Topological transversality and boundary value problems on time scales

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Abstract

This work presents some results for the existence of solutions to boundary value problems on time scales. The ideas rely on the topological transversality of A. Granas.

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1. Introduction

This paper examines the existence of solutions to \( n \)-th-order dynamic equations on “time scales.”

The theory of time scales was introduced by Hilger [14] in an attempt to unify ideas from continuous and discrete calculus. For a function \( y(t) \), with \( t \) in a so-called time scale \( T \) (which is an arbitrary closed subset of \( \mathbb{R} \)), Hilger defined the generalized derivative \( y^\Delta(t) \). If the time scale is, say, \( \mathbb{R} \) then \( y^\Delta(t) = y'(t) \), the familiar derivative from calculus. If the time scale is, say, \( \mathbb{Z} \) then \( y^\Delta(t) = \Delta y(t) \), the familiar forward difference. (There are many more time scales than just \( \mathbb{R} \) and \( \mathbb{Z} \).)

From Hilger’s seminal work, dynamic equations on time scales of the form

\[
f(t, y, y^\Delta, \ldots, y^{\Delta^{n}}) = 0, \quad t \in T,\]

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have been researched, with the theory currently developing at a rapid rate (see [5,6] and references therein). Once again, if $T = \mathbb{R}$ then (1) is a differential equation, while if $T = \mathbb{Z}$ then (1) is a difference equation.

An important class of dynamic equations are boundary value problems (BVPs), due to their striking applications to almost all areas of science, engineering and technology. By researching BVPs on time scales the results unify the theory of differential and difference equations (and removes obscurity from both areas) and provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time.

The study of BVPs on time scales is still in its infancy, with the pioneering existence results to be found in [1,2,4,8,9]. These papers have relied on methods such as the Schauder fixed point theorem, the nonlinear alternative of Leray–Schauder or on disconjugacy to prove the existence of solutions to second-order BVPs on time scales subject to linear, separated boundary conditions.

In this paper we employ more recent methods due to Granas [11] to prove the existence of solutions. Granas’ method is commonly known as topological transversality and relies on the idea of an essential map. For BVPs involving ordinary differential equations, the method has been highly useful for proving the existence of solutions (see [12] and references therein and [7]) and this has been a major motivation in using topological transversality in the more general time scale setting.

Many difficulties occur when considering dynamic equations in the time scale setting. For example, basic tools from calculus such as Fermat’s theorem, Rolle’s theorem and the intermediate value theorem may not necessarily hold and fundamental concepts such as chain, product and quotient rules and certain smoothness properties all need to be modified.

The paper is organized as follows.

In Section 2 we introduce the necessary definitions associated with time scales and topological preliminaries.

In Section 3 we present some general existence theorems for solutions to \( n \)th-order BVPs on time scales by utilizing the topological transversality method. A key assumption of these general theorems is that all solutions to certain families of BVPs are bounded a priori.

In Section 4 sufficient (and easily verifiable) conditions (involving maximum principles on time scales) are formulated so that all solutions to the BVP under consideration will satisfy certain a priori bounds. The boundary conditions of these BVPs may be very general, including nonlinear and mixed variations.

In Section 5 we apply the results from Sections 3 and 4 to provide detailed existence results for a wide range of second-order BVPs on time scales.

We briefly remark that some of the results herein generalize the workings of Granas et al. [12] when considering the special case $T = \mathbb{R}$.

For more on the theory of time scales we refer the reader to the monographs [5,6,16] and references therein.

2. Preliminaries

To understand the idea of time scales some definitions are needed.
Definition. A time scale \( T \) is a nonempty closed subset of \( \mathbb{R} \).

Since a time scale may or may not be connected, the concept of a jump operator is required.

**Definition.** Define the forward (backward) jump operator \( \sigma(t) \) at \( t \) for \( t < \sup T \) (\( \rho(t) \) at \( t \) for \( t > \inf T \)) by
\[
\sigma(t) = \inf\{\tau > t : \tau \in T\} \quad (\rho(t) = \sup\{\tau < t : \tau \in T\}) \quad \text{for all } t \in T.
\]

For simplicity and clarity denote \( \sigma^2(t) = \sigma(\sigma(t)) \) and \( y^{\Delta}(t) = y(\sigma(t)) \).

Throughout this work the assumption is made that \( T \) has the topology that it inherits from the standard topology on the real numbers \( \mathbb{R} \). Also assume throughout that \( a < b \) are points in \( T \) with \([a, b] = \{t \in T : a \leq t \leq b \}\).

The jump operators \( \sigma \) and \( \rho \) allow the classification of points in a time scale in the following way: If \( \sigma(t) > t \) then call the point \( t \) right-scattered; while if \( \rho(t) < t \) then say \( t \) is left-scattered. If \( \sigma(t) = t \) then call the point \( t \) right-dense; while if \( \rho(t) = t \) then say \( t \) is left-dense.

If \( T \) has a left-scattered maximum at \( m \) then define \( T^k = T - \{m\} \). Otherwise \( T^k = T \).

**Definition.** Fix \( t \in T \) and let \( y : T \rightarrow \mathbb{R} \). Define \( y^{\Delta}(t) \) to be the number (if it exists) with the property that given \( \epsilon > 0 \) there is a neighbourhood \( U \) of \( t \) with
\[
\left| \left[ y(\sigma(t)) - y(s) \right] - y^{\Delta}(t)\left[ \sigma(t) - s \right] \right| < \epsilon \left| \sigma(t) - s \right| \quad \text{for all } s \in U.
\]

Call \( y^{\Delta}(t) \) the (delta) derivative of \( y(t) \) at the point \( t \).

The following theorem is due to Hilger [14].

**Theorem 1.** Assume that \( y : T \rightarrow \mathbb{R} \) and let \( t \in T^k \).

(i) If \( y \) is differentiable at \( t \) then \( y \) is continuous at \( t \).

(ii) If \( y \) is continuous at \( t \) and \( t \) is right-scattered then \( y \) is differentiable at \( t \) with
\[
y^{\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.
\]

(iii) If \( y \) is differentiable and \( t \) is right-dense then
\[
y^{\Delta}(t) = \lim_{s \to t^+} \frac{y(t) - y(s)}{t - s}.
\]

(iv) If \( y \) is differentiable at \( t \) then \( y(\sigma(t)) = y(t) + (\sigma(t) - t)y^{\Delta}(t) \).

**Definition.** Define \( f \in C_{rd}(T; \mathbb{R}) \) as right-dense continuous if at all \( t \in T \) then
\[
\lim_{s \to t^+} f(s) = f(t) \quad \text{at every right-dense point } t \in T,
\]
\[
\lim_{s \to t^-} f(s) \quad \text{exists and is finite at every left-dense point } t \in T.
\]
Some topological ideas are now introduced. For more detail we refer the reader to [12]. Let $D$ be a convex subset of a Banach space $E$ with $C \subset D$ being open in $D$.

**Definition.** We say a map $F$ has a fixed point if $F(y) = y$ for some $y$. Otherwise say $F$ is fixed point free.

**Definition.** A compact map $F : \bar{C} \to D$ that is fixed point free for all $y \in \partial C$ is called essential when all compact maps $G : \bar{C} \to D$ that agree with $F$ on $\partial C$ have a fixed point $y \in C$.

**Theorem 2.** Let $p$ be an arbitrary point in $C$ and $F : \bar{C} \to D$ be the constant map $F(y) = p$ (that is fixed point free on $\partial C$). Then $F$ is essential.

**Theorem 3.** Let $C$, $D$ and $E$ be as above. Suppose:

(i) $H_{\lambda}(y)$ is a compact map $H : \bar{C} \times [0, 1] \to D$ for each $\lambda \in [0, 1]$;

(ii) $H_0(y)$ is essential;

(iii) $H_{\lambda}(y)$ is fixed point free on $\partial C$ for all $\lambda \in [0, 1]$.

Then there exists at least one $y \in C$ satisfying $y = H_{\lambda}(y)$ for all $\lambda \in [0, 1]$. In particular, $H_1(y)$ has at least one fixed point $y \in C$.

3. General existence theorems

Consider the following general $n$th-order BVP on a time scale $\mathbb{T}$:

\[(Ly)(t) = f(t, y^{\sigma}(t)), \quad t \in [a, b],\]  \hspace{1cm} (2)

\[U_i(y) = \sum_{j=0}^{n-1} [a_{ij} y^{\Delta^j}(a) + b_{ij} y^{\Delta^j}(\sigma^{n-j}(b))] = 0, \quad i = 1, \ldots, n,\]  \hspace{1cm} (3)

where

\[(Ly)(t) = \sum_{j=1}^{n} a_j(t) y^{\Delta^j}(t) + a_0(t)y^{\sigma}(t), \quad a_n(t) \neq 0 \text{ for } t \in [a, b],\]

and each $a_j$ and $f$ are continuous, and $y \in B$ denotes the set of linear, homogeneous boundary conditions (3).

Now consider the family of BVPs

\[(My)(t) = g(t, y(\sigma(t)), \lambda), \quad t \in [a, b], \lambda \in [0, 1],\]  \hspace{1cm} (4)

\[y \in B,\]  \hspace{1cm} (5)

where

\[(My)(t) = \sum_{j=1}^{n} b_j(t) y^{\Delta^j}(t) + b_0(t)y^{\sigma}(t), \quad b_n(t) \neq 0 \text{ for } t \in [a, b],\]
and each $b_j$ and $g$ are continuous with $g(t, y'(t), 0) = 0$.

**Definition.** Let $C^n_{rd}([a, b])$ denote the space of functions

$$C^n_{rd}([a, b]) = \{ y : y \in C([a, \sigma(b)]), \ldots, y^{n-1} \in C([a, \sigma(b)]), y^n \in C_{rd}([a, b]) \},$$

and equip this space with the norm $\| \cdot \|_n$ by

$$\| y \|_n = \max \left\{ \sup_{t \in [a, \sigma_n(b)]} |y(t)|, \ldots, \sup_{t \in [a, b]} |y^n(t)| \right\}.$$  

A solution to the BVP (2), (3) (or to the BVP (4), (5)) is a function $y \in C^n_{rd}([a, b])$ that satisfies the given BVP.

**Theorem 4.** Let $L$, $M$, $f$ and $g$ be as above. Assume:

(i) Problems (2), (3) and (4), (5) are equivalent when $\lambda = 1$;
(ii) The operator $(M, B)$ is invertible as a map from $C^n_{rd} \rightarrow C_{rd}$;
(iii) There is a constant $P > 0$, independent of $\lambda$, such that all solutions to (4), (5) satisfy

$$\| y \|_n < P$$

for all $\lambda \in [0, 1]$.

Then the BVP (2), (3) has at least one solution.

**Proof.** We will use Theorem 3. Define

$$S_B = \{ y : y \in C^n_{rd}([a, b]), y \in B \},$$

and see that $S_B$ is a convex subset of the Banach space $C^n_{rd}([a, b])$. Now let

$$K_P = \{ y : y \in S_B, \| y \|_n < P \}.$$

Define $T_\lambda : C_{rd} \rightarrow C_{rd}$ for $\lambda \in [0, 1]$ by

$$(T_\lambda v)(t) = g(t, v(t), \lambda).$$

Now let $k : S_B \rightarrow C_{rd}$ be the completely continuous embedding of $S_B$ into $C_{rd}$. Then

$$H_\lambda = M^{-1} T_\lambda k$$

defines a mapping $H_\lambda : K_P \rightarrow S_B$. We see that fixed points of $H_\lambda$ correspond to solutions of (4), (5). By (iii) we have that $H_\lambda$ is fixed point free on $\partial K_P$. Assumption (ii) and the complete continuity of $k$ imply that $H_\lambda$ is compact. Since $H_0 = 0 \in K_P$ we have that $H_0$ is essential. Thus by Theorem 3, $H_1$ has a fixed point $y \in K_P$ and by (i), this is a solution to (2), (3). This concludes the proof.

Now consider the general $n$th-order BVP subject to inhomogeneous boundary conditions in the following form:

$$(Ly)(t) = f(t, y'(t)), \quad t \in [a, b],$$

$$U_i(y) = r_i, \quad i = 1, \ldots, n,$$

where $L$, $f$ and $U_i$ are as above and $r_i$ are constants.
Consider the corresponding family of BVPs
\[(My)(t) = g(t, y^{(n)}(t), \lambda), \quad t \in [a, b], \quad \lambda \in [0, 1], \tag{8}\]
\[U_i(y) = r_i, \quad i = 1, \ldots, n, \tag{9}\]
where \(M\) and \(g\) are as above.

Arguing similar to Theorem 4 we can obtain the following result.

**Theorem 5.** Let \(L, M, f\) and \(g\) be as above. Assume:

(i) Problems (6), (7) and (8), (9) are equivalent when \(\lambda = 1;\)
(ii) The operator \((M, B)\) is invertible as a map from \(C_n^{rd} \to C_{rd};\)
(iii) There is a constant \(M > 0,\) independent of \(\lambda,\) such that all solutions to (8), (9) satisfy \(\|y\|_n < M\) for all \(\lambda \in [0, 1].\)

Then the BVP (6), (7) has at least one solution.

Next, consider the general \(n\)th-order BVP subject to nonlinear boundary conditions in the following form:
\[(Ly)(t) = f(t, y^{(n)}(t)), \quad t \in [a, b], \tag{10}\]
\[U_i(y) = V_i(y), \quad i = 1, \ldots, n, \tag{11}\]
where \(L, f\) and \(U_i\) are as above, and
\[V_i(y) = \phi_i(y(a), \ldots, y^{\Delta^{n-1}}(a), y^{(n)}(\sigma(b)), y^\Delta(\sigma^{n-1}(b)), \ldots, y^{\Delta^{n-1}}(\sigma(b)))\]
with each \(\phi_i\) continuous. Here the boundary conditions \(V_i\) may be nonlinear and mixed. Again the approach is to compare the BVP (10), (11) with the family of BVPs
\[(My)(t) = g(t, y^{(n)}(t), \lambda), \quad t \in [a, b], \quad \lambda \in [0, 1], \tag{12}\]
\[U_i(y) = W_i(y, \lambda), \quad i = 1, \ldots, n, \tag{13}\]
where \(M\) and \(g\) are as above and
\[W_i(y, \lambda) = \psi_i(y(a), \ldots, y^{\Delta^{n-1}}(a), y^{(n)}(\sigma(b)), y^\Delta(\sigma^{n-1}(b)), \ldots, y^{\Delta^{n-1}}(\sigma(b)), \lambda)\]
with each \(\psi_i\) continuous and \(W_i(y, 0) = 0.\) By modifying the space of solutions and under similar assumptions as in the treatment of the earlier BVP (4), (5) we obtain the following result.

**Theorem 6.** Let \(L, M, f, g, U_i, V_i\) and \(W_i\) be as above. Assume:

(i) Problems (10), (11) and (12), (13) are equivalent when \(\lambda = 1;\)
(ii) The operator \((M, B)\) is invertible as a map from \(C_n^{rd} \to C_{rd};\)
(iii) There is a constant \(P > 0,\) independent of \(\lambda,\) such that all solutions to (12), (13) satisfy \(\|y\|_n < P\) for all \(\lambda \in [0, 1].\)

Then the BVP (10), (11) has at least one solution.
**Proof.** The proof is similar to that of Theorem 4 with suitable modifications in certain spaces needed.

Let

\[ KP = \{ y : y \in \text{Crd}[a, \sigma^2(b)], \|y\|_n < P \}. \]

Define \( T_\lambda : \text{Crd} \rightarrow \text{Crd} \times \mathbb{R}^n \) for \( \lambda \in [0, 1] \) by

\[ (T_\lambda v)(t) = (g(t, v(t), \lambda), W_1(v, \lambda), \ldots, W_n(v, \lambda)). \]

Now define \( M_1 : \text{Crd} \rightarrow \text{Crd} \times \mathbb{R}^n \) by

\[ (M_1 y)(t) = ((My)(t), U_1(y), \ldots, U_n(y)). \]

and, as previously, let \( k : \text{Crd} \rightarrow \text{Crd} \) be the completely continuous embedding of \( \text{Crd} \) into \( \text{Crd} \). By (ii), \( M_1 \) is a continuous, linear, one-to-one map of \( \text{Crd} \) onto \( \text{Crd} \times \mathbb{R}^n \) and therefore has a continuous inverse \( M_1^{-1} \). Now let

\[ H_\lambda = M_1^{-1} T_\lambda k, \]

and we obtain a mapping \( H_\lambda : K_P \rightarrow \text{Crd} \). By (iii) we have that \( H_\lambda \) is fixed point free on \( \partial K_P \). Assumption (ii) and the complete continuity of \( k \) imply that \( H_\lambda \) is compact. By using Theorem 3 the result follows. This concludes the proof. \( \Box \)

4. A priori bounds on solutions

The following maximum principle shall be very useful throughout the rest of the paper and can be found in [10] (see also [4]).

**Lemma 1.** If a function \( r : T \rightarrow \mathbb{R} \) has a local maximum at a point \( c \in [a, \sigma^2(b)] \) then \( r^{\Delta}(c) \leq 0 \) and

\[ r^{\Delta^2}(\rho(c)) \leq 0, \]  

provided \( c \) is not simultaneously left-dense and right-scattered and that \( r^{\Delta}(c) \) and \( r^{\Delta^2}(\rho(c)) \) both exist.

Consider the second-order dynamic equation

\[ y^{\Delta^2}(t) = f(t, y^\sigma), \quad t \in [a, b], \]  

subject to some prescribed boundary conditions, where \( f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R} \).

**Assumption A.** Let \( R > 0 \) be a constant such that (15) satisfies

\[ y^\sigma f(t, y^\sigma) > 0 \quad \text{for all } t \in [a, b] \text{ and } |y^\sigma| \geq R. \]  

**Remark.** Notice that if Assumption A holds then both \( \lambda f(t, y^\sigma) \) and

\[ \lambda f(t, y^\sigma) + (1 - \lambda)y^\sigma \]  

satisfy (16) for all \( \lambda \in (0, 1] \).
Lemma 2. Let Assumption A hold. If $y$ is a solution to (15) and $|y|$ does not achieve its maximum value on $[a, \sigma^2(b)]$ at $t = a$ or at $t = \sigma^2(b)$ then $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. Assume that the conclusion of the lemma is false. In particular, assume that $y(c) \geq R$ for some $c \in [a, \sigma^2(b)]$ (the case for $y(c) \leq -R$ for some $c \in [a, \sigma^2(b)]$ is handled similar to the following). Since we have a continuous function $r(t) = y(t) - R$, it must have a nonnegative maximum in $[a, \sigma^2(b)]$. By hypothesis, this maximum must occur in $(a, \sigma^2(b))$. Choose $c \in (a, \sigma^2(b))$ such that $r(c) = \max\{r(t), t \in [a, \sigma^2(b)]\} \geq 0$ and $r(t) < r(c)$ for $c < t < \sigma^2(b)$.

First show that the point $c$ cannot be simultaneously left-dense and right-scattered. Assume the contrary by letting $\rho(c) = c < \sigma(c)$. If $r^\Delta(c) \geq 0$ then $r(\sigma(c)) \geq r(c)$ and this contradicts (18). If $r^\Delta(c) < 0$ then

$$\lim_{t \to c^-} r^\Delta(t) = r^\Delta(c) < 0.$$  

Therefore there exists $\delta > 0$ such that $r^\Delta(t) < 0$ on $(c - \delta, c]$. Hence $r(t)$ is strictly decreasing on $(c - \delta, c]$ and this contradicts the way $c$ was chosen.

Therefore the point $c$ cannot be simultaneously left-dense and right-scattered.

By Lemma 1 we must have $r^\Delta(c) \leq 0$ and

$$r^\Delta^2(\rho(c)) \leq 0.$$  

We have

$$r^\Delta^2(\rho(c)) = f(\rho(c), y^\Delta(\rho(c))) > 0$$  

by (16),

which contradicts (19). Therefore $y(t) < R$ for $t \in [a, \sigma^2(b)]$. (Notice at $c$ that $y^\sigma(\rho(c)) = y(c) \geq R$, since $c$ is not simultaneously left-dense and right-scattered.) The case showing $y(t) \geq -R$ for all $t \in [a, \sigma^2(b)]$ follows similarly. This concludes the proof. □

Lemma 3. Let Assumption A hold. Then every solution $y$ to the homogeneous Dirichlet BVP (15),

$$y(a) = 0 = y(\sigma^2(b)), \quad (20)$$

satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. Immediate from Lemma 2. □

Lemma 4. Let Assumption A hold. Then every solution $y$ to the homogeneous Neumann BVP (15),

$$y^\Delta(a) = 0 = y^\Delta(\sigma(b)), \quad (21)$$

satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$. 

Proof. Let $y$ be a solution such that $|y(a)|$ is the maximum value of $|y|$. If $|y(a)| \geq R$ then (16) gives
\[ y^\sigma(a)f(a, y^\sigma(a)) = y^\sigma(a)y^A(a) > 0. \] (22)

If $a$ is right-scattered then the boundary conditions give $y^A(a) = 0$ with $y^\sigma(a) = y(a)$. We show that $|y^\sigma(a)|$ cannot be the maximum value of $|y|$. If $y(a) = y^\sigma(a) \geq R$ then (22) gives $y^A(a) = y^A(\rho(\sigma(a))) > 0$ and thus $y$ cannot have a maximum at $\sigma(a)$. Thus $y$ cannot have a maximum at $a$. If $y(a) \leq -R$ then $y(a)y^A(a) < 0$ and a similar argument shows that $y^\sigma(a) = y(a)$ is not the minimum value of $|y|$ on $[a, \sigma^2(b)]$, a contradiction.

Now if $a$ is right-dense then (22) becomes
\[ y(a)f(a, y(a)) = y(a)y''(a) > 0. \] (23)

If $y(a) \geq R$ then $y''(a) > 0$, so $y'(t)$ is strictly increasing near $t = a$. Then $y'(t) > y'(a) = 0$ for $t > a$ near zero, so $y$ is strictly increasing near zero. If $y(a) \leq -R$ then we conclude that $y$ is strictly decreasing near zero. Hence $|y(a)|$ is not the maximum value of $|y|$ on $[a, \sigma^2(b)]$, a contradiction.

Arguing for the case when $y$ is a solution such that $|y(b)|$ is the maximum value of $|y|$ follows in a similar manner to the above. Since Assumption A holds, all of the conditions of Lemma 2 hold and the result follows. \[\Box\]

Lemma 5. Let Assumption A hold. Then every solution $y$ to the periodic BVP (15),
\[ y(a) = y(\sigma^2(b)), \quad y^A(a) = y^A(\sigma(b)), \] (24)
satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. If $y^A(a) = y^A(\sigma(b)) \neq 0$, then it follows from $y(a) = y(\sigma^2(b))$ that $|y|$ cannot achieve its maximum at $t = a$ or $t = \sigma^2(b)$, and since Assumption A holds, the result follows from Lemma 2. If $y^A(a) = y^A(\sigma(b)) = 0$ then $y$ satisfies the homogeneous Neumann boundary conditions and from Lemma 4, $|y(a)|, |y(\sigma^2(b))| \leq R$. Since Assumption A holds, the result follows from Lemma 4. \[\Box\]

Lemma 6. Let Assumption A hold. If $\alpha, \beta, \gamma, \delta$ are nonnegative constants (with $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$) then every solution $y$ to the homogeneous Sturm–Liouville BVP (15),
\[ -\alpha y(a) + \beta y^A(a) = 0 = \gamma y(\sigma^2(b)) + \delta y^A(\sigma(b)), \] (25)
satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. If $|y(t)|$ has a maximum at $t = a$, then $0 \geq y(a)y^A(a)$ and using (25) we obtain
\[ 0 \geq y(a)\beta y^A(a) = \alpha[y(a)]^2 > 0, \]
a contradiction for $\alpha$ and $\beta$ both nonzero. If $\alpha$ or $\beta$ are zero then we obtain Neumann and Dirichlet boundary conditions which we treated in Lemmas 3 and 4. In any case, $|y(t)|$ does
not have its maximum at \( t = a \). Similarly, if \( |y(\sigma^2(b))| \) is the maximum value of \( |y(t)| \), then \( y(\sigma^2(b))y^\Delta(\sigma(b)) \geq 0 \) and using the boundary conditions we reach a contradiction. Since Assumption A holds, \( |y(t)| < R \) by Lemma 2. This concludes the proof. □

Note that Lemma 6 provides a priori bounds on solutions to (15) subject to homogeneous Nicolletti boundary conditions
\[
y^\Delta(a) = 0 = \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)),
\]
or subject to the homogeneous Cordineanu boundary conditions
\[
-\alpha y(a) + \beta y^\Delta(a) = 0 = y^\Delta(\sigma(b)).
\]

The question now arises on how to ensure a priori bounds on solutions when \( |y(t)| \) does achieve its maximum value on \([a, \sigma^2(b)]\) at \( t = a \) or at \( t = \sigma^2(b) \)? This is particularly relevant when considering BVPs with inhomogeneous boundary conditions.

**Lemma 7.** Let Assumption A hold and let \( M \geq 0 \) be a constant. If \( y \) is a solution to the inhomogeneous Dirichlet BVP (15),
\[
y(a) = A, \quad y(\sigma^2(b)) = B
\]
with \( \max\{|y(a)|, |y(\sigma^2(b))|\} \leq M \), then
\[
|y(t)| < \max\{M, R\} + 1 \quad \text{for all } t \in [a, \sigma^2(b)].
\]

**Proof.** See that \( \max\{|y(a)|, |y(\sigma^2(b))|\} < M + 1 \) and since Assumption A holds, the result easily follows. □

**Lemma 8.** Let Assumption A hold. If \( \alpha, \gamma > 0 \) and \( \beta, \delta \geq 0 \), then every solution \( y \) to the Sturm–Liouville BVP (15),
\[
-\alpha y(a) + \beta y^\Delta(a) = u, \quad \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) = v, \tag{27}
\]
satisfies
\[
|y(t)| < \max\{|u|/\alpha, |v|/\gamma, R\} + 1 \quad \text{for all } t \in [a, \sigma^2(b)].
\]

**Proof.** If \( |y(t)| \) has a maximum at \( t = a \), then \( 0 \geq y(a)y^\Delta(a) \) and using the boundary conditions, we obtain
\[
0 \geq y(a)y^\Delta(a) = a\left[\frac{u}{\alpha y(a)} - 1\right],
\]
and rearranging we get \( |y(a)| \leq |u|/\alpha \).

Similarly, if \( |y(t)| \) has a maximum at \( t = \sigma^2(b) \) then \( y(\sigma^2(b))y^\Delta(\sigma(b)) \geq 0 \) and using the boundary conditions we obtain \( |y(\sigma^2(b))| \leq |v|/\gamma \). Since Assumption A holds we have \( |y(t)| < R \) for all \( t \in (a, \sigma^2(b)) \) and the result follows. This concludes the proof. □

By piecing together parts of the previous lemmas we have the following results. The proofs are omitted for brevity.
Lemma 9. Let Assumption A hold. If $\gamma > 0$ and $\delta \geq 0$, then every solution $y$ to the Nicolletti BVP (15),

$$y^\Delta(a) = 0, \quad \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) = v,$$

satisfies

$$|y(t)| < \max\{|y(0)/\gamma, R\} + 1 \quad \text{for all } t \in [a, \sigma^2(b)].$$

Lemma 10. Let Assumption A hold. If $\alpha > 0$ and $\beta \geq 0$, then every solution $y$ to the Cordineanu BVP (15),

$$-\alpha y(a) + \beta y^\Delta(a) = u, \quad y^\Delta(\sigma(b)) = 0,$$

satisfies

$$|y(t)| < \max\{|u|/\alpha, R\} + 1 \quad \text{for all } t \in [a, \sigma^2(b)].$$

The above results will be needed for applications at the end of the paper. We now turn our attention to BVPs with nonlinear boundary conditions.

Lemma 11. Let Assumption A hold. If the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy

$$r \phi(r) > 0, \quad r \psi(r) < 0 \quad \text{for all } r \neq 0,$$

then every solution $y$ to the BVP (15),

$$y^\Delta(a) = \phi(y(a)), \quad y^\Delta(\sigma(b)) = \phi(y(\sigma^2(b))),$$

satisfies

$$|y(t)| < R \quad \text{for all } t \in [a, \sigma^2(b)].$$

Proof. Assume $|y(c)| \geq R$ for some $c \in [a, \sigma^2(b)]$. If $|y(a)|$ is the maximum value of $|y(t)|$ on $[a, \sigma^2(b)]$ then $0 \geq y(a)y^\Delta(a)$. Using the boundary conditions we obtain

$$0 \geq y(a)y^\Delta(a) = y(a)\phi(y(a)) > 0,$$

a contradiction. Similarly, if $|y(\sigma^2(t))|$ is the maximum value of $|y(t)|$ on $[a, \sigma^2(b)]$, then $0 \leq y(\sigma^2(b))y^\Delta(\sigma(b))$ and using the boundary conditions leads to another contradiction. Therefore $|y(t)|$ does not have a maximum at $t = a$ or at $t = \sigma^2(b)$. Since Assumption A holds the result follows from Lemma 2. This concludes the proof. □

Lemma 12. Let Assumption A hold and let $M \geq 0$ be a constant. If the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\max\{\phi(0), |\psi(0)|\} = M, \quad r \phi(r) > 0, \quad r \psi(r) < 0 \quad \text{for all } r \neq 0,$$

then every solution $y$ to the BVP (15),

$$y(a) = \phi(y^\Delta(a)), \quad y(\sigma^2(b)) = \phi(y^\Delta(\sigma(b))),$$

satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$. 

Proof. Arguing as in the proof of Lemma 11, if $|y(a)|$ is the maximum value of $|y(t)|$ on $[a, \sigma^2(b)]$ then $0 \geq y(a)y^\Delta(a)$. Using the boundary conditions we obtain

$$0 \geq y(a)y^\Delta(a) = y(a)\phi(y(a)) > 0 \quad \text{for } y^\Delta(a) \neq 0,$$

a contradiction. If $y^\Delta(a) = 0$ then $|y(a)| = |\phi(0)| = M$ and the bound follows. Similar methods give the a priori bounds on $|y(\sigma^2(b))|$. Since Assumption A holds, the result follows from Lemma 2. This concludes the proof. \(\square\)

Lemma 13. Let Assumption A hold. If functions $\phi, \psi : \mathbb{R}^4 \to \mathbb{R}$ satisfy

$$r\phi(r, r_1, r_2, r_3) > 0, \quad r\psi(r, r_1, r_2, r_3) < 0 \quad \text{for } r \geq R, r_1, r_2, r_3,$$

then every solution $y$ to the BVP (15),

$$y^\Delta(a) = \phi(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))), \quad (31)$$

$$y^\Delta(\sigma(b)) = \psi(y(a), y^\Delta(a), y(\sigma^2(b)), y^\Delta(\sigma(b))), \quad (32)$$

satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. Similar to that of Lemma 11. \(\square\)

Lemma 14. Let Assumption A hold. If functions $\phi, \psi : \mathbb{R}^4 \to \mathbb{R}$ are continuous and satisfy

$$r\phi(r, r_1, r_2, r_3) > 0, \quad r\psi(r, r_1, r_2, r_3) < 0 \quad \text{for } r \neq R, r_1, r_2, r_3,$$

then every solution $y$ to the BVP (15),

$$y(a) = \phi(y^\Delta(a), y(a), y(\sigma^2(b)), y^\Delta(\sigma(b))), \quad (33)$$

$$y(\sigma^2(b)) = \phi(y^\Delta(\sigma(b)), y(\sigma^2(b)), y^\Delta(a), y(a)), \quad (34)$$

satisfies $|y(t)| < R$ for all $t \in [a, \sigma^2(b)]$.

Proof. Similar to that of Lemma 12. The continuity of $\phi$ and $\psi$ ensure that

$$\phi(0, r_1, r_2, r_3) = 0 = \psi(0, r_1, r_2, r_3). \quad \square$$

Remark 1. Since $f$ in (15) does not depend on $y^\Delta$, once we have obtained a priori bounds on $|y|$, then a priori bounds for $|y^\Delta|$ and $|y^{\Delta^2}|$ follow immediately.

5. Existence of solutions for $n = 2$

In this section some existence results are presented for second-order BVPs. The proofs rely on the a priori bounds on solutions of Section 4 and the general existence theorems of Section 3.

Assumption B. Let $f(t, p)$ be a continuous map from $[a, \sigma^2(b)] \times \mathbb{R}$ to $\mathbb{R}$.
Corollary 1. Let the conditions of Lemma 4 and Assumption B hold. Then the homogeneous Neumann BVP (15), (20) has at least one solution.

Proof. Consider
\[ g(t, y^\sigma, \lambda) = \lambda \left( f(t, y^\sigma) - y^\sigma \right) \] for \( \lambda \in [0, 1] \),
and \((My)(t) = y^{\Delta^2}(t) - y^\sigma(t)\). Now since
\[ (My)(t) = 0, \quad y^\Delta(a) = 0 = y^\Delta(b) \]
has only the zero solution, \((M, B)\) is one-to-one. Now consider the family of BVPs
\[ (My)(t) = y^{\Delta^2}(t) - y^\sigma(t) = \lambda \left[ f(t, y^\sigma) - y^\sigma \right] = g(t, y^\sigma, \lambda), \quad (35) \]
\[ y^\Delta(a) = 0 = y^\Delta(b), \quad (36) \]
and the challenge remains to show all solutions to (35), (36) are bounded a priori. Firstly, if \( \lambda = 0 \) then the solution to the BVP is the zero solution, so assume \( \lambda \in (0, 1] \). Rearranging (35) we obtain
\[ y^{\Delta^2}(t) = \lambda f(t, y^\sigma) + (1 - \lambda)y^\sigma(t), \]
and this satisfies Assumption A. The family of BVPs satisfy all of the conditions of Lemma 4 and thus all solutions to (35), (36) satisfy \(|y(t)| < R\) for all \( t \in [a, \sigma^2(b)] \) with \( R \) independent of \( \lambda \). Therefore all the conditions of Theorem 4 are satisfied and we conclude that the homogeneous Neumann problem has a solution. This concludes the proof. \( \square \)

Corollary 2. Let the conditions of Lemma 5 and Assumption B hold. Then the periodic BVP (15), (24) has at least one solution.

Proof. Again, consider
\[ g(t, y^\sigma, \lambda) = \lambda \left( f(t, y^\sigma) - y^\sigma \right) \] for \( \lambda \in [0, 1] \),
and \((My)(t) = y^{\Delta^2}(t) - y^\sigma(t)\). Now since
\[ (My)(t) = 0, \quad y(a) = y(\sigma^2(b)), \quad y^\Delta(a) = y^\Delta(b) \]
has only the zero solution, \((M, B)\) is one-to-one. Now consider the family of BVPs
\[ (My)(t) = y^{\Delta^2}(t) - y^\sigma(t) = \lambda \left[ f(t, y^\sigma) - y^\sigma \right] = g(t, y^\sigma, \lambda), \quad (37) \]
\[ y(a) = y(\sigma^2(b)), \quad y^\Delta(a) = y^\Delta(b), \quad (38) \]
and the challenge remains to show all solutions to (37), (38) are bounded a priori. Firstly, if \( \lambda = 0 \) then the solution to the BVP is the zero solution, so assume \( \lambda \in (0, 1] \). Rearranging (37) we obtain
\[ y^{\Delta^2}(t) = \lambda f(t, y^\sigma) + (1 - \lambda)y^\sigma(t), \]
and this satisfies Assumption A. The family of BVPs satisfy all of the conditions of Lemma 5 and thus all solutions to (37), (38) satisfy \(|y(t)| < R\) for all \( t \in [a, \sigma^2(b)] \) with \( R \) independent of \( \lambda \). Therefore all the conditions of Theorem 4 are satisfied and we conclude that the periodic problem has a solution. This concludes the proof. \( \square \)
Corollary 3. Let the conditions of Lemma 8 and Assumption B hold. Then the inhomogeneous Sturm–Liouville BVP (15), (27) has at least one solution.

Proof. Consider
\[ g(t, y^\sigma, \lambda) = \lambda f(t, y^\sigma) \quad \text{for } \lambda \in [0, 1], \]
and \((M) (t) = y^{\Delta^2}(t)\). Now since
\[ (M) (t) = 0, \quad -\alpha y(a) + \beta y^\Delta(a) = 0 = \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) \]
has only the zero solution, \((M, B)\) is one-to-one. Now consider the family of BVPs
\[ (My)(t) = y^{\Delta^2}(t) = \lambda f(t, y^\sigma) = g(t, y^\sigma, \lambda), \quad \text{(39)} \]
\[ -\alpha y(a) + \beta y^\Delta(a) = u, \quad \text{(40)} \]
\[ \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) = v, \quad \text{(41)} \]
and the challenge remains to show all solutions to (39)–(41) are bounded a priori. Firstly, if \(\lambda = 0\) then the solution to the BVP is \(y = Ct + D\), so assume \(\lambda \in (0, 1]\). We see that \(\lambda f(t, y^\sigma)\) satisfies Assumption A. The family of BVPs satisfy all of the conditions of Lemma 8 and thus all solutions to (39)–(41) satisfy \(|y(t)| < M\) for all \(t \in [a, \sigma^2(b)]\) with \(M\) independent of \(\lambda\). Therefore all the conditions of Theorem 5 are satisfied and we conclude that the Sturm–Liouville problem has a solution. \(\square\)

Corollary 4. Let the conditions of Lemma 11 and Assumption B hold. In addition, assume \(\phi\) and \(\psi\) are continuous. Then the BVP (15), (29) has at least one solution.

Proof. Consider
\[ g(t, y^\sigma, \lambda) = \lambda \left(f(t, y^\sigma) - y^\sigma\right) \quad \text{for } \lambda \in [0, 1], \]
and \((M) (t) = y^{\Delta^2}(t) - y^\sigma(t)\). Now since
\[ (M) (t) = 0, \quad y^\Delta(a) = 0 = y^\Delta(b) \]
has only the zero solution, \((M, B)\) is one-to-one. Now consider the family of BVPs
\[ (My)(t) = y^{\Delta^2}(t) - y^\sigma(t) = \lambda \left[f(t, y^\sigma) - y^\sigma\right] = g(t, y^\sigma, \lambda), \quad \text{(42)} \]
\[ y^\Delta(a) = \lambda \phi(y(a)), \quad y^\Delta(b) = \lambda \psi(y(\sigma^2(b))), \quad \text{(43)} \]
and the challenge remains to show all solutions to (42), (43) are bounded a priori. Firstly, if \(\lambda = 0\) then the solution to the BVP is the zero solution, so assume \(\lambda \in (0, 1]\). Rearranging (42) we obtain
\[ y^{\Delta^2}(t) = \lambda f(t, y^\sigma) + (1 - \lambda)y^\sigma. \]
The family of BVPs satisfy all of the conditions of Lemma 11 and all solutions to (42), (43) satisfy \(|y(t)| < R\) for all \(t \in [a, \sigma^2(b)]\) with \(R\) independent of \(\lambda\). Therefore all the conditions of Theorem 6 are satisfied and we conclude that the problem has a solution. This concludes the proof. \(\square\)
Corollary 5. Let the conditions of Lemma 12 and Assumption B hold. In addition, assume \( \phi \) and \( \psi \) are continuous. Then the BVP (15), (30) has at least one solution.

Proof. Since \( \phi \) and \( \psi \) are continuous, we see that \( \phi(0) = 0 = \psi(0) \). Consider

\[
g(t, y^\sigma, \lambda) = \lambda f(t, y^\sigma) \quad \text{for } \lambda \in [0, 1],
\]

and \((My)(t) = y^{\Delta^2}(t)\). Now since

\[
(My)(t) = 0, \quad y(a) = y(\sigma^2(b))
\]

has only the zero solution, \((M, B)\) is one-to-one. Now consider the family of BVPs

\[
(My)(t) = y^{\Delta^2}(t) = \lambda f(t, y^\sigma),
\]

\[
y(a) = \lambda \phi(y^{\Delta}(a)), \quad y(\sigma^2(b)) = \lambda \psi(y^{\Delta}(\sigma(b))),
\]

and the challenge remains to show all solutions to (44), (45) are bounded a priori. Firstly, if \( \lambda = 0 \) then the solution to the BVP is the zero solution, so assume \( \lambda \in (0, 1] \). Rearranging (44) we obtain

\[
y^{\Delta^2}(t) = \lambda f(t, y^\sigma) + (1 - \lambda)y^\sigma.
\]

Thus family of BVPs satisfy all of the conditions of Lemma 11 and all solutions to (44), (45) satisfy \( |y(t)| < R \) for all \( t \in [a, \sigma^2(b)] \) with \( R \) independent of \( \lambda \). Therefore all the conditions of Theorem 6 are satisfied and we conclude that the problem has a solution. This concludes the proof.

Corollary 6. Let the conditions of Lemma 13 and Assumption B hold. In addition, assume \( \phi \) and \( \psi \) are continuous. Then the BVP (15), (31), (32) has at least one solution.

Proof. Similar to that of Corollary 4.

Corollary 7. Let the conditions of Lemma 14 and Assumption B hold. In addition, assume \( \phi \) and \( \psi \) are continuous. Then the BVP (15), (33), (34) has at least one solution.

Proof. Similar to that of Corollary 5.

6. Application

Consider the Nicolletti BVP [17] arising during the analysis of heat and mass transfer in a porous catalyst, given by

\[
y^{\Delta^2}(t) = py(t) \exp\left[\frac{qr(1 - y^\sigma(t))}{1 + r(1 - y^\sigma(t))}\right], \quad t \in [a, b],
\]

\[
y^{\Delta}(a) = 0, \quad y(\sigma^2(b)) = 1,
\]

where \( p, q \) and \( r \) are positive constants relating to Thiele’s modulus, dimensionless energy of activation and heat evolution, respectively. We claim that the BVP has a solution \( y \) satisfying \( |y(t)| < 1 + 1/2r \).
We see that $f(t, p)$ is a continuous map from $[a, b] \times [-1 - 1/2r, 1 + 1/2r]$ to $\mathbb{R}$.

In addition, we see that Assumption A holds for $R = 1 + 1/2r$. The result follows by an application of Lemma 9 and Theorem 5.

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**References**