Sudoku, logic and proof

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As you probably know, a Sudoku puzzle is a 9×9 grid divided into nine 3×3 subgrids. Some of the cells in the grid contain a symbol: usually the symbols are the numbers $1, 2, \ldots, 9$. The idea is to put a symbol into each empty cell so that every symbol appears exactly once in each row, exactly once in each column and exactly once in each of the nine 3×3 subgrids. An example of a Sudoku puzzle is given below: it was the daily puzzle on `http://sudoku.com.au` on 9 June 2011, when I wrote this article.

		1			**4**	**9**		**2**
6	**5**				**9**			
2				**8**	**7**	**1**	**4**	
9	**1**				**6**	**5**		
	6		**3**		**2**		**7**	
		8	**9**				**1**	**3**
	8	**2**	**4**	**5**				**6**
			7				**3**	**8**
3		**7**	**2**			**4**		

Figure 1: A Sudoku puzzle

Sudoku puzzles are extremely popular, addictive and fun. Wikipedia [4] describes Sudoku as "a logic-based, combinatorial number-placement puzzle". I want to focus on the links between Sudoku, logic and proof, but as a combinatorialist it is my duty to tell you a bit about the combinatorics of Sudoku first.

A *latin square* is an $n \times n$ grid containing the symbols $1, 2, \ldots, n$ such that every symbol occurs exactly once in every row and exactly once in every column. So a completed Sudoku puzzle is a 9×9 latin square with an extra property: namely that the nine 3×3 subsquares contain each symbol exactly once. Using brute-force computer enumeration it has been shown that there are

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different 9×9 Sudoku grids [2] (that's about 6.7×10^{21}), but many of these only differ by some kind of symmetry: for example, writing the rows in a different order, or the columns in a different order, or swapping the symbols around, or rotating the grid. If we agree that these symmetries produce Sudoku puzzles that are essentially the same, then there are

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essentially different Sudoku grids [2]. That's about 5.5 billion different completed puzzles.

Now let's talk about Sudoku and mathematical logic. Sudoku puzzles became popular outside of Japan around 2004, when they started appearing in daily newspapers in the United Kingdom and then around the world. Many newspapers felt the need to reassure nervous readers that they did not need to use mathematics in order to solve the puzzle. (Sadly, many people do not understand the difference between "mathematics" and "arithmetic"!) For example, *The Australian* newspaper used to print this text next to Sudoku puzzles:

> The game requires no mathematics and can be solved by logic alone.

I imagine mathematicians around the world tearing their hair out when confronted with such statements. A far better message would be to tell the world that Sudoku puzzles use logic and that *logic is mathematics.* (Therefore, since Sudoku puzzles are fun, it follows that mathematics can be fun too!)

Every mathematical proof requires careful logic, and you can learn a lot about mathematical logic by solving Sudoku puzzles. To illustrate this, let's start to solve our example Sudoku. Have a look at the cell marked by the black square in Figure 2. We know

		1			**4**	**9**		**2**
6	**5**				**9**			
2				**8**	**7**	**1**	**4**	
9	**1**		■		**6**	**5**		
	6		**3**		**2**		**7**	
		8	**9**				**1**	**3**
	8	**2**	**4**	**5**				**6**
			7				**3**	**8**
3		**7**	**2**			**4**		

Figure 2: Direct proof

that it has to contain a symbol, and looking at the symbols in the same row or column of that cell, we see that the only possible symbol for that cell is 8. Therefore, that cell must contain 8. This is an example of a *direct proof*, where we consider everything that we know is true and draw a logical conclusion. The conclusion that we draw is called a *deduction*. We might write a careful proof as follows:

> The marked cell must contain a symbol from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. However, it cannot contain a symbol from $\{1, 5, 6, 9\}$, because these symbols occur in the same row as the marked cell. Similarly, it cannot contain a symbol from $\{2, 3, 4, 7, 9\}$ as these symbols occur in the same column as the marked cell. Therefore the cell must contain the symbol 8.

Now we can fill in the marked cell, and using the same kind of direct proof we can also fill in the centre cell (for example).

This gives the grid shown in Figure 3, which also contains three marked cells which we'll use below. Next we will illustrate another kind of proof technique, called *proof by cases*. (It is also called *proof by exhaustion of cases* but I think that sounds far too tiring.)

		1			4	9		2
6	5				9			
2				8	7	1	4	
9	1		8	♠	6	5		
	6		3	1	2		7	
■	■	8	9				1	3
	8	2	4	5				6
			7				3	8
3		7	2			4		

Figure 3: Proof by cases

> Consider the 3×3 subgrid which contains the cells marked with the black squares in Figure 3. This subgrid does not yet contain a 7, and the only cells where a 7 could be placed are the two cells marked by the black squares. So the 7 must be in the marked cell on the left (call this Case 1) or the marked cell on the right (call this Case 2). In either case, the cell marked with the spades symbol ♠ must contain 7, since this is the only cell in the centre 3×3 subgrid which does not have a 7 in its row.

The important point is that we were able to conclude *with certainty* that there must be a 7 in the cell marked ♠, even though we do not know which of Case 1 or Case 2 holds.

It is enough to know that *either* Case 1 *or* Case 2 must be true, and that our desired conclusion holds in either case.

The proof by cases argument leads to the grid shown in Figure 4. Now we fill

		1			4	9		2
6	5				9			
2				8	7	1	4	
9	1		8	7	6	5		
	6		3	1	2		7	
		8	9				1	3
	8	2	4	5				6
			7				3	8
3		7	2			4		

Figure 4: The partially completed solution

in three more cells (using direct proofs) to reach the grid shown in Figure 5, which also contains a cell marked with a black square, a cell marked with a ♠ symbol and a cell marked with a ♣ symbol. We will use these cells to illustrate our next proof

8		1			4	9		2
6	5				9	♠	■	
2				8	7	1	4	
9	1		8	7	6	5	♣	
	6		3	1	2		7	
		8	9	4	5		1	3
	8	2	4	5				6
			7				3	8
3		7	2			4		

Figure 5: Proof by contradiction

technique, *proof by contradiction*. Here we wish to prove that something is true, and yet we start off by assuming that the *opposite* is true. We argue logically until we obtain a *contradiction*, which is something impossible that cannot be true. The only way to explain this contradiction is that our original assumption was incorrect. For example, consider the cell marked by the black square in Figure 5. We wish to show that it must contain 8. Our proof by contradiction proceeds as follows:

> For a contradiction, suppose that the cell in Figure 5 marked by a black square does *not* contain 8. The second row does not yet contain an 8, and the only possible cells in the second row where an 8 may be placed are the cells marked with the black square and the ♠ symbol. Therefore, by our assumption, the cell marked with the ♠ symbol must contain 8. Now consider the 3×3 subgrid containing the cell marked with the ♣ symbol. This subgrid does not yet contain an 8, and the only cell in this subgrid which does not have an 8 in its column is the cell marked with the ♣ symbol. So this cell must contain an 8. But then the symbol 8 appears twice in the the row containing the ♣ symbol, which is a *contradiction*. Therefore our original assumption (that the cell marked with the black square does not contain 8) is *incorrect*. We conclude that the cell marked with the black square must contain 8.

(There is a branch of mathematical philosophy called *constructivism* which does not trust proof by contradiction. But most mathematicians are untroubled by these concerns.)

After our proof by contradiction we arrive at the grid shown in Figure 6.

8		1			4	9		2
6	5				9		8	
2				8	7	1	4	
9	1		8	7	6	5		
	6		3	1	2		7	
		8	9	4	5		1	3
	8	2	4	5				6
			7				3	8
3		7	2			4		

Figure 6: A partial solution

So far we have covered most proof techniques used by mathematicians, but not quite all: *proof by contrapositive* and the *principle of mathematical induction* are the main ones we have missed. (These are both extremely useful in mathematics but do not easily apply to solving Sudoku puzzles.)

		1			4	9		2
6	5				9			
2				8	7	1	4	
9	1		8	4	6	5		
	6		3	1	2		7	
		8	9	■	♣		1	3
	8	2	4	5				6
			7				3	8
3		7	2			4		

Figure 7: An incorrect partial solution

Finally, let's discuss what happens when you make a mistake in Sudoku, and why it's important *never* to guess. In Sudoku, as in any mathematical proof, any wrong guess or error means that every subsequent step you take is unreliable. For example, suppose that when we filled in the cell marked with the ♠ symbol in Figure 3, we made a mistake (or an incorrect guess) and placed a 4 there. Then we would have the grid shown in Figure 7. From here, by direct proof, we would conclude that the cell marked with a black square in Figure 7 must contain 7. Then we would complete the centre 3×3 grid by placing a 5 in the cell marked with the clubs symbol ♣ in Figure 7. By comparing this with Figure 6 we see that our first deduction was incorrect but our second deduction was correct! From a false assumption you can correctly prove some true statements and correctly prove some false statements, meaning that *everything* proved from a false assumption is unreliable. Even if you realise later that a mistake has been made, it can be very difficult to find the original mistake: often you just have to throw everything out and start again. This shows why it is so important, in Sudoku as in mathematical proof, to proceed carefully and make sure that every step you take is correct.

If you would like to learn more about mathematical proofs, a good reference is Franklin & Daoud's *Proof in Mathematics: an Introduction* [1]. Meanwhile, keep on puzzling!

References

[1] J. Franklin and A. Daoud, *Proof in Mathematics: an Introduction*, Kew Books, Sydney, Australia, 2011. PDF files of each chapter from an earlier edition are also available from `http://web.maths.unsw.edu.au/~jim/proofs.html`.

[2] `http://www.afjarvis.staff.shef.ac.uk/sudoku/`, accessed on 9 June 2011.

[3] `http://sudoku.com.au`, accessed on 9 June 2011.

[4] `http://en.wikipedia.org/wiki/Sudoku`, accessed on 9 June 2011.