U–Alg. University Algebra: Semester 1, Year 1

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Part 1: Complex numbers.

This document is designed to accompany the author's YouTube videos.

Motivation

Complex numbers are an extension of the well–known real numbers. Complex numbers (and complex–valued functions) find useful applications in: signal analysis; fluid mechanics; relativity; and quantum mechanics. A solid understanding of basic calculations with complex numbers empower us to do more significant and interesting things with them.

Building the intuition.

Example: Define the complex numbers z and w by $z := 2 - 5i$ and $w := 1 + 2i$. Calculate

$$\text{(a) } \frac{1+7i}{w}; \quad \text{(b) } 4\overline{z}w; \quad \text{(c) } \operatorname{Arg}(w - 3i).$$

The bigger picture

- If division by a complex number is involved then apply the "rationalization" approach by using the complex conjugate.
- Multiplication of complex numbers can be carried out by simply expanding the brackets in the normal way.
- For addition and subtraction of complex numbers, the calculations involve the corresponding real and imaginary parts.

Learn by doing – try the following

Example: Define the complex numbers u and v by $u := 2 - 5i$ and $v := 1 + 2i$. Calculate

$$\text{(a) } \frac{1+6i}{u}; \quad \text{(b) } \overline{u}v; \quad \text{(c) } \mathrm{Arg}(v - i).$$

Motivation

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Building the intuition.

Example: Let $z := 2e^{i\pi/6}$. Calculate: z^3; z^{-1}; and $-3z$. In addition, plot your calculated complex numbers on the same Argand diagram.

The bigger picture

- The polar form gives us compact way of writing a complex number that gives us virtually immediate information about its geometric nature.
- The polar form of a complex number can be very convenient for simplifying multiplication and power operations.
- Complex numbers of the type $-re^{i\theta}$ can be written in standard from via

$$-re^{i\theta} = re^{i(\theta \pm \pi)}.$$

Learn by doing – try the following

Example: Let $z := 3e^{i\pi/7}$. Calculate: z^2; z^{-1}; and $-2z^{-1}$. In addition, plot your calculated complex numbers on the same Argand diagram.

Motivation

Complex numbers are an extension of the well–known real numbers. Complex numbers (and complex–valued functions) find useful applications in: signal analysis; fluid mechanics; relativity; and quantum mechanics. A solid understanding of how to sketch regions in the complex plane empower us to do more significant and interesting things with them.

Building the intuition.

Example: Sketch the region in the complex plane defined by all those complex numbers z such that

$$|z - 2i| < 1, \qquad \text{and} \qquad 0 < \text{Arg}(z - 2i) \leq \frac{3\pi}{4}.$$

The bigger picture

- Don't be daunted when graphing complicated regions! Work through the problem systematically.
- After coming up with your region, test one or two points in the region to see if they possess the desired properties.
- With some practice you will recognize the mathematical expressions for regions in the complex plane very quickly.

Learn by doing – try the following

Example: Sketch the region in the complex plane defined by all those complex numbers z such that

$$|z - i| < 3, \qquad \text{and} \qquad 0 < \text{Arg}(z - i) \leq \frac{\pi}{2}.$$

Example: Sketch the region in the complex plane defined by all those complex numbers z such that

$$0 < \text{Arg}(z - i) \leq \frac{\pi}{2}.$$

How to solve polynomial equations that involve complex numbers: Dr Chris Tisdell

Motivation

In the analysis of applied problems, we often need to solve equations, with the solution(s) being the desired information that we seek. Equations involving complex numbers can require more care to solve.

Building the intuition

Example: Find all of the (complex) fourth roots of $8(-1+\sqrt{3}i)$. You may leave your answer in polar form.

The bigger picture

- The approach involving the polar form is probably the easiest and most elegant method for finding the roots of complex polynomials, although there are alternatives for solving quadratic equations.
- Sketching the solutions can provide some geometric insights into the problem and its solution.

Learn by doing – try the following

Example: Find all of the (complex) sixth roots of $-27i$. You may leave your answer in polar form.
Example: Find the square roots of $1 - \sqrt{3}i$ by writing $z = a + bi$ and solving

$$(a + bi)^2 = 1 - \sqrt{3}i$$

for a and b.

Motivation

The relationship between $\cos\theta$, $\sin\theta$ and $e^{i\theta}$ can be applied to simplify trigonometric expressions. These simplifications can then be more easily worked with, for example, in integration.

Building the intuition

Example: Apply the identity

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

to write $\sin^5\theta$ in terms of $\sin n\theta$ for $n = 1, 2, \ldots$

The bigger picture

- The basic identities are:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right);$$

  and come from the basic expression $e^{i\theta} = \cos\theta + i\sin\theta$.

- Applying the above identities may help us to evaluate difficult integrals like

$$\int \sin^5\theta\ d\theta.$$

Learn by doing – try the following

Example: Apply the identity

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

to write $\cos^4\theta$ in terms of $\cos n\theta$ for $n = 1, 2, \ldots$ Hence evaluate

$$\int \cos^4\theta\ d\theta.$$