Section 2: Vector functions of one variable.

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S1: Motivation

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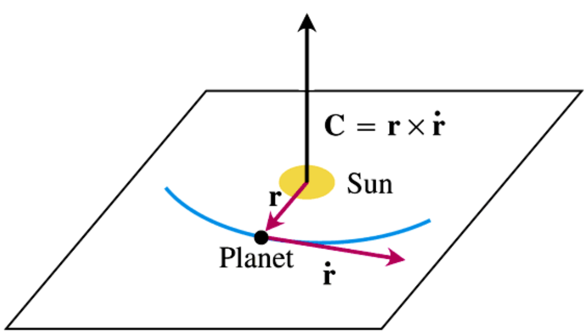
Images from: "Thomas' Calculus" by Wier, Hass &, Giordano, Pearson Education Inc, 2008.

S1: Motivation.

Many of the problems that scientists and engineers study involve motion in 3D space, for example, the mathematics of space flight or the orbit of planets.

A real–valued function is of limited value for understanding motion in 3D and can change in just one of two ways: it can increase; or it can decrease.

On the other hand, "vector"–valued functions provide the perfect setting to understand applied problems in 3D. A vector–valued function can change, not just in magnitude, but also in direction, and the rate of change is not just a single number but is itself a vector.



S2: Curves, vector functions & parametrizations.

In first–year, you learnt about describing curves in the plane, for example, the cycloid.

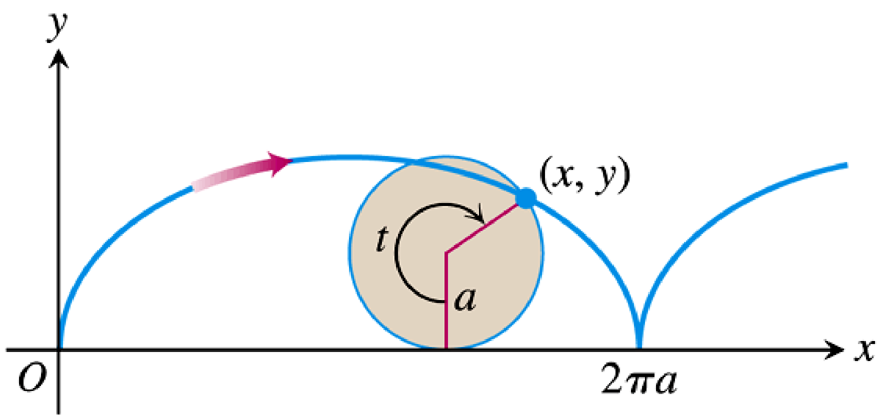


FIGURE The cycloid $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

Observe that the co–ordinates of the points on the curve are in the form $x = f(t), \quad y = g(t), \quad t \in I$, where I is an interval.

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

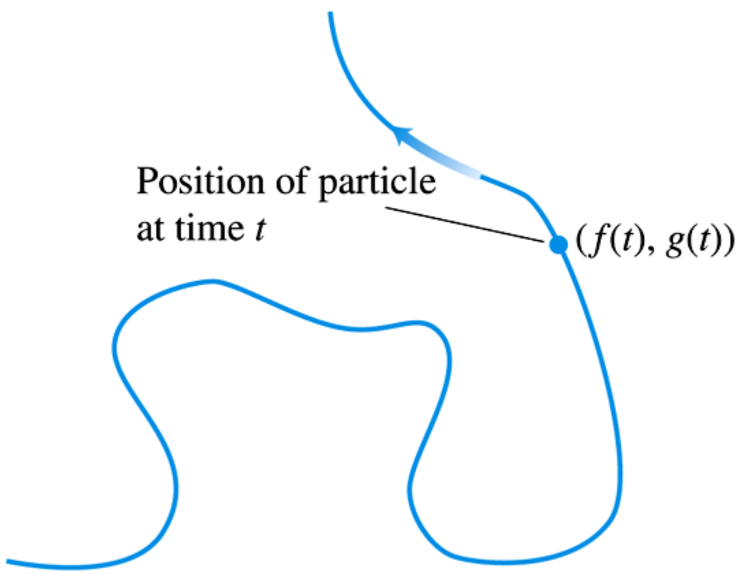


FIGURE The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

If a particle travels through space over some time interval I of time t then the co–ordinates of the particle can be described by

$$x = f(t), \quad y = g(t), \quad z = h(t), \qquad t \in I. \tag{1}$$

The set of points

$$(x, y, z) = (f(t), g(t), h(t)), \qquad t \in I$$

form a curve $\mathcal{C}$ in space that we call the path of the particle.

We say that (1) (including I) parametrize (traces out) the curve $\mathcal{C}$ in space.

If $P(f(t), g(t), h(t))$ is the particle's co–ordinates at time t, then we can form a position vector $\vec{OP}$ that will depend on time t.

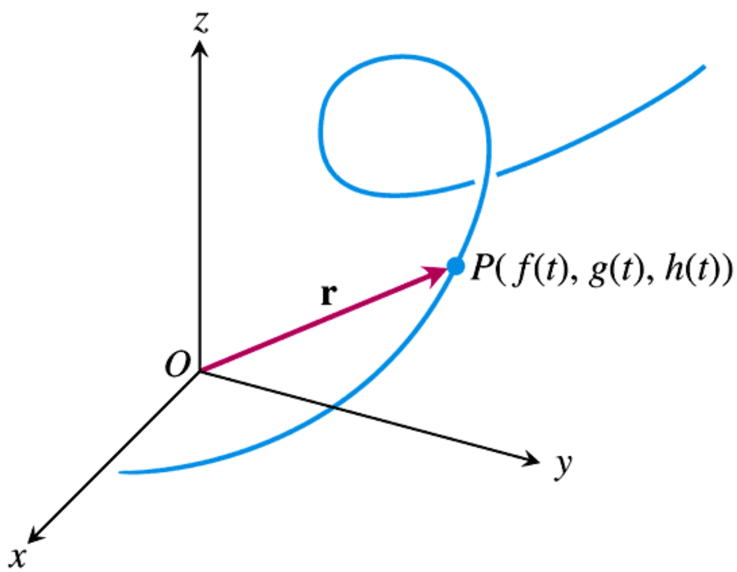


FIGURE The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

Thus, we have formed

$$\begin{aligned}\mathbf{r}(t) = \vec{OP} &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \\ &= (f(t), g(t), h(t)), \quad t \in I,\end{aligned}$$

and $\mathbf{r} = \mathbf{r}(t)$ is known as a vector–valued function (or just a "vector function") of one variable.

We say that $\mathbf{r}$ (and I) parametrize (traces out) the curve $\mathcal{C}$ in space.

For example

$$\begin{aligned}\mathbf{r}(t) &:= t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad t \in [0,1] \\ \mathbf{r}(t) &:= \cos\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}, \quad t \in \mathbb{R} \\ \mathbf{r}(t) &:= \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t \in [0, 2\pi)\end{aligned}$$

are all vector functions of one variable.

Sometimes we refer to real–valued functions as "scalar functions" to distinguish them from vector functions.

Lines

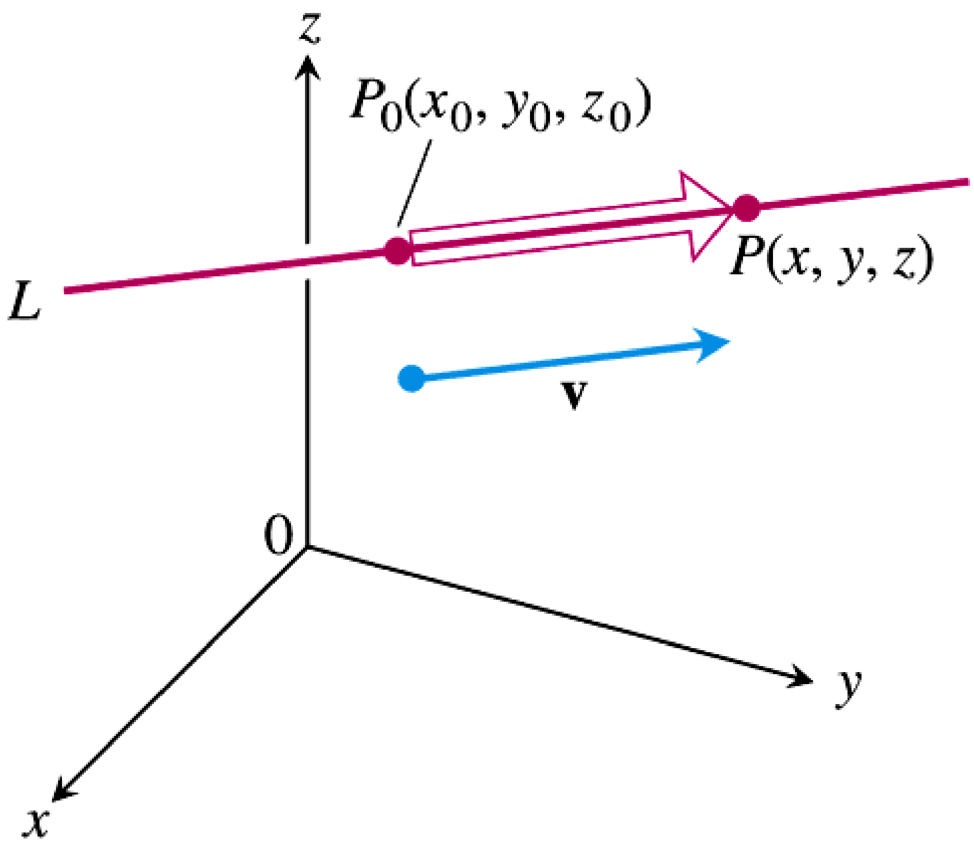


FIGURE A point P lies on L through P_0 parallel to **v** if and only if $\overrightarrow{P_0P}$ is a scalar multiple of **v**.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The variable t above is known as a parameter.

If we think of the line as the path travelled by a particle with initial postion $\mathbf{r}_0$ and t representing time then we can write our equation as

$$\mathbf{r}(t) = \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$$

and so see that the position of the particle at any time t is just:

the initial position ($\mathbf{r}_0$) plus its distance moved (speed x time $= t \times |\mathbf{v}|$) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight–line motion.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

EXAMPLE Parametrizing a Line Through a Point Parallel to a Vector

Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

EXAMPLE Parametrizing a Line Through Two Points

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t.

Note that there can be many ways of parametrizing lines (and curves). For example, the equations

$$x = -3 + 4t^5, \quad y = 2 - 3t^5, \quad z = 4 + 7t^5$$

also parametrize the line in the above example.

EXAMPLE Parametrizing a Line Segment

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution We begin with equations for the line through P and Q, taking them, in this case, from above

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1.$$

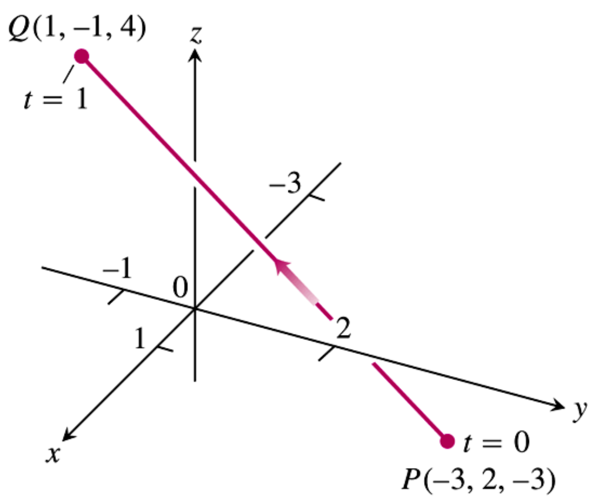
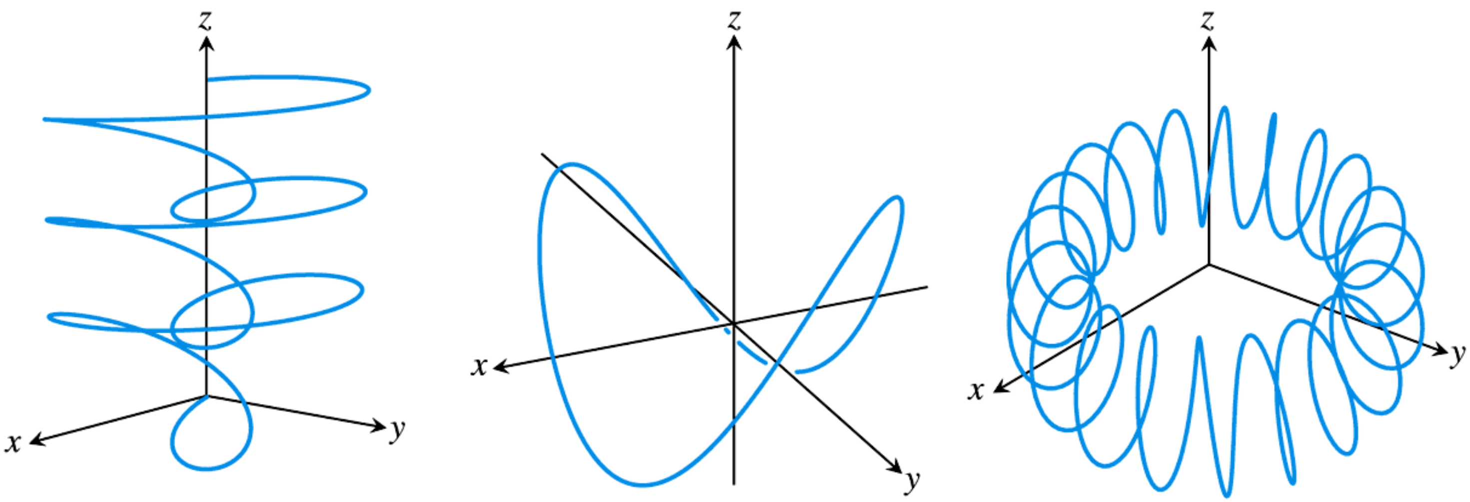


FIGURE Example derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

Other curves.



$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$

(a)

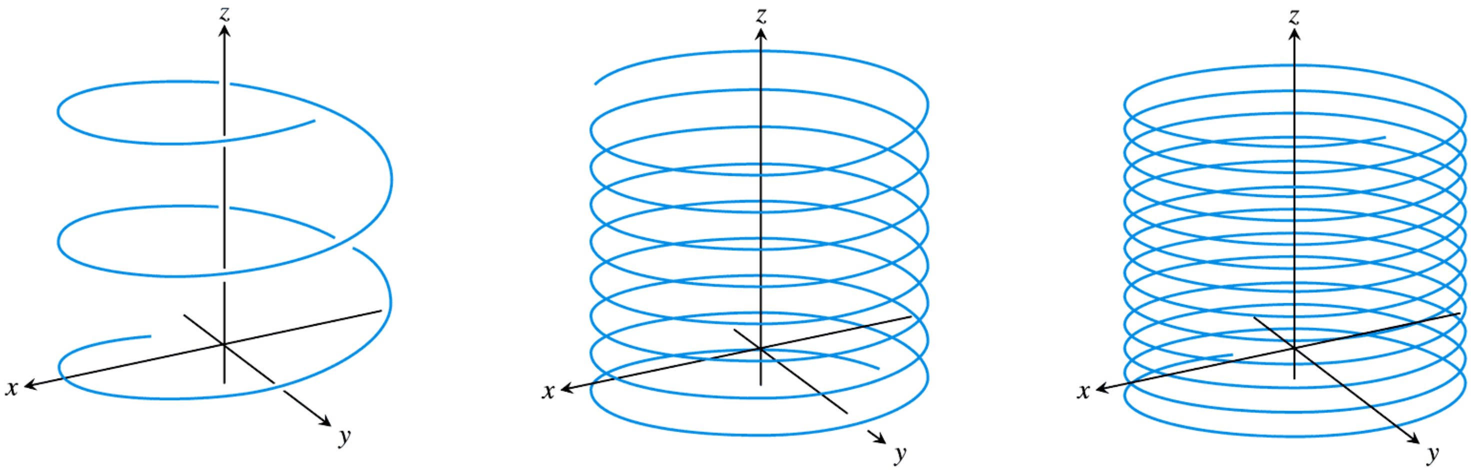
$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$

(b)

$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$

(c)

FIGURE Computer-generated space curves are defined by the position vectors $\mathbf{r}(t)$.



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\quad$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\quad$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

FIGURE Helices drawn by computer.

Ex: Identify and sketch the curve in the plane parametrized by

$$\mathbf{r}(t) := 4t\mathbf{i} + 32t^2\mathbf{j}, \quad t \in \mathbb{R}.$$

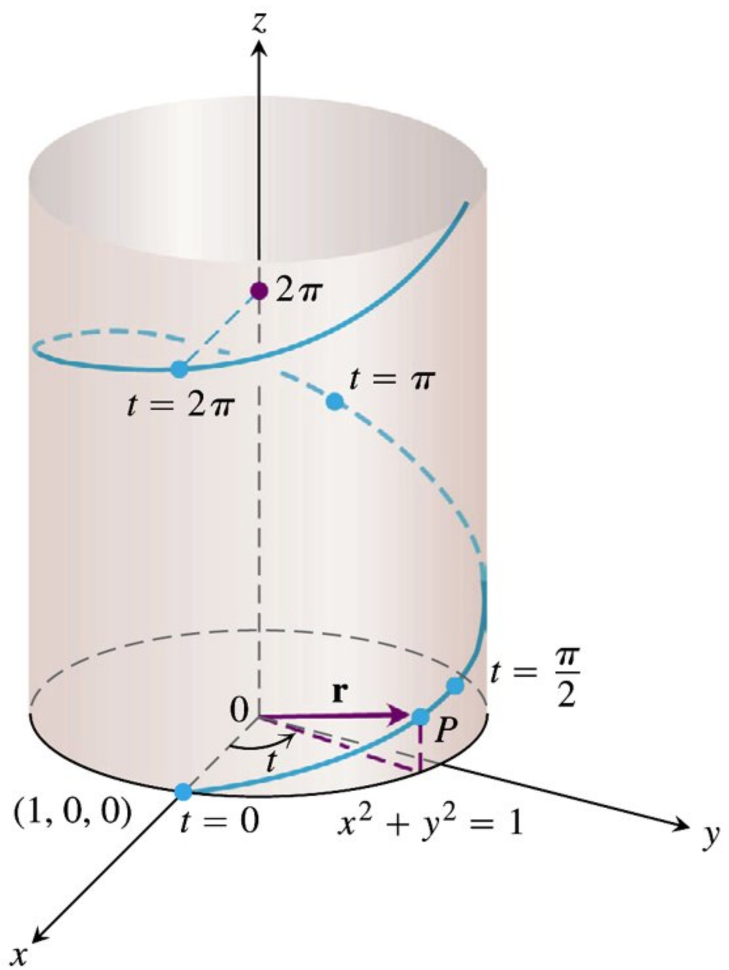


FIGURE The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

Ex: Identify and sketch the curve in the plane parametrized by

$$\mathbf{r}(t) := t\mathbf{i} + (t+2)\mathbf{j}, \quad t \in \mathbb{R}.$$

Ex: Identify the curve in space parametrized by

$$\mathbf{r}(t) := (2t - 1)\mathbf{i} + (t + 2)\mathbf{j} + t\mathbf{k}, \quad t \in \mathbb{R}.$$

Ex: Identify and sketch the curve in space parametrized by

$$\mathbf{r}(t) := t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}, \quad t \in [0, \pi].$$

Ex: Parametrize, in two ways, the curve $\mathcal{C}$ on which a particle travels anticlockwise around the top half of the unit circle in the plane. How could we parametrize clockwise movement?

Planes

Two–dimensional surfaces that sit in $\mathbb{R}^3$ can also be parametrized. We briefly examine planes.

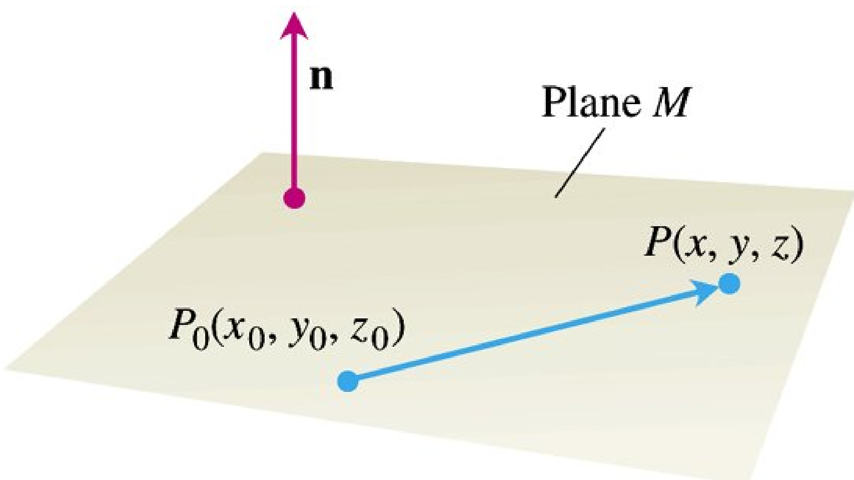


FIGURE The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D,$ where
	$D = Ax_0 + By_0 + Cz_0$

The "parametric" vector form of the equation for the plane containing a point P_0 and parallel to the vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{OP_0} + s\mathbf{u} + t\mathbf{v}, \qquad s, t \in \mathbb{R}.$$

Here we assume that $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

S3: Limits of vector functions.

Limits of vector functions may be calculated in a componentwise fashion, ie

$$\begin{aligned}\lim_{t\to t_0} \mathbf{r}(t) &= \lim_{t\to t_0} (f(t)\mathbf{i} + g(t)\mathbf{j} + k(t)\mathbf{k}) \\ &= \left(\lim_{t\to t_0} f(t)\right)\mathbf{i} + \left(\lim_{t\to t_0} g(t)\right)\mathbf{j} + \left(\lim_{t\to t_0} h(t)\right)\mathbf{k}\end{aligned}$$

provided each of these limits actually exist.

Ex: If $\mathbf{r}(t) := t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $t \in \mathbb{R}$ then calculate

$$\lim_{t\to 1} \mathbf{r}(t).$$

DEFINITION Limit of Vector Functions

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \quad \Rightarrow \quad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

Ex: If

$$\mathbf{r}(t) := t\mathbf{i} + t\mathbf{j} + t\sin t\mathbf{k}, \quad t \in \mathbb{R}$$

then formally prove that

$$\lim_{t \to 0} \mathbf{r}(t) = \mathbf{0}.$$

Ex: If

$$\mathbf{r}(t) := \frac{\sin t}{t}\mathbf{i} + \frac{\cos t}{t+1}\mathbf{j} + t^2\mathbf{k}, \quad \text{for all } t \neq 0$$

then calculate

$$\lim_{t \to 0} \mathbf{r}(t).$$

Ex: If

$$\mathbf{r}(t) := \frac{\sin^2 t}{t^2}\mathbf{i} + \frac{\cos^2 t}{t^2}\mathbf{j} + e^{-t}\mathbf{k}, \quad \text{for all } t > 0$$

then calculate

$$\lim_{t\to\infty} \mathbf{r}(t).$$

DEFINITION Continuous at a Point

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Note that a vector function $\mathbf{r}$ will be continuous if and only if each of its component functions f, g, h are continuous.

Ex: Briefly explain why the vector function

$$\mathbf{r}(t) := t^2\mathbf{i} + t^3\mathbf{j} + (1-t)\mathbf{k}, \quad t \in \mathbb{R}$$

is continuous (for all $t \in \mathbb{R}$).

Ex: Consider

$$\mathbf{r}(t) := \frac{\sin \pi t}{t-1}\mathbf{i} + \frac{\sin 2\pi t}{t-1}\mathbf{j} + \frac{\sin 3\pi t}{t-1}\mathbf{k}, \qquad \text{for all } t \neq 1.$$

What (vector) value can we define $\mathbf{r}(1)$ to be that will force $\mathbf{r}$ to be continuous at $t = 1$?

S4: Derivative of a vector function

Since we have defined limits of vector functions in a componentwise fashion, we can also define the derivative of a vector function in a similar componentwise manner.

DEFINITION Derivative

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

Ex: If $\mathbf{r}(t) := t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ for all $t \in \mathbb{R}$ then calculate $\mathbf{r}'$.

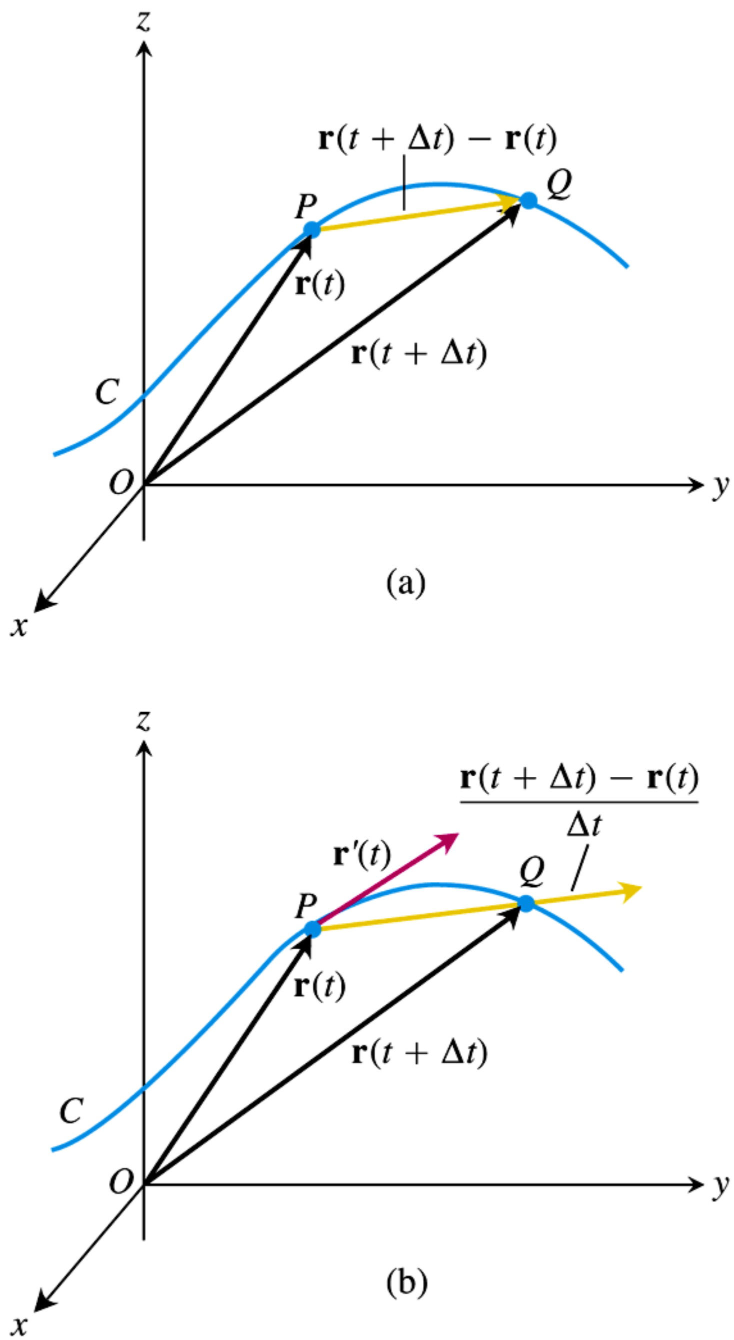


FIGURE As $\Delta t \to 0$, the point Q approaches the point P along the curve C. In the limit, the vector $\overrightarrow{PQ}/\Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

We say that the curve $\mathcal{C}$ traced out by the vector function $\mathbf{r}$ is "smooth" if the derivative $\mathbf{r}'$ is continuous and never zero.

A smooth curve will have no sharp corners or cusps.

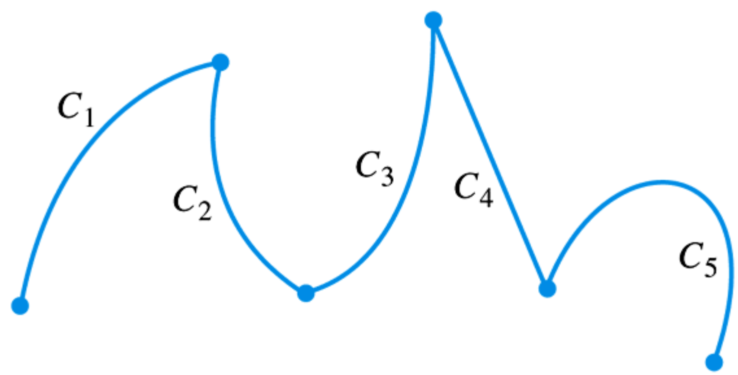


FIGURE A piecewise smooth curve made up of five smooth curves connected end to end in continuous fashion.

Ex: Show the the curve $\mathcal{C}$ with paramertrization

$$\mathbf{r}(t) := (t^2 + 1)\mathbf{i} + t\mathbf{j} + \cos t\mathbf{k}, \quad t \geq 0$$

is smooth.

When $\mathbf{r}'$ is continuous and nonzero, it forms a tangent vector that points in the direction of motion.

DEFINITIONS Velocity, Direction, Speed, Acceleration

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Applications matter! In the following, compute the glider's velocity and acceleration vectors. What is the glider's speed at any time t?

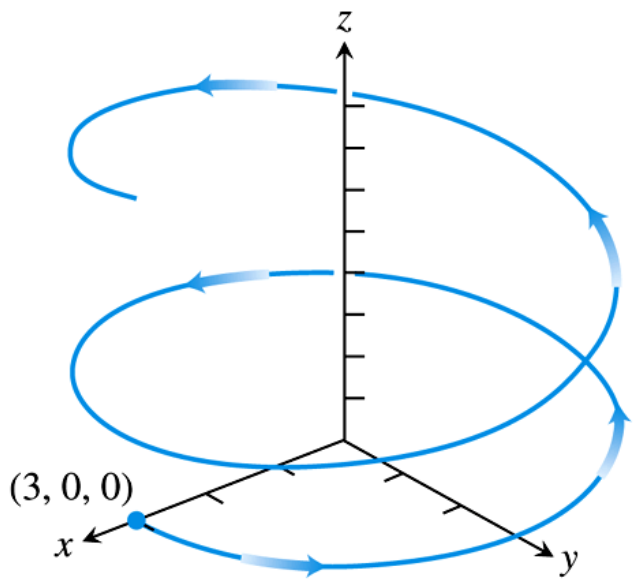


FIGURE The path of a hang glider with position vector $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

S5: Integral of a vector function

DEFINITION **Indefinite Integral**

The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

Ex: If $\mathbf{r}(t) := (t^2+1)\mathbf{i} + t\mathbf{j} + \cos t\mathbf{k}, \quad t \geq 0$ then compute $\int \mathbf{r}(t)\,dt$.

DEFINITION Definite Integral

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

EXAMPLE Evaluating Definite Integrals

$$\begin{aligned}\int_0^{\pi} ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt &= \left(\int_0^{\pi} \cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi} dt\right)\mathbf{j} - \left(\int_0^{\pi} 2t\,dt\right)\mathbf{k} \\ &= \Big[\sin t\Big]_0^{\pi}\,\mathbf{i} + \Big[t\Big]_0^{\pi}\,\mathbf{j} - \Big[t^2\Big]_0^{\pi}\,\mathbf{k} \\ &= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k} \\ &= \pi\mathbf{j} - \pi^2\mathbf{k}\end{aligned}$$

Riemann sums & integrals of vector–valued functions

In first–year you learnt that the (Riemann) integral is just the limit of Riemann sums. The same is true for integrals of vector–valued functions in the sense that can form vector–valued Riemann sums and look at their (vector–valued) limits.

For example, let $\mathbf{r} : [a, b] \to \mathbb{R}^2$. Slice up the interval $[a, b]$ to form a partition $\mathcal{P} := \{t_0, t_1, \ldots, t_n\}$ where n is some positive integer. If $\Delta t_i := t_{i+1} - t_i$ and if we choose any $t_i^* \in [t_i, t_{i+1}]$ then consider the (vector) Riemann sum

$$\mathbf{S}_n := \sum_{i=0}^{n-1} \mathbf{r}(t_i^*)\Delta t_i.$$

If $\lim \mathbf{S}_n$ exists as $n \to \infty$ (and, say, the limit equals some vector $\mathbf{L}$) independently of how we choose $\mathcal{P}$ and t_i^* then we say $\mathbf{r}$ is integrable on $[a, b]$.

Every continuous vector function $\mathbf{r}$ on $[a, b]$ is integrable on $[a, b]$.

Independent learning ex: Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) := (t, t, t)$. Calculate the integral of $\mathbf{r}$ over $[0, 1]$ by forming the lower Riemann sum and taking the limit.

Independent learning ex: In first–year you learnt powerful ideas such as the fundamental theorem(s) of calculus. Can these ideas be extended to vector functions of one variable? If so, then how?