Non-degeneracy of Positive Solutions to Homogeneous Second Order Differential Systems and Its Applications*

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Abstract: This paper considers the Dirichlet problem of homogeneous and inhomogeneous second order ordinary differential systems. A non-degeneracy result is proven for positive solutions of homogeneous systems. Sufficient and necessary conditions for the existence of multiple positive solutions for inhomogeneous systems are obtained by making use of the nondegeneracy and uniqueness results of homogeneous systems.

Key words: Second ordinary differential system, Dirichlet problem, positive solution, nondegeneracy, multiplicity result.

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1 Introduction

In this paper we consider the following weakly coupled homogeneous second order semilinear ordinary differential system:

\[
\begin{cases}
-u''(x) = v^p(x), & x \in (-\ell, \ell) \\
-v''(x) = u^q(x), & x \in (-\ell, \ell) \\
u(x), v(x) > 0, & x \in (-\ell, \ell) \\
u(-\ell) = u(\ell) = v(-\ell) = v(\ell) = 0
\end{cases}
\] (1.1)

where \( p, q > 1 \); and its corresponding inhomogeneous version:

\[
\begin{cases}
-u''(x) = v^p + \lambda f(x), & x \in (-\ell, \ell) \\
-v''(x) = u^q + \lambda g(x), & x \in (-\ell, \ell) \\
u(x) > 0, v(x) > 0, & x \in (-\ell, \ell) \\
u(-\ell) = u(\ell) = v(-\ell) = v(\ell) = 0
\end{cases}
\] (1.2)

The aim of this paper is two-fold: firstly, to prove that any solution pair \((u, v)\) of problem (1.1) is nondegenerate; and secondly, to identify the exact conditions which ensure the existence of multiple solutions of problem (1.2) by making use of the nondegeneracy and uniqueness results of problem (1.1).

Problem (1.1) and (1.2) are the one dimensional cases of the following well known Lane-Emden systems respectively

\[
\begin{cases}
-\Delta u(x) = v^p(x), & x \in \Omega \\
-\Delta v(x) = u^q(x), & x \in \Omega \\
u(x), v(x) > 0, & x \in \Omega \\
u(x) = v(x) = 0, & x \in \partial \Omega
\end{cases}
\] (1.3)

and

\[
\begin{cases}
-\Delta u(x) = v^p(x) + \lambda f(x), & x \in \Omega \\
-\Delta v(x) = u^q(x) + \lambda g(x), & x \in \Omega \\
u(x), v(x) > 0, & x \in \Omega \\
u(x) = v(x) = 0, & x \in \partial \Omega
\end{cases}
\] (1.4)

where \( \Omega \) is a domain in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is the Laplacian on \( \Omega \).
Lane-Emden systems (1.3) and (1.4) arise naturally from the study of various nonlinear phenomena, such as: pattern formation; population evolution; chemical reactions; and so on (see [3]), and have attracted much attention in recent years. In the literature, properties of solutions to (1.3) such as: \textit{a priori} estimates; existence results; Liouville-type theorems; some symmetric and uniqueness results; have been obtained, see [3, 4, 6-13]. Multiplicity results have also been proved for problem (1.4) under certain conditions (see [1]).

Almost all the papers we mentioned above are about problem (1.3) and (1.4) with dimension $n \geq 2$. The method used in these papers can also be used to prove the corresponding results for the one dimensional cases problem (1.1) and (1.2). Hence, we will neglect the study of \textit{a priori} estimates, existence results etc. for problem (1.1) and (1.2). Instead, we pay particular attention to a so-called nondegeneracy property of solutions of problem (1.1) and its applications to problem (1.2). The nondegeneracy of solutions of problem (1.1) is defined as follows.

\textbf{Definition} A solution pair $(u(x), v(x))$ of problem (1.1) is called non-degenerate if the following linearized problem

\[
\begin{aligned}
-\varphi'' &= pv^{p-1}\psi, \quad x \in (-\ell, \ell) \\
-\psi'' &= qu^{q-1}\varphi, \quad x \in (-\ell, \ell) \\
\varphi(-\ell) &= \varphi(\ell) = \psi(-\ell) = \psi(\ell) = 0
\end{aligned}
\]

(1.5)

admits only the trivial solution pair $(\varphi(x), \psi(x)) \equiv (0, 0)$.

We know from the study of scalar Lane-Emden equations that the nondegeneracy property is of significance. It has many applications in the analysis of uniqueness, bifurcation, and exact multiplicity results. However, there is no nondegeneracy result for problem (1.1) and (1.3) until now. A nondegeneracy result is badly wanted for problem (1.3), however we believe it would be very difficult to prove. In fact, a general nondegeneracy result has still not been obtained for the scalar Lane-Emden equation until now, let alone for Lane-Emden system (1.3). This situation lead us to study a simple case of problem (1.3), that is, problem (1.1).

Now, we are in a position to state our first result of the present paper.

\textbf{Theorem 1.1} Any solution pair $(u(x), v(x))$ of problem (1.1) is non-degenerate.

Next, we will present some results about the problem (1.2). For simplicity of notation,
we denote by $\mathcal{M}$ the set of $f(x) \in C[-\ell, \ell] \setminus \{0\}$ such that the following problem

\[
\begin{cases}
-\varphi''(x) = f(x), & x \in (-\ell, \ell) \\
\varphi(-\ell) = \varphi(\ell) = 0 \\
\varphi \geq 0
\end{cases}
\tag{1.6}
\]

is solvable.

It can be proved by a same argument as that used in [1] that problem (1.2) has at least two solutions when $f(x), g(x) \in \mathcal{M}$ and $\lambda$ small enough. Here, we will prove that $f(x), g(x) \in \mathcal{M}$ is also necessary for the existence of multiple solutions for problem (1.2). This is the second result of this paper and it can be stated as the following.

**Theorem 1.2** There exists a positive number $\lambda_0$ such that problem (1.2) has at least two solution pairs for any $\lambda \in (0, \lambda_0)$ if and only if $f(x), g(x) \in \mathcal{M}$.

Concerning the exact number of the solution pairs of problem (1.2), we have the following result.

**Theorem 1.3** There exists a positive number $\lambda_0$ such that problem (1.2) has at most two solution pairs for any $\lambda \in (0, \lambda_0)$.

**Theorem 1.4** If $f(x), g(x) \in C[-\ell, \ell] \setminus \mathcal{M}$, then there exists a positive number $\lambda_0$ such that problem (1.2) has at most one solution pair for any $\lambda \in (0, \lambda_0)$.

Finally, we conclude the introduction with a plan of this paper. Section 2 is devoted to prove Theorem 1.1. The proofs of Theorem 1.2, 1.3 and 1.4 are presented in Section 3. In Section 4, we give some examples of functions which ensure the solvability, or unsolvability, of problem (1.6)

## 2 The Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We need the following lemma.

**Lemma 2.1** Let $(u(x), v(x))$ be a solution pair of problem (1.1) and $(\varphi(x), \psi(x))$ be a solution pair of problem (1.5). Then we have

\[
\int_{-\ell}^{\ell} v^p \psi \, dx = \int_{-\ell}^{\ell} u^p \varphi \, dx = 0.
\]
Proof Multiplying the first equation in problem (1.5) by \(v(x)\) and integrating the resultant equation on \([-\ell, \ell]\), we obtain

\[
\int_{-\ell}^{\ell} pv^p \psi \, dx = -\varphi'v \bigg|_{-\ell}^{\ell} + \varphi v' \bigg|_{-\ell}^{\ell} - \int_{-\ell}^{\ell} v'' \varphi \, dx.
\]

By the boundary conditions of \(\varphi, v\) and the first equation in problem (1.1), we have

\[
\int_{-\ell}^{\ell} u^q \varphi \, dx = p \int_{-\ell}^{\ell} v^p \psi \, dx. \tag{2.1}
\]

Similarly, we also have

\[
\int_{-\ell}^{\ell} v^p \psi \, dx = q \int_{-\ell}^{\ell} u^q \varphi \, dx. \tag{2.2}
\]

From (2.1) and (2.2), we deduce that

\[
(pq - 1) \int_{-\ell}^{\ell} u^q \varphi \, dx = 0,
\]

\[
(pq - 1) \int_{-\ell}^{\ell} v^p \psi \, dx = 0.
\]

Taking \(pq > 1\) into account, we arrive at

\[
\int_{-\ell}^{\ell} u^q \varphi \, dx = \int_{-\ell}^{\ell} v^p \psi \, dx = 0.
\]

\[\blacksquare\]

Proof of Theorem 1.1 Let \((u(x), v(x))\) be a solution pair of problem (1.1) and \((\varphi(x), \psi(x))\) be an arbitrary solution pair of problem (1.5). We are going to prove that \((\varphi(x), \psi(x)) \equiv (0, 0)\). To this end, we let

\[
\xi(x) = (x + \ell) u'(x), \quad \eta(x) = (x + \ell) v'(x).
\]

Then a simple calculation implies that

\[
\xi'(x) = u'(x) - (x + \ell) v^p,
\]

\[
-\xi''(x) = 2v^p + pv^{p-1} \eta(x), \tag{2.3}
\]

\[
\eta'(x) = v'(x) - (x + \ell) u^q,
\]

\[
-\eta''(x) = 2u^q + qu^{q-1} \xi(x).
\]
Now, we compute

\[ 2\ell u'(\ell)\psi'(\ell) = \int_{-\ell}^{\ell} (\xi(x)\psi'(x))' \, dx \]
\[ = \int_{-\ell}^{\ell} \xi'(x)\psi'(x) \, dx + \int_{-\ell}^{\ell} \xi(x)\psi''(x) \, dx \]
\[ = -\int_{-\ell}^{\ell} \xi''(x)\psi(x) \, dx + \int_{-\ell}^{\ell} \xi(x)\psi''(x) \, dx \]
\[ = \int_{-\ell}^{\ell} (2v^p + pv^{p-1}\eta(x)) \psi(x) \, dx - q \int_{-\ell}^{\ell} \xi(x)u^{q-1}\varphi(x) \, dx \]
\[ = p \int_{-\ell}^{\ell} v^{p-1}\eta(x)\psi(x) \, dx - q \int_{-\ell}^{\ell} u^{q-1}\xi(x)\varphi(x) \, dx \]

and

\[ 2\ell v'(\ell)\varphi'(\ell) = \int_{-\ell}^{\ell} (\eta(x)\varphi'(x))' \, dx \]
\[ = \int_{-\ell}^{\ell} \eta'(x)\varphi'(x) \, dx + \int_{-\ell}^{\ell} \eta(x)\varphi''(x) \, dx \]
\[ = -\int_{-\ell}^{\ell} \eta''(x)\varphi(x) \, dx + \int_{-\ell}^{\ell} \eta(x)\varphi''(x) \, dx \]
\[ = q \int_{-\ell}^{\ell} u^{q-1}\xi(x)\varphi(x) \, dx - p \int_{-\ell}^{\ell} v^{p-1}\eta(x)\psi(x) \, dx \]

Consequently, we have

\[ u'(\ell)\psi'(\ell) + v'(\ell)\varphi'(\ell) = 0. \tag{2.4} \]

Since \((u(x), v(x))\) is a solution pair of problem (1.1), it follows from Hopf’s boundary point lemma that

\[ u'(\ell) < 0, \quad v'(\ell) < 0. \]

At this stage, (2.4) implies the following two possibilities:

(i) \(\psi'(\ell) = \varphi'(\ell) = 0\); or

(ii) \(\psi'(\ell) \cdot \varphi'(\ell) < 0\).

We will prove by contradiction that the possibility (ii) cannot occur. Otherwise, we may assume, without loss of generality, that \(\varphi'(\ell) < 0\). Thus, there must exist a number \(\delta > 0\) such that

\[ \varphi(x) > 0, \quad \forall x \in (\ell - \delta, \ell). \]

On the other hand, it follows from Lemma 2.1 that \(\varphi(x)\) must change sign on \((-\ell, \ell)\). This implies that there exists at least one point \(x^* \in (-\ell, \ell)\) such that \(\varphi(x^*) = 0\). Let

\[ x_0 = \max\{x^* \in (-\ell, \ell) \mid \varphi(x^*) = 0\}. \]
Then, we have
\[ \varphi(x_0) = \varphi(\ell) = 0 \] (2.5)
and
\[ \varphi(x) > 0, \quad \forall x \in (x_0, \ell). \] (2.6)
This, combined with the second equation in problem (1.5), implies that
\[ \psi''(x) < 0, \quad \forall x \in (x_0, \ell). \]
Consequently, we have
\[ \psi'(\ell) - \psi'(x) = \int_x^\ell \psi''(x) \, dx < 0, \quad \forall x \in (x_0, \ell). \]
That is \( \psi'(x) > \psi'(\ell) \).

Noticing that \( \varphi'(\ell) < 0 \) and \( \psi'(\ell) \cdot \varphi'(\ell) < 0 \), we then have
\[ \psi'(x) > \psi'(\ell) > 0, \quad \forall x \in (x_0, \ell). \]
Integrating the above inequality on \((x, \ell)\), we obtain
\[ \psi(\ell) - \psi(x) = \int_x^\ell \psi'(x) \, dx > 0, \quad \text{for any } x \in (x_0, \ell). \]
Hence
\[ \psi(x) < \psi(\ell) = 0, \quad \forall x \in (x_0, \ell). \]
From this and (2.5), we know that \( \varphi(x) \) satisfies
\[
\begin{aligned}
-\varphi'' &= p\nu^{p-1}\psi(x) < 0, \quad x \in (x_0, \ell) \\
\varphi(x_0) &= \varphi(\ell) = 0.
\end{aligned}
\]
By the strong maximum principle, we have
\[ \varphi(x) < 0, \quad \forall x \in (x_0, \ell). \]
This contradicts (2.6). Hence, possibility (ii) cannot occur, and we arrive at
\[ \varphi'(\ell) = \psi'(\ell) = 0. \]
Similarly, by using the auxiliary functions
\[ \xi(x) = (\ell - x) u'(x), \quad \eta(x) = (\ell - x) v'(x), \]
we can prove that
\[ \varphi'(-\ell) = \psi'(-\ell) = 0. \]
Now the conclusion of Theorem 1.1 follows immediately from the uniqueness result of the following initial value problem

\[
\begin{align*}
-\varphi'' &= pv^{p-1}\psi, \\
-\psi'' &= qu^{q-1}\varphi, \\
\varphi(-\ell) &= \psi(-\ell) = 0, \\
\varphi'(-\ell) &= \psi'(-\ell) = 0.
\end{align*}
\]

This completes the proof of Theorem 1.1. \qed

3 The Proof of Theorems 1.2, 1.3 and 1.4

In this section we will prove Theorem 1.2, 1.3 and 1.4. With this in mind, we present some lemmas which are needed in the proof of these Theorems.

Lemma 3.1 Problem (1.1) has exactly one solution pair for any $p, q > 1$.

Lemma 3.1 is proven in [2]. We can also give a new and simple proof of Lemma 3.1 by making use of Theorem 1.1. For a similar discussion, we refer the interested reader to [5] for scalar equations.

Lemma 3.2 There exists a positive number $A$ small enough such that problem (1.2) has at most one solution pair $(u(x), v(x))$ which satisfies

\[\|u(x)\|_{L^\infty} + \|v(x)\|_{L^\infty} \leq A.\]

Proof Let $\lambda_1$ be the first eigenvalue of the eigenvalue problem

\[
\begin{align*}
-\varphi'' &= \lambda \varphi, \\
\varphi(-\ell) &= \varphi(\ell) = 0
\end{align*}
\]

and choose $A$ so small that

\[pA^{p-1} + qA^{q-1} < \lambda_1.\]  \hspace{1cm} (3.1)

Then, we can conclude that problem (1.2) has at most one solution pair $(u(x), v(x))$ which satisfies

\[\|u(x)\|_{L^\infty} + \|v(x)\|_{L^\infty} \leq A.\]  \hspace{1cm} (3.2)
In fact, if \((u_1(x), v_1(x))\) and \((u_2(x), v_2(x))\) are two solutions of problem (1.2) which satisfy (3.2), then

\[
U(x) = u_1(x) - u_2(x) \\
V(x) = v_1(x) - v_2(x)
\]

satisfy

\[
\begin{cases}
-U'' = p \xi^{p-1}(x)V \\
-V'' = q \eta^{q-1}(x)U \\
U(-\ell) = U(\ell) = 0 \\
V(-\ell) = V(\ell) = 0
\end{cases}
\]

where \(\xi(x)\) and \(\eta(x)\) are functions between \(v_1(x)\) and \(v_2(x)\) and \(u_1(x)\) and \(u_2(x)\) respectively. Consequently, we have

\[
\int_{-\ell}^{\ell} |U'|^2 \, dx = p \int_{-\ell}^{\ell} \xi^{p-1}UV \, dx \leq pA^{p-1} \int_{-\ell}^{\ell} |U| \, dx,
\]

\[
\int_{-\ell}^{\ell} |V'|^2 \, dx = q \int_{-\ell}^{\ell} \eta^{q-1}UV \, dx \leq qA^{q-1} \int_{-\ell}^{\ell} |U| \, dx.
\]

By the Poincare and Hölder inequalities we have

\[
\lambda_1 \int_{-\ell}^{\ell} U^2 \, dx \leq pA^{p-1} \left( \int_{-\ell}^{\ell} U^2 \, dx \right)^{\frac{1}{2}} \left( \int_{-\ell}^{\ell} V^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\lambda_1 \int_{-\ell}^{\ell} V^2 \, dx \leq qA^{q-1} \left( \int_{-\ell}^{\ell} U^2 \, dx \right)^{\frac{1}{2}} \left( \int_{-\ell}^{\ell} V^2 \, dx \right)^{\frac{1}{2}}
\]

Hence, we have

\[
\lambda_1 \left[ \left( \int_{-\ell}^{\ell} U^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{-\ell}^{\ell} V^2 \, dx \right)^{\frac{1}{2}} \right] \leq (pA^{p-1} + qA^{q-1}) \left[ \left( \int_{-\ell}^{\ell} U^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{-\ell}^{\ell} V^2 \, dx \right)^{\frac{1}{2}} \right].
\]

Since \(\lambda_1 > pA^{p-1} + qA^{q-1}\), we have

\[
\left( \int_{-\ell}^{\ell} U^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{-\ell}^{\ell} V^2 \, dx \right)^{\frac{1}{2}} = 0.
\]

This implies that

\[
U(x) \equiv V(x) \equiv 0.
\]

That is \((u_1(x), v_1(x)) \equiv (u_2(x), v_2(x))\). This concludes the proof of Lemma 3.2. \qed
Lemma 3.3  For any fixed $\lambda_0 > 0$, there exists a positive constant $M > 0$ independent of $\lambda \in (0, \lambda_0)$ such that any solution pair $(u_\lambda(x), v_\lambda(x))$ of problem (1.2) with respect to $\lambda$ satisfies

$$\|u_\lambda(x)\|_{L^\infty} + \|v_\lambda(x)\|_{L^\infty} \leq M.$$  

Lemma 3.3 can be proved in a similar way as that used in [3] due to the following Liouville-type theorem. Hence, we omit its proof here.

Lemma 3.4  (see [4]) The only non-negative solution of problem

$$\begin{cases} 
-u'' = v^p, & x \in (-\infty, \infty) \\
-v'' = u^q, & x \in (-\infty, \infty) 
\end{cases}$$

is $(u(x), v(x)) \equiv (0, 0)$.

Lemma 3.5  Suppose that there exists a $\lambda_0 > 0$ such that problem (1.2) has at least two distinct solution pairs $(u^1_\lambda(x), v^1_\lambda(x))$ and $(u^2_\lambda(x), v^2_\lambda(x))$ for any $\lambda \in (0, \lambda_0)$. Then at least one of these solution pairs satisfies

$$\lim_{\lambda \to 0} (\|u_\lambda(x)\|_{L^\infty} + \|v_\lambda(x)\|_{L^\infty}) = 0.$$  

Proof  Without loss of generality, we may assume that

$$\|u^1_\lambda\|_{L^\infty} + \|v^1_\lambda\|_{L^\infty} \leq \|u^2_\lambda\|_{L^\infty} + \|v^2_\lambda\|_{L^\infty}. \quad (3.3)$$

What we want to do is to prove that

$$\lim_{\lambda \to 0} (\|u^1_\lambda\|_{L^\infty} + \|v^1_\lambda\|_{L^\infty}) = 0.$$  

If this conclusion is not valid, then we can find a sequence $\{\lambda_j\}$ such that $\lambda_j \to 0$ as $j \to \infty$ and

$$\lim_{j \to \infty} (\|u^1_{\lambda_j}\|_{L^\infty} + \|v^1_{\lambda_j}\|_{L^\infty}) = A > 0.$$  

By (3.3) we also have

$$\lim_{j \to \infty} (\|u^2_{\lambda_j}\|_{L^\infty} + \|v^2_{\lambda_j}\|_{L^\infty}) \geq A > 0.$$  

On the other hand, by Lemma 3.3 we know that there exists a positive number $M$ independent of $\lambda_j$ such that

$$\|u^k_{\lambda_j}\|_{L^\infty} + \|v^k_{\lambda_j}\|_{L^\infty} \leq M, \quad k = 1, 2.$$  

Hence

$$\left\|\left(u^k_{\lambda_j}\right)'\right\|_{L^\infty} + \left\|\left(v^k_{\lambda_j}\right)'\right\|_{L^\infty} + \left\|\left(u^k_{\lambda_j}\right)''\right\|_{L^\infty} + \left\|\left(v^k_{\lambda_j}\right)''\right\|_{L^\infty} \leq C, \quad k = 1, 2.$$  

for some positive constant $C$ independent of $\lambda_j$. Consequently, up to a subsequence, we may assume that

\[
\begin{align*}
    u^k_{\lambda_j} &\to u^k \not\equiv 0 \ 	ext{uniformly on } (-\ell, \ell), \text{ for } k = 1, 2, \\
v^k_{\lambda_j} &\to v^k \not\equiv 0 \ 	ext{uniformly on } (-\ell, \ell), \text{ for } k = 1, 2.
\end{align*}
\]

It is easy to see that $(u^1(x), v^1(x))$ and $(u^2(x), v^2(x))$ are solutions of problem (1.1). Hence, by Lemma 3.1, we have

\[
(u^1(x), v^1(x)) \equiv (u^2(x), v^2(x)) \not\equiv (u_0(x), v_0(x)). \tag{3.4}
\]

Let

\[
\begin{align*}
    U_j(x) &= u^1_{\lambda_j}(x) - u^2_{\lambda_j}(x), \\
    V_j(x) &= v^1_{\lambda_j}(x) - v^2_{\lambda_j}(x).
\end{align*}
\]

Then $U_j(x) \not\equiv 0$, $V_j(x) \not\equiv 0$, since $(u^1_{\lambda_j}(x), v^1_{\lambda_j}(x))$ and $(u^2_{\lambda_j}(x), v^2_{\lambda_j}(x))$ are distinct. Moreover, it is easy to see that $U_j(x)$ and $V_j(x)$ satisfy

\[
\begin{align*}
    -U''_j(x) &= p \int_0^1 \left[ tv^1_{\lambda_j}(x) + (1-t)v^2_{\lambda_j}(x) \right]^{p-1} dt \cdot V_j(x), \quad x \in (-\ell, \ell) \\
    -V''_j(x) &= q \int_0^1 \left[ tu^1_{\lambda_j}(x) + (1-t)u^2_{\lambda_j}(x) \right]^{q-1} dt \cdot U_j(x), \quad x \in (-\ell, \ell) \\
    U_j(-\ell) &= U_j(\ell) = 0 \\
    V_j(-\ell) &= V_j(\ell) = 0
\end{align*}
\]

Set $M_j = \|U_j\|_{L^\infty} + \|V_j\|_{L^\infty}$ and $\Phi_j = \frac{U_j}{M_j}$, $\Psi_j = \frac{V_j}{M_j}$. Then $(\Phi_j, \Psi_j)$ satisfies

\[
\begin{align*}
    -\Phi''_j &= p \int_0^1 \left[ tv^1_{\lambda_j}(x) + (1-t)v^2_{\lambda_j}(x) \right]^{p-1} \Psi_j, \quad x \in (-\ell, \ell) \\
    -\Psi''_j &= q \int_0^1 \left[ tu^1_{\lambda_j}(x) + (1-t)u^2_{\lambda_j}(x) \right]^{q-1} \Phi_j, \quad x \in (-\ell, \ell) \\
    \|\Phi_j\|_{L^\infty} + \|\Psi_j\|_{L^\infty} &= 1 \\
    \Phi_j(-\ell) &= \Phi_j(\ell) = \Psi_j(-\ell) = \Psi_j(\ell) = 0
\end{align*}
\]

Obviously, up to a subsequence, we may assume that

\[
\begin{align*}
    \Phi_j &\to \Phi \ 	ext{uniformly on } [-\ell, \ell], \\
    \Psi_j &\to \Psi \ 	ext{uniformly on } [-\ell, \ell].
\end{align*}
\]
Passing to the limit in (3.5) and taking into account (3.4) we have

\[
\begin{align*}
-\Phi'' &= pv_0^{p-1}\Psi \\
-\Psi'' &= qu_0^{q-1}\Phi \\
\|\Phi\|_{L^\infty} + \|\Psi\|_{L^\infty} &= 1 \\
\Phi(-\ell) &= \Phi(\ell) = \Psi(-\ell) = \Psi(\ell) = 0
\end{align*}
\]

This contradicts Theorem 1.1, since \((u_0, v_0)\) is the unique solution of problem (1.1). \(\square\)

**Proof of Theorem 1.2** Since the “if” part of Theorem 1.2 can be proven in a similar way to that in [1], we omit its proof here. Hence, to complete the proof of Theorem 1.2, we have only to prove the following claim.

**Claim:** If there exists a positive number \(\lambda_0\) such that problem (1.2) has at least two solution pairs for any \(\lambda \in (0, \lambda_0)\), then necessarily \(f(x), g(x) \in \mathcal{M}\).

Now, we turn to prove Claim 1. Let \((u^1_\lambda(x), v^1_\lambda(x))\) and \((u^2_\lambda(x), v^2_\lambda(x))\) be two distinct solution pairs with respect to parameter \(\lambda \in (0, \lambda_0)\). By Lemma 3.5, the \(L^\infty\)-norm of one of these two solutions tends to zero as \(\lambda \to 0\). Without loss of generality, we may assume that

\[
\lim_{\lambda \to 0} \left( \|u^1_\lambda\|_{L^\infty} + \|v^1_\lambda\|_{L^\infty} \right) = 0. \quad (3.6)
\]

Let

\[
\lambda U_\lambda = u^1_\lambda, \\
\lambda V_\lambda = v^1_\lambda
\]

Then \((U_\lambda, V_\lambda)\) satisfies

\[
\begin{align*}
-U''_\lambda &= (u^1_\lambda)^{p-1}V_\lambda + f(x) \\
-V''_\lambda &= (v^1_\lambda)^{q-1}U_\lambda + g(x) \\
U_\lambda(-\ell) &= U_\lambda(\ell) = 0 \\
V_\lambda(-\ell) &= V_\lambda(\ell) = 0
\end{align*}
\]

By (3.6), (3.7), Poincare’s inequalities and Hölder inequality, we can easily see that there exists a positive constant \(C\) independent of \(\lambda\) such that

\[
\int_{-\ell}^{\ell} |U''_\lambda|^2 + \int_{-\ell}^{\ell} |V''_\lambda|^2 \leq C, \quad \text{for } \lambda \text{ small enough.}
\]

Since

\[
U_\lambda(x) = \int_{-\ell}^{x} U'_\lambda dx,
\]

12
and
\[ V_\lambda(x) = \int_{-\ell}^{x} V'_{\lambda} \, dx, \]
we have
\[ |U_\lambda|_{L^\infty} \leq C_1 \text{ independent of } \lambda, \]
\[ |V_\lambda|_{L^\infty} \leq C_1 \text{ independent of } \lambda. \]

Passing to the limit in (3.7) we can find \((\varphi, \psi) \geq (0,0)\) such that \(\varphi\) and \(\psi\) are non-negative solutions of problem (1.6), that is \(f, g \in \mathcal{M}\). This completes the proof of Claim 1 and hence completes the proof of Theorem 1.2. \(\square\)

**Proof of Theorem 1.3** If problem (1.2) has at least three solution pairs \((u^1_\lambda, v^1_\lambda), (u^2_\lambda, v^2_\lambda), (u^3_\lambda, v^3_\lambda)\), then by Lemma 3.2 we know that there exists a positive number \(A > 0\) small enough such that at least two of the above solution pairs satisfy
\[ \lim_{\lambda \to 0} \left( \|u_\lambda\|_{L^\infty} + \|v_\lambda\|_{L^\infty} \right) \geq A > 0. \]
Now, by a similar argument to that used in the proof of Lemma 3.5, we can arrive at a contradiction. \(\square\)

**Proof of Theorem 1.4** Theorem 1.4 is a direct consequences of Theorem 1.2. It can be proven by contradiction by making use of Theorem 1.2. \(\square\)

### 4 Some Examples

In this section, we will give some examples of functions which ensure the solvability, or unsolvability of problem (1.6). At first, it is trivial to see that problem (1.6) is solvable for any \(0 \neq f(x) \geq 0\) and unsolvable for any \(0 \neq f(x) \leq 0\). Hence, by Theorem 1.2, 1.3 and 1.4, we have the following corollaries

**Corollary 4.1** If \(f(x), g(x) \geq 0\), then there exists a positive number \(\lambda_0\) such that problem (1.2) has exactly two solution pairs for any \(\lambda \in (0, \lambda_0)\).

**Corollary 4.2** If \(f(x) \geq 0\) and \(g(x) \leq 0\), or \(f(x) \leq 0\) and \(g(x) \geq 0\), then there exists a positive number \(\lambda_0\) such that problem (1.2) has at most one solution pairs for any \(\lambda \in (0, \lambda_0)\).
To find a nontrivial example of sign changing function which ensure the solvability of problem (1.6), we first let

\[ f_c(x) = \begin{cases} 1 & x \in [-2\ell, -c) \\ -1 & x \in [-c, c] \\ 1 & x \in (c, 2\ell] \end{cases} \]

and consider the following problem

\[ \begin{align*}
-\varphi''(x) &= f_c(x), & x &\in (-\ell, \ell) \\
\varphi(-\ell) &= \varphi(\ell) = 0.
\end{align*} \tag{4.1} \]

It is easy to check that the weak solution of the above problem is

\[ \varphi_c(x) = \begin{cases} \frac{1}{2}(x + \ell)(\ell - x - 4c) & x \in [-\ell, -c) \\
\frac{1}{2}(\ell - 4c) + c^2 + \frac{1}{2}x^2 & x \in [-c, c] \\
\frac{1}{2}(\ell - x)(\ell + x - 4c) & x \in (c, \ell] \end{cases} \]

Obviously, \( \varphi_c(x) > 0 \) when \( c \leq \frac{\ell}{4} \), and \( \varphi_c(x) \) changes sign when \( c > \frac{\ell}{4} \).

Next, we let \( f_c^\varepsilon(x) \) be the regularization of \( f_c(x) \), that is

\[ f_c^\varepsilon(x) = \varepsilon^{-1} \int_{-2\ell}^{2\ell} \rho \left( \frac{x - y}{\varepsilon} \right) f_c(y) dy \]

with

\[ \rho(x) = \begin{cases} \text{dexp}\left(\frac{1}{x^2 - 1}\right) & \text{for } |x| \leq 1 \\
0 & \text{for } |x| \geq 1 \end{cases} \]

and \( d \) being chosen so that \( \int_{-\infty}^{\infty} \rho(x) dx = 1 \). It is well known that for any interval \( I \subset (-2\ell, 2\ell) \) we have \( f_c^\varepsilon(x) \in C^\infty(I) \) and \( f_c^\varepsilon(x) \) converges uniformly on \( I \) to \( f_c(x) \). Hence, if we denote by \( \varphi_c^\varepsilon(x) \) the solution of problem (4.1) with \( f_c(x) \) being replaced by \( f_c^\varepsilon(x) \), then it is easy to prove that \( \varphi_c^\varepsilon(x) \) converges uniformly on \([-\ell, \ell]\) to \( \varphi_c(x) \). This implies that for \( \varepsilon \) small enough \( \varphi_c^\varepsilon(x) > 0 \) when \( c \leq \frac{\ell}{4} \), and \( \varphi_c^\varepsilon(x) \) changes sign when \( c > \frac{\ell}{4} \). Hence, problem (1.6) with \( f_c^\varepsilon(x) \) being replaced by \( f_c^\varepsilon(x) \) is solvable for \( c \leq \frac{\ell}{4} \), and is unsolvable for \( c > \frac{\ell}{4} \) when \( \varepsilon \) is small enough. Consequently, we have

**Corollary 4.3** For sufficiently small \( \varepsilon > 0 \), if \( f(x) = f_{c_1}^\varepsilon(x) \), \( g(x) = f_{c_2}^\varepsilon(x) \) in problem (1.2), and \( c_1, c_2 \leq \frac{\ell}{4} \), then there exists a positive number \( \lambda_0 \) such that problem (1.2) has exactly two solution pairs for any \( \lambda \in (0, \lambda_0) \).

**Corollary 4.4** For sufficiently small \( \varepsilon > 0 \), if \( f(x) = f_{c_1}^\varepsilon(x) \), \( g(x) = f_{c_2}^\varepsilon(x) \) in problem (1.2), and \( c_1 > \frac{\ell}{4} \), or \( c_2 > \frac{\ell}{4} \), then there exists a positive number \( \lambda_0 \) such that problem (1.2) has at most one solution pairs for any \( \lambda \in (0, \lambda_0) \).
References


