Lecture 2: Recurrence relations and population dynamics

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UNSW

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We will approach this down-to-middle-earth question using a type of mathematical model called a *recurrence relation*. Those of you studying MATH1081 will see more of the mathematical aspects of this.
Our goal is not so much to deal with the mathematics, but to see where the concept can be useful.

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This is a *recurrence relation* for the variable/sequence P_n which expresses P_{n+1} in terms of 'earlier' values of the variable, namely P_n.

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Question

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Year, n	1	2	3	4	5	6
Population, P_n	5468	5810	6140	6510	6912	7411

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Conclusion: The model seems reasonably accurate as the ratio hovers around the constant 1.06 which we can take for a. (Alternatively, we can average the ratios above in some way). In other words, the population increases by about 6% a year.

Comparing the mathematical model with reality

Once we fix a value for a, say 1.06, we can also test the model by comparing the actual population P_n to the model's predicted value $\tilde{P}_n = P_1 1.06^{n-1}$.

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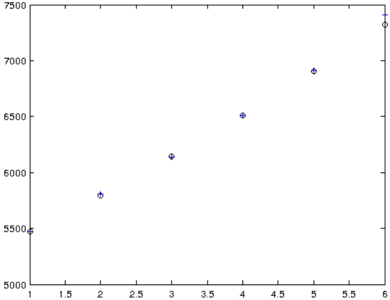

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$$P_{n+1} = aP_n - bP_nO_n$$
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for some real positive constants a, b, c, d.

This is a coupled system of recurrence relations in two variables.

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Warning Don't try to solve this explicitly! However, remember that given parameter and initial values, we can get the computer to work out as many values as we like.

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Hypothesis Let's suppose we have prior data which give the growth rates of hobbit and orc populations at 6% and 4% respectively. Then $a = 1.06, c = 1.04$.

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$$0 = \Delta O_e = O_e(.04 - dP_e)$$

so $d = .04/2322 \approx 1.72 \times 10^{-5}$.

Perturbation from equilibrium

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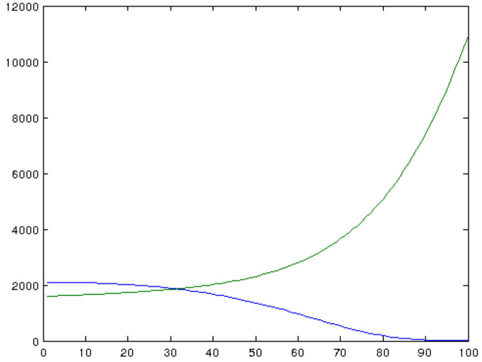
We can use the new recurrence relations with the aid of an appropriate computer package. It's easiest to plot the result over the next 100 years.

Conclusion

hobbits = blue, orcs = green.



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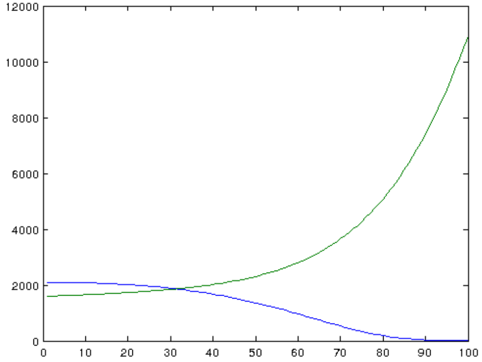

It seems that the hobbit population will be killed off and the orc population will grow exponentially.

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In our example, we posit exponential population growth if the hobbits are left to themselves.
GROUP WORK: Is this reasonable? Discuss.