Difference Equations in Banach Spaces

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Abstract—Difference equations which discretely approximate boundary value problems for second-order ordinary differential equations are analysed. It is well known that the existence of solutions to the continuous problem does not necessarily imply existence of solutions to the discrete problem and, even if solutions to the discrete problem are guaranteed, they may be unrelated and inapplicable to the continuous problem.

Analogues to theorems for the continuous problem regarding a priori bounds and existence of solutions are formulated for the discrete problem. Solutions to the discrete problem are shown to converge to solutions of the continuous problem in an aggregate sense.

An example which arises in the study of the finite deflections of an elastic string under a transverse load is investigated. The earlier results are applied to show the existence of a solution; the sufficient estimates on the step size are presented. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $E$ be a Banach space with norm $\| \cdot \|$. Let $A, B \in E$, let $f$ be completely continuous from $[0,1] \times E \times E$ to $E$ and continuous from $[0,1] \times E \times E$ to $F$, where $F$ is equipped with the weak topology. Consider the two-point boundary value problem (BVP)
\[ y'' = f(x, y, y'), \quad 0 \leq x \leq 1, \quad (1) \]
\[ y(0) = A, \quad y(1) = B, \quad (2) \]
and its discrete approximation
\[ D^2 y_{k+1} = f(x_k, y_k, D y_k), \quad k = 1, \ldots, n - 1, \quad (3) \]
\[ y_0 = A, \quad y_n = B, \quad (4) \]
where \( f \) is nonlinear, the step size \( h = 1/n \) and grid points \( x_k = kh \) for \( k = 0, \ldots, n \). We denote the first (backward) difference quotient by \( D y_k = (y_k - y_{k-1})/h \) for \( k = 1, \ldots, n \), so that \( D^2 y_{k+1} = (y_{k+1} - 2y_k + y_{k-1})/h^2 \) for \( k = 1, \ldots, n - 1 \).

In the case where \( E \) is finite-dimensional, Agarwal [1], Gaines [2], Lasota [3], and others [4, 5] have researched discretized BVPs and the “effect” that this discretization can have on possible solutions, when compared with solutions to the original BVP. For example, the continuous BVP may have a solution, while its discretization may have no solution at all (see [1]). Also, solutions to the discrete problem may become large and inapplicable to the continuous problem as \( h \to 0 \) (see [2]).

This work shows that the results in [4] on \( a \ priori \) bounds, existence and convergence of solutions to (3), (4) when \( E = \mathbb{R}^d \), easily extend to Banach and Hilbert spaces provided the appropriate requirements, such as the above continuity conditions on \( f \), are satisfied. Conditions which guarantee \( a \ priori \) bounds on first difference quotients of possible solutions to (3) in Banach spaces are presented. These bounds (which are independent of the step-size \( h \)) are used to formulate some existence theorems for solutions to (3), (4) in Hilbert spaces.

Since the \( a \ priori \) bounds on first difference quotients are independent of the step-size, solutions to the discrete BVP are shown to converge to solutions of the continuous BVP in an aggregate sense. In the case \( E = \mathbb{R} \), these results are special cases of [2] and [6] and when \( E = \mathbb{R}^d \), these results coincide with some workings from [4]. However, the main advantage of this paper’s results is that they apply to infinite systems of BVPs, which have applications to the method of lines for elliptic PDE in the continuous case (see [7]).

An example of a discrete approximation to a BVP arising from a partial differential-integral equation is analyzed. The existence of a solution is shown. As an application, a problem which arises in the study of the finite deflections of an elastic string under a transverse load is investigated. The results are applied to show the existence of a solution; the sufficient estimates on the step size are presented. This paper was motivated by the research of Agarwal and O’Regan [8], Gaines [2], and also Schmitt and Thompson [9].

We use the following notation. Let \( E^{n+1} = E \times \cdots \times E \). Let \( \| \hat{y} \| = \max \{ \| y_k \|, y_k \in E : k = 0, \ldots, n \} \). For a set \( U \), let \( \partial U \) denote the boundary of \( U \) and let \( \hat{U} \) denote its closure. Denote the space of \( m \) times continuously differentiable functions mapping from \( A \) to \( B \) by \( C^m(A; B) \) endowed with the usual maximum norm. If \( B = \mathbb{R} \), then the \( B \) is omitted. A solution to (1) is a \( y \in C^2([0, 1]; E) \) satisfying (1) for all \( x \in [0, 1] \). A solution to (3) is a vector \( \hat{y} = (y_0, \ldots, y_n) \in E^{n+1} \) (with each \( y_k \in E \)) satisfying (3) for all \( k = 1, \ldots, n-1 \). The value of the \( k \)th component, \( y_k \), of a solution \( \hat{y} \) of (3) is expected to approximate \( y(x_k) \), for some solution \( y \) of (1).

2. \( A \ priori \) bounds

This section presents some \( a \ priori \) bound theorems for first differences quotients of solutions to the discrete problem. The extension to systems of equations is a nontrivial one; for instance, see the classic example due to Heinz [10] for the continuous case when \( E \) is finite dimensional.

The following growth condition was studied by Schmitt and Thompson [9] for the continuous problem and provides a tool to obtain bounds on the derivatives of solutions to (1) in terms of bounds on the solutions. We prove a similar result for difference equations.

The subsequent lemma is a discrete analogue to [9, Lemma 2.1], extending [4, Theorem 1] in the process.

**Lemma 1.** Let \( R > 0 \) be a constant. If there exists a positive, nondecreasing, real-valued function \( \Phi(s) \) such that \( \hat{y} \in E^{n+1} \) satisfies

\[
\| \hat{y} \| \leq R, \quad \| D^2 y_{k+1} \| \leq \Phi(\| D y_k \|), \quad \text{for all} \ k = 1, \ldots, n - 1, \quad (5)
\]
and
\[ \lim_{s \to \infty} \frac{s^2}{\Phi(s)} = +\infty, \] (6)
then there exists a constant \( N \) (depending on \( \Phi \) and \( R \) and independent of \( h \)) such that \( \|Dy_k\| \leq N \) for each \( k = 1, \ldots, n \).

**Proof.** The proof is similar to that in [4]. For completeness, we include it. Choose \( Q > 0 \) such that \( s^2/\Phi(s) > 9R \) for \( s > Q \), and set \( N = \max\{Q, 17R\} \).

Let \( M = \max\{\|Dy_k\| : k = 1, \ldots, n\} \). Thus, \( M = \|Dy_p\| \) for some \( p, 1 \leq p \leq n \). First, notice that \( \|Dy_k\| = \|y_k - y_{k-1}\|/h \leq 2R/h = 2Rn \), for all \( k = 1, \ldots, n \), so that \( M \leq 2Rn \). We bootstrap on this bound to show that \( M \leq N \), which is independent of \( n \).

If \( p \leq n/2 \), then for any natural number \( \mu \) satisfying \( 1 \leq \mu \leq n/2 \) we have \( p + \mu \leq n \), and using a discrete Taylor expansion we obtain
\[ y_{p+\mu} = y_p + \mu hDy_p + \sum_{s=p}^{p+\mu-1} (p + \mu - s)h^2D^2y_{s+1}. \]

Rearranging and taking norms of both sides, we get
\[ |\mu||Dy_p|h \leq 2R + h^2\Phi(\|Dy_p\|) \sum_{s=p}^{p+\mu-1} (p + \mu - s), \]

since \( \Phi \) is nondecreasing. Thus,
\[ M \leq \frac{2R}{(h|\mu|)} + \Phi(M)h|\mu|\frac{|\mu + 1|}{(2|\mu|)} \] (7)
Suppose \( M > N \), then \( M^2/\Phi(M) > 9R \), and we obtain
\[ M < \frac{2R}{(h|\mu|)} + \frac{M^2}{(18R)h|\mu|} + \frac{M^2}{(18R)h} \leq l(h|\mu|), \] (8)
where \( l(s) = 2R/s + M^2s/(18R) + M^2h/(18R) \). If \( 8R/M \geq 1/2 \), then \( M \leq 16R \leq N \), a contradiction so \( 8R/M < 1/2 \). Moreover \( M \leq 2R/h \), so that \( h \leq 2R/M \). Now it is easy to see that for \( s > 0 \), \( l(s) \geq l(6R/M) = 7M/9 \), and for \( 6R/M \leq s \leq 6R/M + 2R/M = 8R/M \), \( l(s) \leq l(6R/M) + M^22R/(18RM) = M \). We show that there is \( \mu \) with \( 1 \leq \mu \leq n/2 \) and \( 6R/M \leq \mu h \leq 6R/M + 2R/M \). Then it follows from (8) that \( M < l(\mu h) \leq M \), a contradiction, and hence,
\[ \|Dy_k\| \leq M \leq N, \quad \text{for all } k = 1, \ldots, n, \]
as required. Since \( 8R/M < 1/2 \), we see that \( \{6R/M, 8R/M\} \subseteq [0, 1/2] \) and \( 8R/M - 6R/M = 2R/M \geq h = 1/n \). It follows that \{\( \mu h : 1 \leq \mu \leq n/2 \}\} \cap \{6R/M, 8R/M\} \neq \emptyset, \) as required.

If \( p > n/2 \), then for any integer \( \mu \) such that \( -n/2 \leq \mu \leq -1 \), we have \( p + \mu \geq 0 \), and using a discrete Taylor expansion we obtain
\[ y_p = y_{p+\mu} - \mu hDy_p + \sum_{s=p+\mu}^{p-1} (p + \mu - s)h^2D^2y_{s+1}. \]

A similar argument to that used in the case \( 1 \leq p \leq n/2 \) shows that \( M \leq N \). Hence, \( \|Dy_k\| \leq N \) for all \( k = 1, \ldots, n \), as required.

**Remark 1.** In applying Lemma 1 to practical examples, the constant \( R \) is fixed by the given equation and thus we may relax (6) to
\[ \lim_{s \to \infty} \frac{s^2}{\Phi(s)} > 9R. \] (9)
3. EXISTENCE RESULTS

Let $E$ be a Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$. We shall need the following result in the proof of our main existence theorem.

**Lemma 2.** Let $R > 0$ and $N > 0$ be constants. Let $f$ be completely continuous from $[0, 1] \times E \times E$ to $E$ and continuous from $[0, 1] \times E \times F$ to $E$, where $F$ is $E$-equipped with the weak topology and let $K_1 \subset E$ be compact and convex. If

$$\langle y, f(x, y, p) \rangle + \|p\|^2 > 0, \quad \text{when } \langle y, p \rangle = 0, \quad \|y\| = R, \quad \|p\| \leq N + 1,$$

then there is a $\gamma > 0$ such that

$$\langle y, f(x, y, p) \rangle + \|p\|^2 \geq 2\gamma > 0,$$

when $|\langle y, p \rangle| \leq \gamma, \quad \|y\| = R, \quad y \in K_1, \quad \|p\| \leq N + 1$.

**Proof.** Assume the result is false. Then there exist sequences $\gamma_i \to 0$ and $(x_i, y_i, p_i) \in [0, 1] \times \{K_1 \cap \partial B_R\} \times \bar{B}_{N+1} = K_2$ such that

$$\langle y_i, f(x_i, y_i, p_i) \rangle + \|p_i\|^2 \leq 2\gamma_i, \quad \text{while } |\langle y_i, p_i \rangle| \leq \gamma_i.$$

Since $K_2$ is a compact in $\mathbb{R} \times E \times F$, there are subsequences which, by abuse of notation, we also label $\gamma_i$ and $(x_i, y_i, p_i)$ such that $(x_i, y_i, p_i)$ converges to $(x, y, p) \in K_2$. Thus, $\liminf_{i \to \infty} \|p_i\|^2 \geq \|p\|^2$ and also $\lim_{i \to \infty} \langle y_i, p_i \rangle = \langle y, p \rangle = 0$. Thus,

$$0 = \lim_{i \to \infty} 2\gamma_i \geq \liminf_{i \to \infty} \left( \langle y_i, f(x_i, y_i, p_i) \rangle + \|p_i\|^2 \right) \geq \langle y, f(x, y, p) \rangle + \|p\|^2,$$

while $\langle y, p \rangle = 0, \quad \|y\| = R, \quad \|p\| \leq N + 1,$ a contradiction. The result follows.

We now present some existence results. The following result is a discrete analogue of [9, Corollary 5.1] and extends [4, Theorem 2] in the process.

**Theorem 1.** Let $R > 0$ be a constant. Let $f$ be completely continuous on $[0, 1] \times F \times E$ and continuous from $[0, 1] \times E \times E$ to $F$, where $F$ is $E$-equipped with the weak topology. In addition, let $f$ satisfy

$$\|f(x, y, p)\| \leq \Phi(\|p\|), \quad \text{for } x \in [0, 1], \quad \|y\| \leq R, \quad p \in E, \quad (10)$$

where $\Phi$ satisfies the conditions of Lemma 1;

$$\langle y, f(x, y, p) \rangle + \|p\|^2 > 0, \quad \text{when } \langle y, p \rangle = 0, \quad \|y\| = R, \quad \|p\| \leq N + 1, \quad (11)$$

where $N$ is the constant given by Lemma 1 and let $\|A\|, \|B\| < R$. Then there exists a solution $\bar{y}$ of problem (3) and (4) with $\|\bar{y}\| < R$, for $h$ sufficiently small.

**Proof.** It may be checked by direct computation that problem (3),(4) has a solution $\bar{y}$ if and only if

$$y_k = h \sum_{i=1}^{n-1} G(x_k, s_i) f(s_i, y_i, Dy_i) + A(1 - x_k) + Bx_k, \quad k = 0, \ldots, n,$$

where

$$G(x, t) = \begin{cases} (x - 1)t, & 0 \leq t \leq x \leq 1, \\ (t - 1)x, & 0 \leq x \leq t \leq 1. \end{cases}$$

Define

$$T(y)_k = h \sum_{i=1}^{n-1} G(x_k, s_i) f(s_i, y_i, Dy_i) + A(1 - x_k) + Bx_k, \quad k = 0, \ldots, n. \quad (12)$$
The problem is thus reduced to showing that $T(\bar{y}) = \bar{y}$ for some $\bar{y} \in E^{n+1}$. We do this by using degree theory. Let

$$\Omega = \{ \bar{y} \in E^{n+1} : \|\bar{y}\| < R, \|Dy_k\| < N + 1, k = 1, \ldots, n \}.$$ 

From the simple properties of the summation operator, and since $f$ is completely continuous, see that $T$ is a compact operator. Now consider

$$(I - \lambda T)(\bar{y}) = 0, \quad \lambda \in [0, 1].$$

(13)

This is equivalent to $\bar{y}$ satisfying

$$D^2y_{k+1} = \lambda f(x_k, y_k, Dy_k), \quad k = 1, \ldots, n - 1,$$

$$y_0 = \lambda A, \quad y_n = \lambda B.$$  (14) (15)

We show that if $(I - \lambda T)(\bar{y}) = 0$ and $\bar{y} \in \Omega$, then $\bar{y} \in \Omega$ and consequently, $\bar{y} \notin \partial\Omega$. First, see that this is trivially satisfied for $\lambda = 0$, and so assume $\lambda \in (0, 1]$. Notice that $\lambda f$ satisfies the inequalities in Theorem 1 for $\lambda \in (0, 1]$. Hence, $\|Dy_k\| \leq N$ for $k = 1, \ldots, n$.

We show that, for a small enough step size, $r_k = \|y_k\|^2$ (where $\bar{y}$ is a solution to (14),(15)) cannot have a maximum with $\|\bar{y}\| = R$. Assume the contrary and see that the restriction $\|A\|$, $\|B\| < R$ forces $r_k$ to attain its maximum at some $k = j$ with $j \in \{1, \ldots, n - 1\}$. Since $f$ is completely continuous,

$$K = f([0, 1] \times B_R \times B_{N+1}) \cup \{A, B, 0\}$$

is compact, and hence, $K_1 = 2cocl\{K\}$ is compact. From (11) and by Lemma 2, there is a $\gamma > 0$ such that

$$(y, \lambda f(x, y, p)) + \|p\|^2 \geq 2\gamma > 0,$$

when $|(\bar{y}, p)| \leq \gamma$, $\|\bar{y}\| = R$, $y \in K_1$, $\|p\| \leq N + 1$.

Now $\|y_k\| = R$ and $\bar{y} \in \Omega$ is a solution so that $\|Dy_j\| \leq N + 1$ for $j = 1, \ldots, n$ and $y_k \in K_1$ by (12) and (13). Using this and arguing as in the proof of Theorem 2 of [5] that, for a small step size, $|\langle y_k, Dy_k \rangle| \leq \lambda \gamma$ and

$$D^2r_{k+1} > \langle y_k, \lambda f(x_k, y_k, Dy_k) \rangle + \|Dy_k\|^2 - \lambda \gamma > 0,$$

when $|\langle y_k, Dy_k \rangle| \leq \lambda \gamma$, $\|y_k\| = R$, $y_k \in K_1$, $\|Dy_k\| \leq N + 1$, and $k = 1, \ldots, n - 1$. Thus, we have $r_k = \|y_k\|^2 < R^2$; that is, $\|\bar{y}\| < R$.

Thus, any solution $\bar{y}$ to (13) satisfies $\bar{y} \in \Omega$. Therefore, $(I - \lambda T)(\bar{y}) \neq 0$, for all $\lambda \in [0, 1]$ and $\bar{y} \in \partial\Omega$. The degree is defined on the bounded, open set $\Omega$ and we have, by the invariance of the degree under homotopy (see [11])

$$d((I - \lambda T)(\bar{y}), \Omega, 0) = d((I - T)(\bar{y}), \Omega, 0) = d(I, \Omega, 0) = 1(\neq 0),$$

since $0 \in \Omega$. Therefore, $T$ has a fixed point and thus there is a solution $\bar{y}$ to (3),(4). This concludes the proof.

**Remark 2.** The condition that: “$f$ is continuous from $[0,1] \times E \times F$ to $E$, where $E$ is equipped with the weak topology”, may be removed from Theorem 1 and the remaining assumptions still guarantee solutions to the discrete problem. However, these solutions may not be related to the
continuous problem. Note that the above condition is always satisfied if \( f(x, y, p) = f(x, y) \) and \( f \) is completely continuous.

The following example is a discrete approximation to [9, Example 5.2], illustrating the applicability of Theorem 1.

**Example.** Consider the partial differential-integral equation

\[
 u_{yy}(x, y) = \int_0^1 p(x, q) u(q, y) \, dq, \quad 0 \leq x \leq 1, \quad 0 < y < 1,
\]

(16)

with the boundary conditions

\[
 u(x, 0) = \phi^0(x), \quad u(x, 1) = \phi^1(x),
\]

(17)

where \( \phi^0(x), \phi^1(x) \in L^2([0, 1]) \) and \( p(x, y) \) is a positive kernel of Hilbert-Schmidt type.

Let \( K \) be the linear operator on \( L^2([0, 1]) \) defined for \( z \in L^2([0, 1]) \) by

\[
 (Kz)(x) = \int_0^1 p(x, q) z(q) \, dq.
\]

For a function of two variables \( u(x, y) \) such that \( u(x, \cdot) \in L^2([0, 1]) \) for each \( y \in [0, 1] \), set \( u(y) = u(x, y) \). Then (16) and (17) can be written as the BVP (where \( E = L^2([0, 1]) \))

\[
 u'' = Ku, \quad 0 \leq y \leq 1,
\]

(18)

\[
 u(0) = \phi^0, \quad u(1) = \phi^1.
\]

(19)

The discrete approximation is given by

\[
 D^2u_{k+1} = Ku_k, \quad k = 1, \ldots, n - 1,
\]

(20)

\[
 u_0 = \phi^0, \quad u_1 = \phi^1.
\]

(21)

The assumption that \( p(x, q) \) is a positive kernel implies that (11) is satisfied. Setting \( \Phi \) as a suitable constant, see that (10) is satisfied for a \( R > 0 \) such that \( ||\phi^0||, ||\phi^1|| < R \). Finally, since \( K \) is completely continuous, see that all the conditions of Theorem 1 are satisfied, and thus the discrete problem (20),(21) has a solution \( \bar{u} \).

### 4. CONVERGENCE OF SOLUTIONS

In this section, we apply our results to formulate some convergence theorems. The following is a generalization of [2, Theorem 2.5].

**Theorem 2.** Let the assumptions of Theorem 1 hold. Given \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that if \( 0 < h < \delta \) and \( \bar{y} \) is a solution of (3),(4), then there is a solution \( y(x) \) of (1),(2) such that

\[
 \max \{ ||y(x, \bar{y}) - y(x)|| : 0 \leq x \leq 1 \} \leq \varepsilon \quad \text{and}
\]

\[
 \max \{ ||v(x, \bar{y}) - y'(x)|| : 0 \leq x \leq 1 \} \leq \varepsilon,
\]

where

\[
 y(x, \bar{y}) = y_k + (x - x_k)Dy_{k+1}, \quad \text{for } x_k \leq x \leq x_{k+1},
\]

and

\[
 v(x, \bar{y}) = \begin{cases} 
 Dy_k + (x - x_k)D^2y_{k+1}, & \text{for } x_k \leq x \leq x_{k+1}, \\
 Dy_1, & \text{for } 0 \leq x \leq x_1.
\end{cases}
\]

**Proof.** The proof is similar to that of [2] and so is omitted.

**Remark 3.** It follows from Theorem 2 that if solutions to the continuous problem (1),(2) are unique, then solutions to (3),(4) converge to solutions of the continuous problem in the sense of Theorem 2.
5. COROLLARIES AND EXAMPLES

Consider the special case $E = \mathbb{R}$. We briefly present some results for the scalar case.

**Definition 1.** Call $\alpha$ ($\beta$) a strict lower (strict upper) solution for (1) if $\alpha$ ($\beta$) $\in C^2([0,1])$

$$\alpha''(x) - f(x, \alpha(x), \alpha'(x)) \geq \gamma, \quad \beta''(x) - f(x, \beta(x), \beta'(x)) \leq \gamma,$$

for some $\gamma > 0$ and all $x \in [0,1]$. In addition, assume $\alpha(0) < A < \beta(0)$ and $\alpha(1) < B < \beta(1)$.

In what follows, $\tilde{\alpha} = (\alpha(0), \alpha(h), \ldots, \alpha(nh))$ where $\alpha$ is a strict lower solution for (1). Define $\tilde{\beta}$ similarly. The inequality $\tilde{\alpha} \leq \tilde{\beta}$ holds if and only if $\alpha_k \leq \beta_k$ for each $k = 0, \ldots, n$.

The following corollary is a special case of [2, Theorem 5.3] and [6, Theorem 3].

**Corollary 1.** Let there exist nondegenerate strict lower and strict upper solutions $\alpha \leq \beta$ for (1) and let $f$ satisfy

$$|f(x,y,p)| \leq \Phi(\|p\|), \quad \text{for } (x,y,p) \in [0,1] \times [\alpha,\beta] \times \mathbb{R},$$

where $\Phi$ satisfies the conditions of Lemma 1. Then there exists a solution $\tilde{y}$ of problems (3) and (4) with $\alpha \leq \tilde{y} \leq \beta$, for $h$ sufficiently small.

**Proof.** The proof follows similar lines to that in [2] and [6] and is omitted.

**Remark 4.** (See [2].) If $f(x,y,p)$ has continuous partial derivatives with respect to $p$ and $\alpha, \beta \in C^3([0,1])$, we can easily estimate $\delta(\varepsilon)$ in Theorem 2. Let $S_1 = \{(x,y,p) : y = \beta(x), |p - \beta'(x)| \leq 1\}$.

If

$$\delta_1(\varepsilon) \leq \frac{\varepsilon}{\|f(x,y,p)\|}, \quad \delta_2(\varepsilon) \leq \frac{\min \left\{ 1/2, \varepsilon / \left( \max_{x \in [0,1]} \left| \frac{\partial f}{\partial p} \right| \right) \right\}}{\max |\beta''(x)|},$$

and $\delta_3(\varepsilon), \delta_4(\varepsilon)$ are defined similarly associated with $\alpha(x)$, then we may take the following:

$$\delta(\varepsilon) = \min_{1 \leq i \leq 4} \delta_i(\varepsilon).$$

As an application of our existence results and Remark 4, we solve the following problem which arises in the study of finite deflections of an elastic string under a transverse load (see [12]).

**Example.** Consider

$$y'' = -\left(1 + a^2 |y|^2\right), \quad 0 \leq x \leq 1, \quad y(0) = 0 = y(1). \quad (22)$$

Say $a^2 = 1/49$. The discrete problem is then

$$D^2 y_{k+1} = -\left(1 + \frac{|D y_k|^2}{49}\right), \quad k = 1, \ldots, n - 1, \quad (24)$$

$$y_0 = 0 = y_n. \quad (25)$$

Choose $\alpha = -2$ and $\beta(x) = 4 - x^2$. It is not difficult to see that these are strict lower and strict upper solutions. Choose $\Phi(s) = 1 + s^2/49$. Then the conditions of Corollary 1 and (9) will be satisfied. Thus, the discrete problem has a solution $\tilde{y}$ satisfying $\tilde{\alpha} \leq \tilde{y} \leq \tilde{\beta}$ for $h$ sufficiently small.

In fact, we can glean some further information about our solution $\tilde{y}$ by the use of a discrete maximum principle. Since $D^2 y_{k+1} < 0$ for all $k = 1, \ldots, n - 1$, then $\tilde{y}$ cannot have a minimum for all $k = 1, \ldots, n - 1$. Thus, $y_k$ must have a minimum at either $k = 0$ or $k = n$. Thus, min $\tilde{y} = 0$, and we may conclude that the nontrivial solution $\tilde{y}$ satisfies $0 \leq \tilde{y} \leq \tilde{\beta}$.

The unique solution to (22), (23) is

$$y(x) = \frac{(\ln |\cos[a(x - 1/2)]/\cos(a/2)|)}{a^2}.$$

By Remark 3, $\tilde{y}$ converges to $y(x)$ in the sense of Theorem 2. The estimates on the step size are $\delta(\varepsilon) \leq \min\{1/2, 49\varepsilon/24\}/2$. 

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