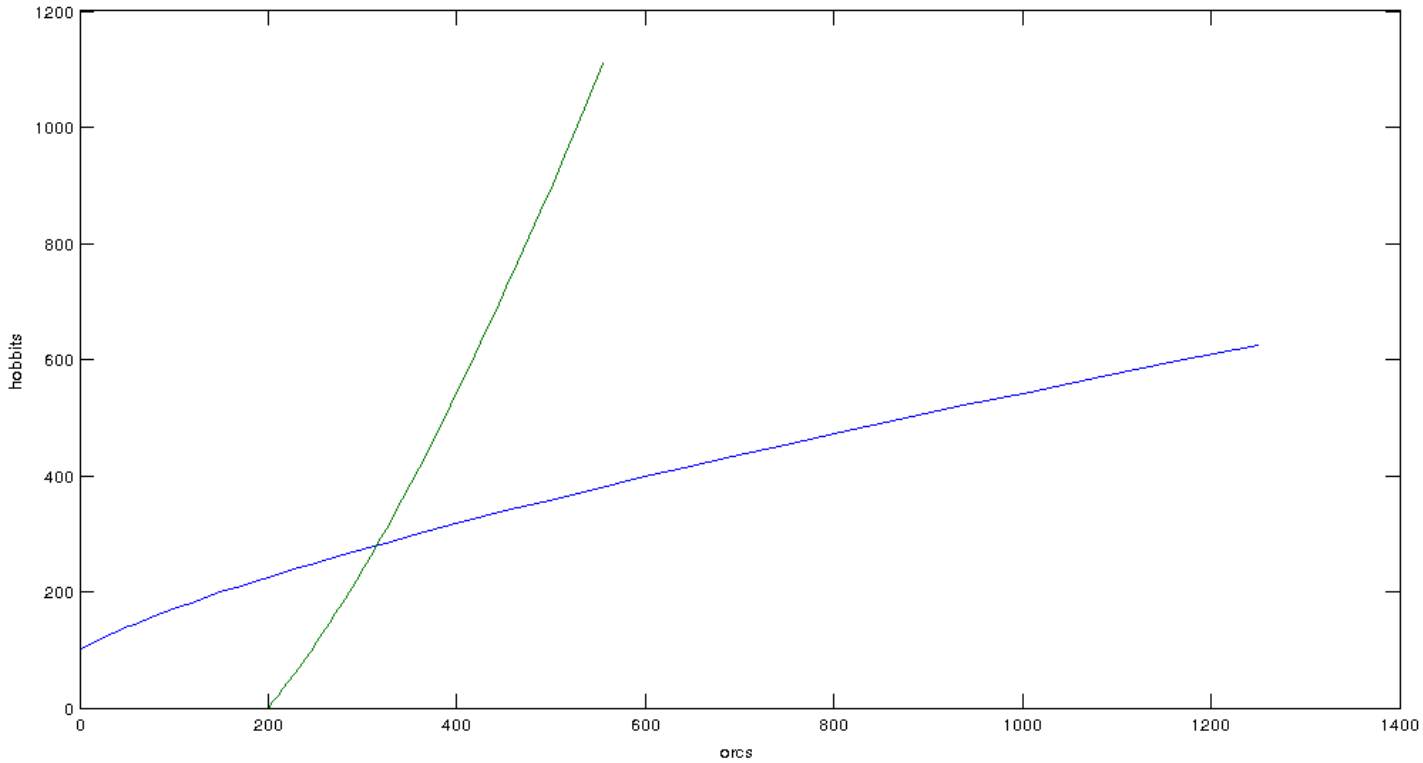

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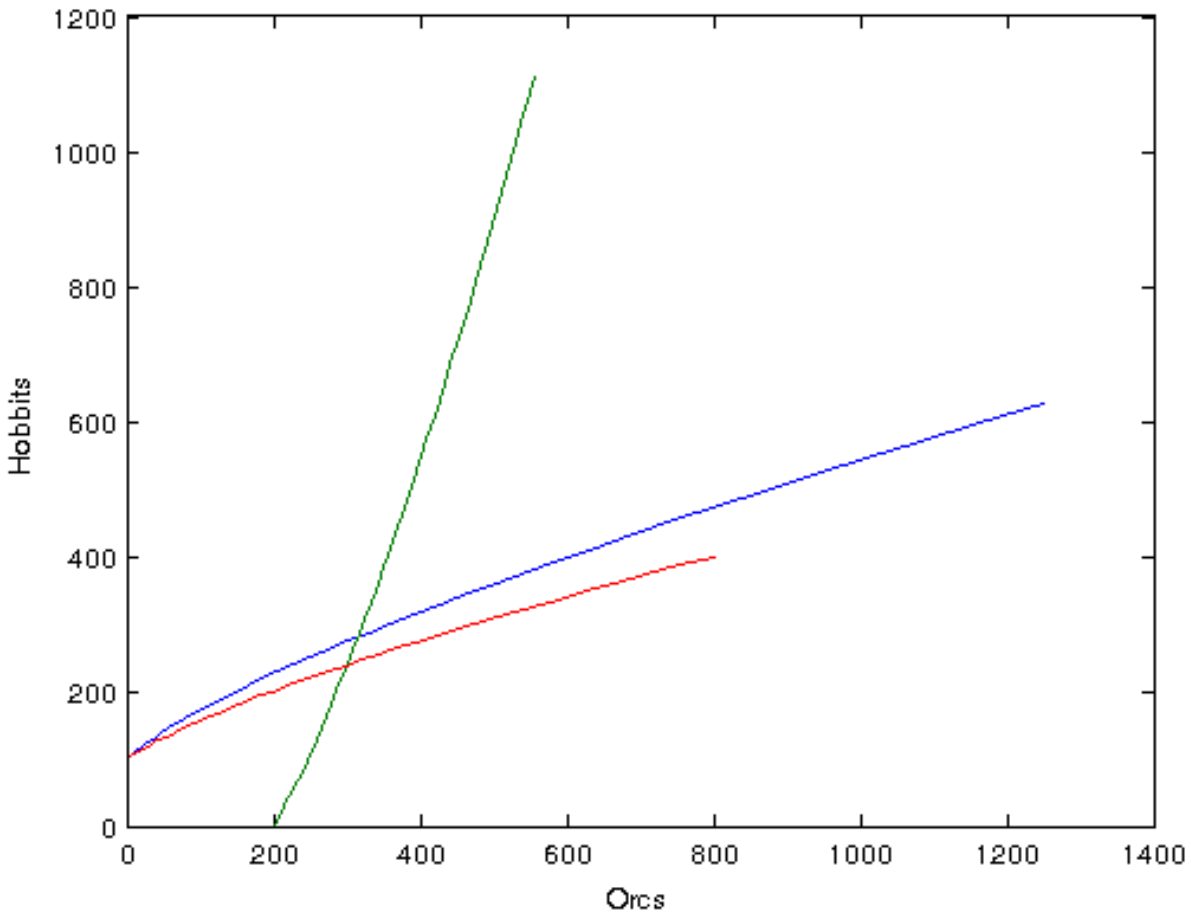
The h-intercept remains the same but from $\frac{do}{dh} = \frac{\ln(h_0/h)-1}{\ln p}$ we see that $\frac{do}{dh}$ increases so $\frac{dh}{do}$ decreases i.e. the curve flattens out.

New graph

Assuming p increases from .4 to .5, we obtain



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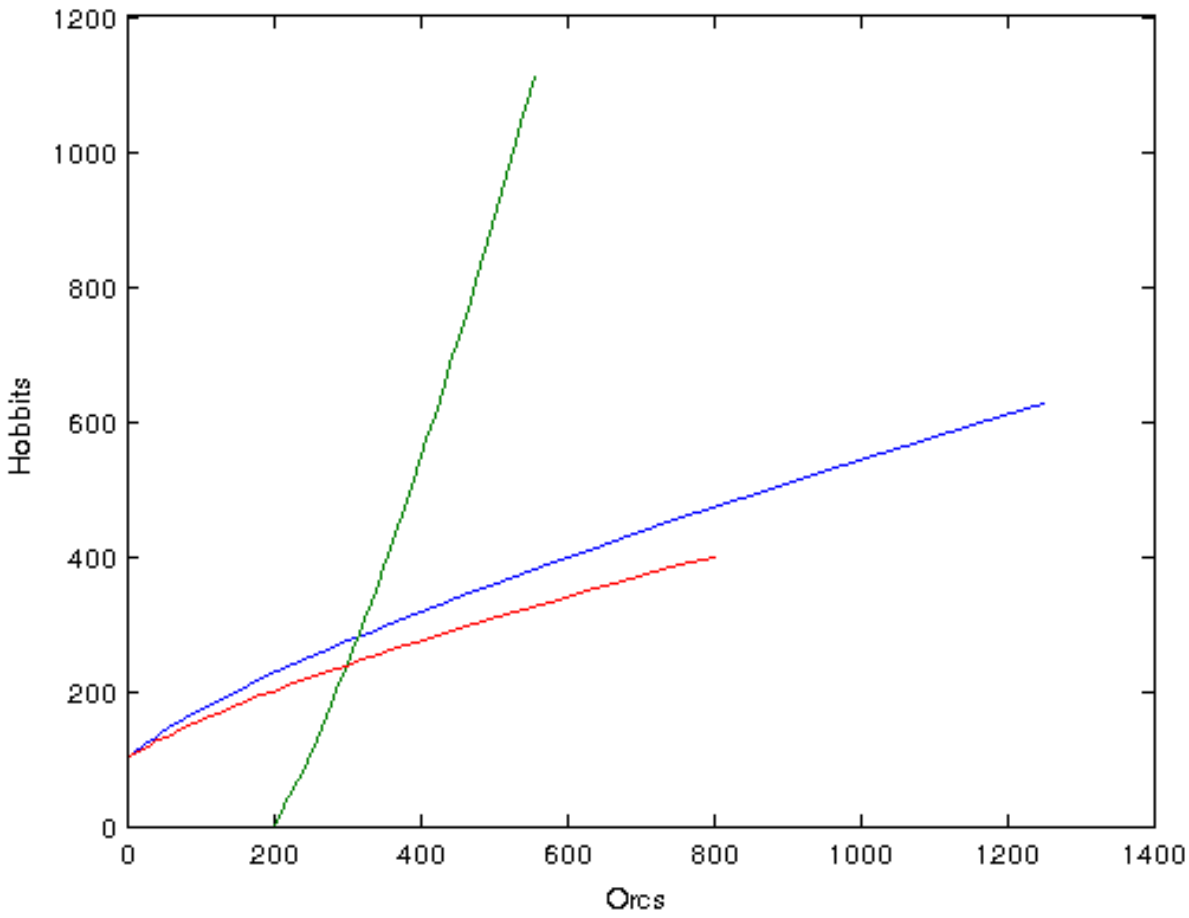

Conclusion The hobbit strategy graph flattens from the blue curve to the red. We see ICBFs reduce for both the hobbits and orcs.