

Section 1: Revision of vectors.

Notes compiled by Chris Tisdell

S1: Motivation

S2: Basic operations with vectors

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S4: The cross product & scalar triple product

Images from: "Thomas' calculus" by Wier, Hass &, Giordano, 2008, Pearson Publishing Inc.

S1: Motivation.

This set of notes provides revision of basic theory of vectors and applications that will be required to study a new area of applied mathematics known as “vector calculus”. Vector calculus is widely used by applied scientists, physicists and engineers.

All of the material herein was taught in first-year at UNSW and more information can be found within the relevant first-year course packs.



Amigos! You must know this basic material!!

1. Basic revision of vectors

DEFINITIONS Vector, Initial and Terminal Point, Length

A **vector** in the plane is a directed line segment. The directed line segment \overrightarrow{AB} has **initial point** A and **terminal point** B ; its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

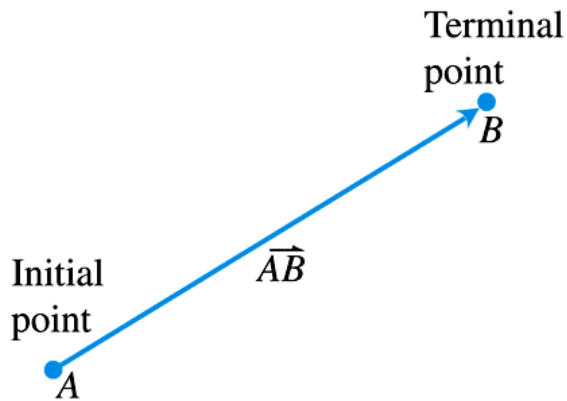


FIGURE The directed line segment \overrightarrow{AB} .

DEFINITION Component Form

If \mathbf{v} is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If \mathbf{v} is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

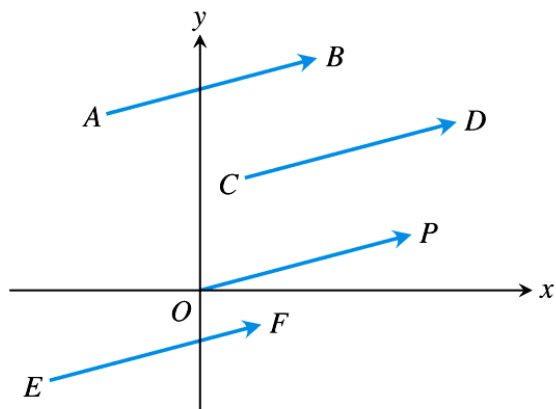


FIGURE The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

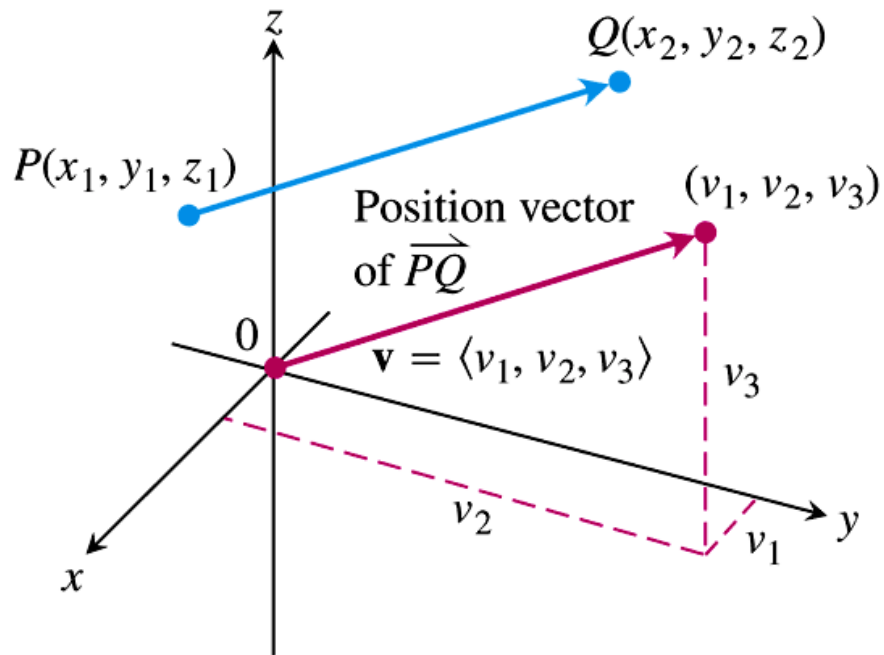


FIGURE A vector \vec{PQ} in standard position has its initial point at the origin. The directed line segments \vec{PQ} and \mathbf{v} are parallel and have the same length.

The **magnitude** or **length** of the vector $\mathbf{v} = \vec{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vector addition & scalar multiplication

DEFINITIONS Vector Addition and Multiplication of a Vector by a Scalar

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

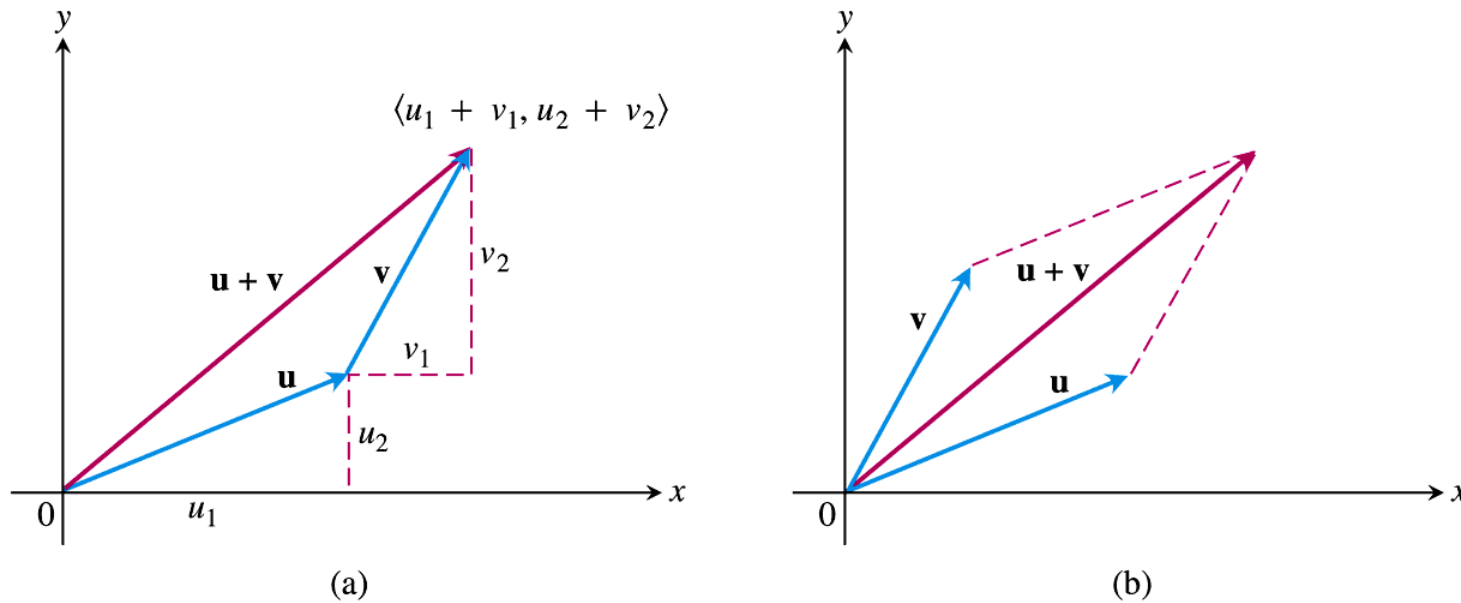
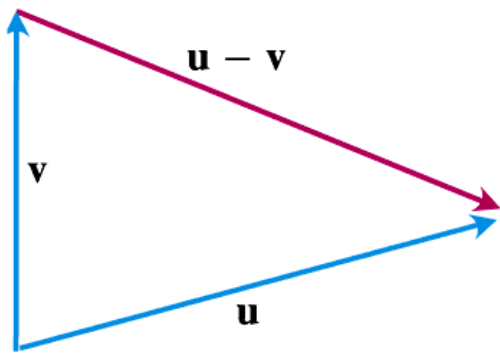
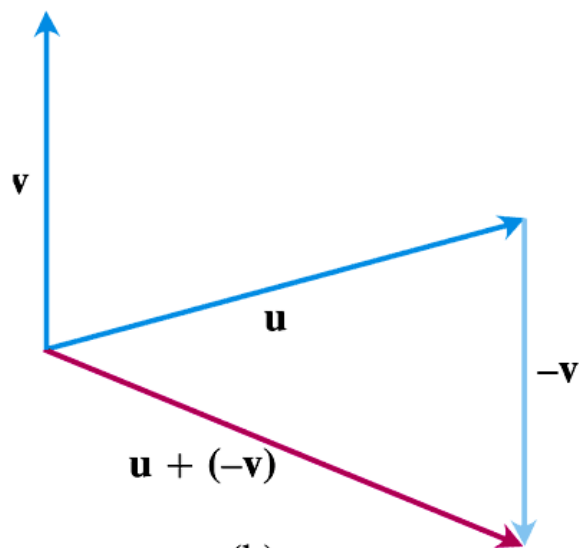


FIGURE (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.



(a)



(b)

FIGURE (a) The vector $\mathbf{u} - \mathbf{v}$, when added to \mathbf{v} , gives \mathbf{u} .
 (b) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

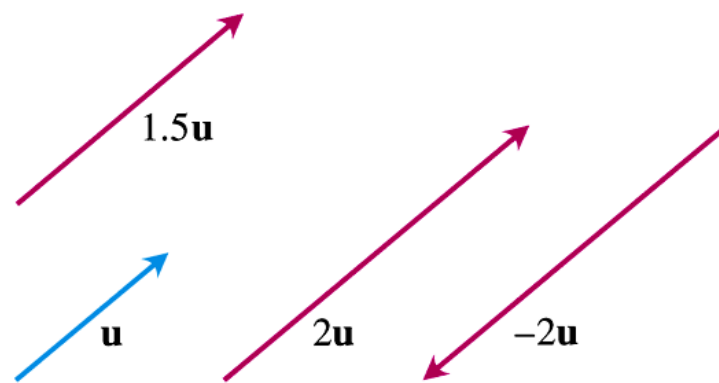


FIGURE Scalar multiples of \mathbf{u} .

EXAMPLE Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} , \mathbf{w} be vectors and a , b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit vectors

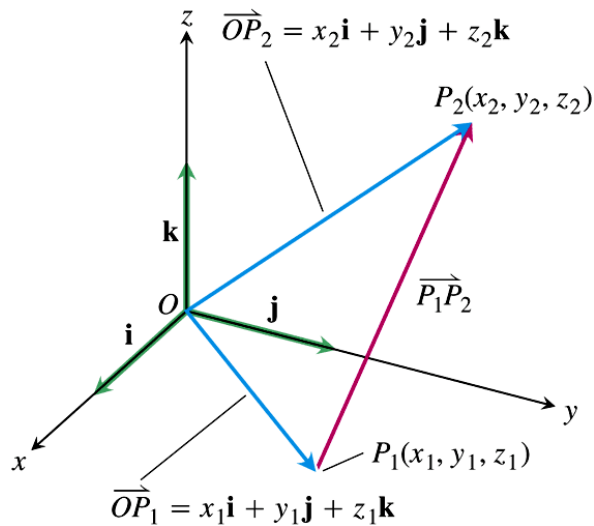


FIGURE The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} in terms of its length and direction.

EXAMPLE Expressing Velocity as Speed Times Direction

If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express \mathbf{v} as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of \mathbf{v} :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as \mathbf{v} :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \right).$$

Length (speed) Direction of motion

S3. The dot product

DEFINITION Dot Product

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“**u dot v**”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE Finding Dot Products

$$\begin{aligned} \text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7 \end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3\mathbf{j} + \mathbf{k} \right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

THEOREM Angle Between Two Vectors

The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

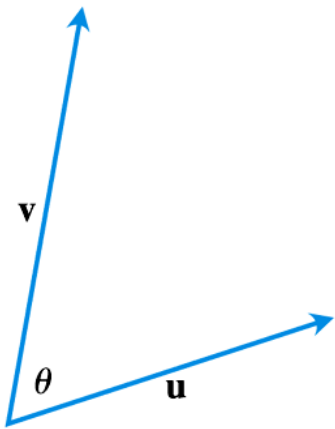


FIGURE The angle between \mathbf{u} and \mathbf{v} .

EXAMPLE Finding the Angle Between Two Vectors in Space

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) \\ &= \cos^{-1} \left(\frac{-4}{(3)(7)} \right) \approx 1.76 \text{ radians.}\end{aligned}$$

The angle formula applies to two-dimensional vectors as well.

DEFINITION Orthogonal Vectors

Vectors \mathbf{u} and \mathbf{v} are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Vector projection of \mathbf{u} onto \mathbf{v} :

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$$

Scalar component of \mathbf{u} in the direction of \mathbf{v} :

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

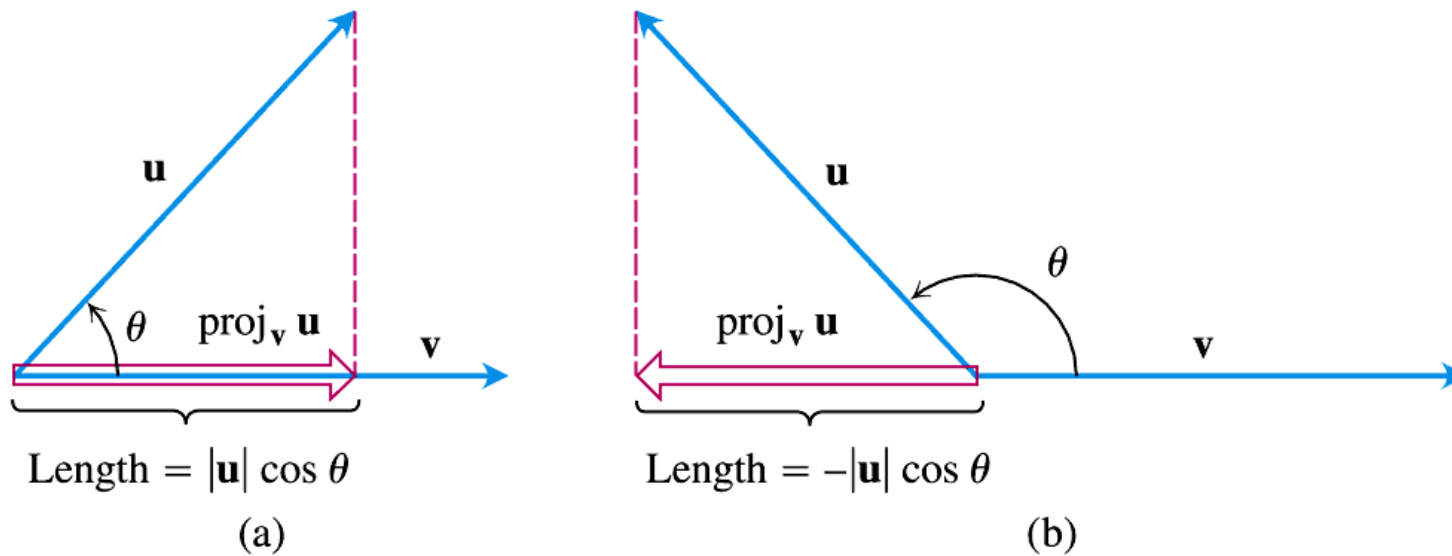


FIGURE The length of $\text{proj}_{\mathbf{v}} \mathbf{u}$ is (a) $|\mathbf{u}| \cos \theta$ if $\cos \theta \geq 0$ and (b) $-|\mathbf{u}| \cos \theta$ if $\cos \theta < 0$.

EXAMPLE Finding the Vector Projection

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

Solution We find $\text{proj}_{\mathbf{v}} \mathbf{u}$

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}.\end{aligned}$$

We find the scalar component of \mathbf{u} in the direction of \mathbf{v}

$$\begin{aligned}|\mathbf{u}| \cos \theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

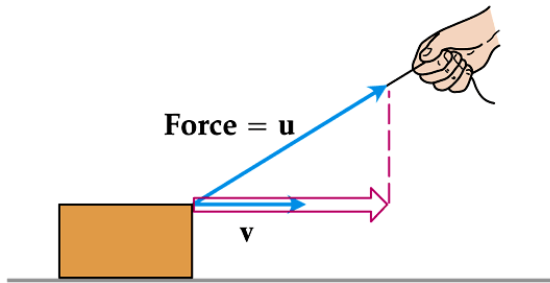


FIGURE If we pull on the box with force \mathbf{u} , the effective force moving the box forward in the direction \mathbf{v} is the projection of \mathbf{u} onto \mathbf{v} .

DEFINITION Work by Constant Force

The **work** done by a constant force \mathbf{F} acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos \theta,$$

where θ is the angle between \mathbf{F} and \mathbf{D} .

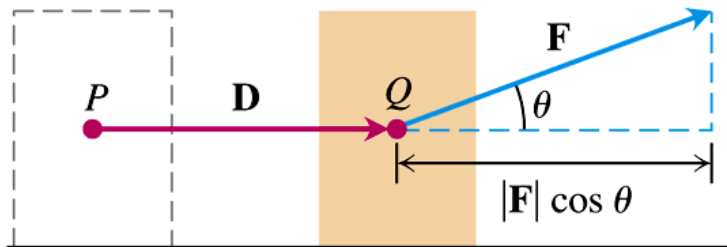
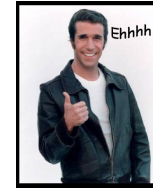


FIGURE The work done by a constant force \mathbf{F} during a displacement \mathbf{D} is $(|\mathbf{F}| \cos \theta) |\mathbf{D}|$.



S4. The cross product

DEFINITION Cross Product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}| |\mathbf{v}| \sin \theta) \mathbf{n}$$

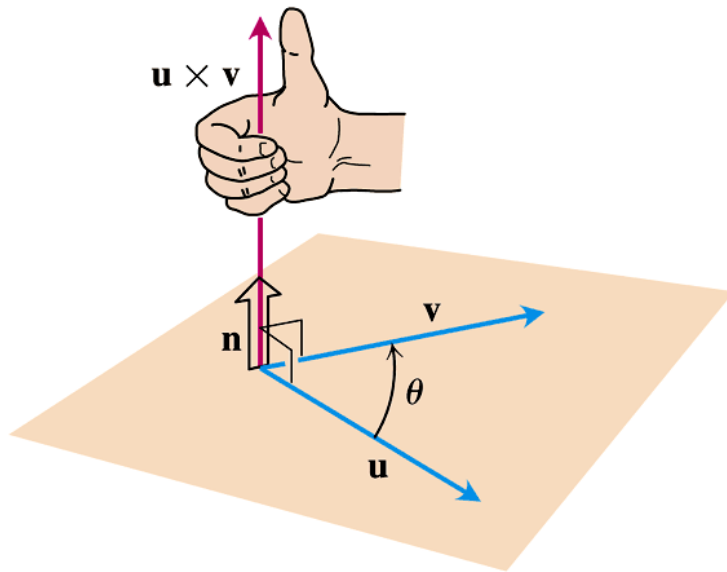


FIGURE The construction of $\mathbf{u} \times \mathbf{v}$.

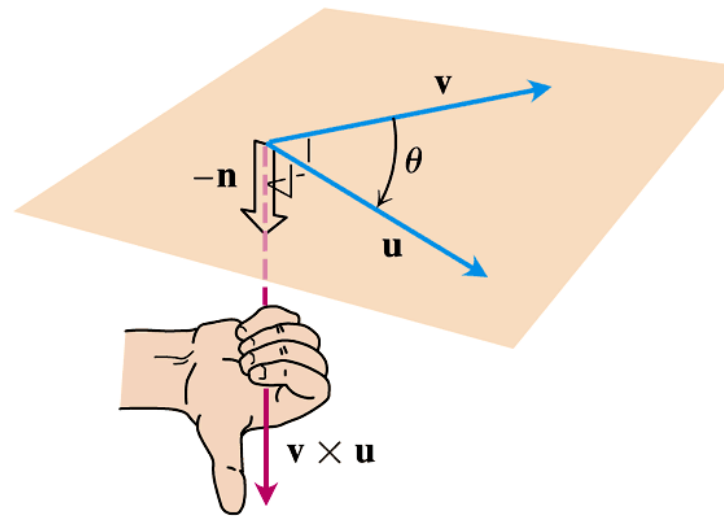


FIGURE The construction of $\mathbf{v} \times \mathbf{u}$.

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Calculating Cross Products Using Determinants

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE Calculating Cross Products with Determinants

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \end{aligned}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

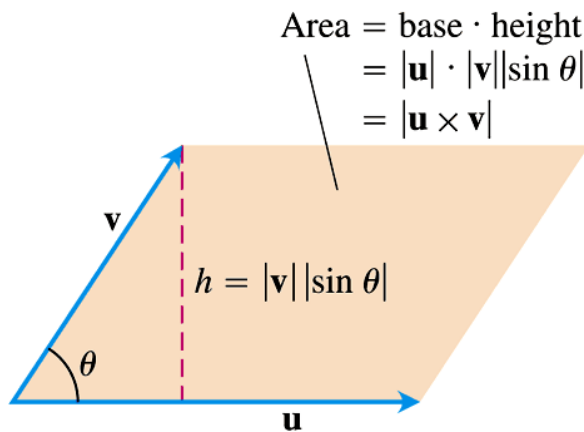
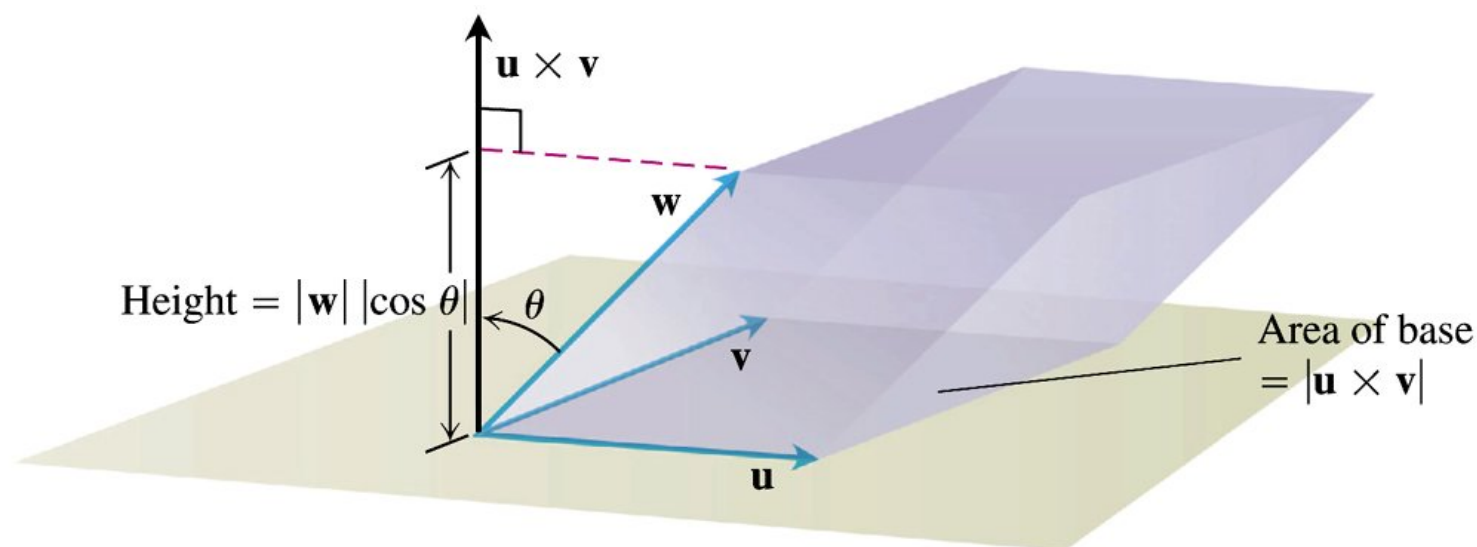


FIGURE The parallelogram determined by \mathbf{u} and \mathbf{v} .

Calculating the Triple Scalar Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



$$\begin{aligned} \text{Volume} &= \text{area of base} \cdot \text{height} \\ &= |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta| \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

FIGURE The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

EXAMPLE Finding the Volume of a Parallelepiped

Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.