

U–Calc. University Calculus: Semester 1, Year 1

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Part 1: Functions, graphs, inequalities and limits.

This document is designed to accompany the author's YouTube videos.

Motivation

Graphs can convey complete information about functions in a simple, visual way. Thus, graphs are a powerful tool for understanding functions and their behaviour.

Graphing reciprocal functions: Building the intuition.

Example: Sketch the graph of $f(x) := \sqrt{x+3}$ and hence sketch the graph of

$$\frac{1}{f(x)} = \frac{1}{\sqrt{x+3}}.$$

The bigger picture

- The function $1/f$ has the same domain as f , excluding the x -points where $f(x) = 0$.
- If $f(c) = 0$ for some c then $x = c$ is a vertical asymptote of $1/f$.
- If f is increasing then $1/f$ is decreasing.
- If

$$f(x) \rightarrow L \quad \text{as } x \rightarrow a \quad \text{then} \quad \frac{1}{f(x)} \rightarrow \frac{1}{L} \quad \text{as } x \rightarrow a.$$

Learn by doing – try the following

Example: Sketch the graph of $f(x) := \sqrt{x-2}$ and hence sketch the graph of

$$\frac{1}{f(x)} = \frac{1}{\sqrt{x-2}}.$$

Motivation

The domain and range of a function gives us: a set of “allowable” inputs; and a set of achieved outputs, respectively. It is important information that is designed to ensure that work associated with the function is well-defined and can be used to learn more about how the function behaves, eg, for graphing purposes.

Building the intuition

Example: Find: the maximal domain; and the range of

$$f(x) := \sqrt{9 - x^2}.$$

The bigger picture

- To construct the domain, try to form (and solve) inequalities that are needed to ensure the function “makes sense”.
- To construct the range try to use f and its domain to form useful inequalities.

Learn by doing – try the following

Examples: Find: the maximal domain; and the range of:

$$f(x) := \sqrt{x^2 - 4}$$

$$g(x) := \frac{1}{\sqrt{x - 1}}.$$

Motivation

Inequalities are used in virtually all areas of mathematics and its applications. A good understanding of inequality techniques empower us to solve more difficult problems where inequalities arise (eg: in optimization; in linear programming; in error bounds for numerical approximation etc).

Building the intuition

Example: Solve

$$\frac{1}{x+1} > -\frac{1}{2}.$$

The bigger picture

- Simplify the original inequality as much as possible through algebraic manipulation, eg, multiplying both sides by a *positive* expression and rearranging.
- Then solve the simpler, equivalent inequality either graphically or algebraically.

Learn by doing – try the following

Example: Solve

$$\frac{1}{x-3} > -1.$$

Motivation

The limit of a function as $x \rightarrow \infty$ (if it exists) describes the function's "long term" behaviour and can be important information in modelling, as this limit enables us to make predictions about the future states of the phenomena involved.

Building the intuition

Example: Let

$$f(x) := \frac{x^2 + \cos x}{x + 2x^2 + \sin x}.$$

Discuss the limiting behaviour of $f(x)$ as $x \rightarrow \infty$.

The bigger picture

- Simplify the original function as much as possible through algebraic manipulation.
- Look to apply basic limit laws or more advanced ideas like the squeeze/pinching/sandwich theorem or L'Hôpital's rule.

Learn by doing – try the following

Example: Let

$$g(x) := \frac{x^4 - \sin x}{x^4 + 1 + \sin x}.$$

Discuss the limiting behaviour of $g(x)$ as $x \rightarrow \infty$.

Motivation

The limit of a function as $x \rightarrow a$ (if this limit exists) describes the function's behaviour "near" the point a . This can be important information in modelling, as this limit enables us to make predictions about the states of the phenomena when x is close to a .

Building the intuition

Example: Let

$$g(x) := \frac{|x^2 + x - 6|}{x - 2}.$$

Discuss the limiting behaviour of $g(x)$ as: (a) $x \rightarrow 2^+$; (b) $x \rightarrow 2^-$; (c) $x \rightarrow 2$.

The bigger picture

- Simplify the original function as much as possible through algebraic manipulation.
- Look to apply basic limit laws or more advanced ideas like the squeeze/pinching/sandwich theorem or L'Hôpital's rule.

Learn by doing – try the following

Example: Let

$$h(x) := \frac{|x^2 + x - 6|}{x + 3}.$$

Discuss the limiting behaviour of $h(x)$ as: (a) $x \rightarrow -3^+$; (b) $x \rightarrow -3^-$; (c) $x \rightarrow -3$.