Section 4: Extreme Values & Lagrange Multipliers.

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S1: Motivation.

Many problems in applied mathematics can be framed within the context of determining maximum or minimum values of a function of several variables.

For example, think of designing a cylindrical silo to maximise its volume from a fixed amount of building material.

At school you studied how to determine extreme values of functions of one variable, ie when

$$y = f(x).$$

In this section we extend these ideas to determine extreme values for functions of *many* variables. In particular, our methods involve applications of partial derivatives

S2. What are local maxima and local minima?

DEFINITIONS   Local Maximum, Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

We group local maxima and local minima under the term “local extrema”.

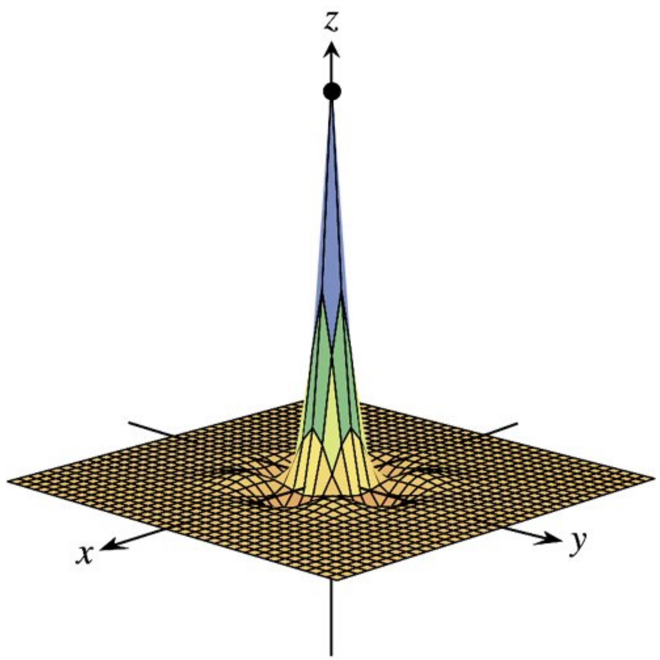


FIGURE The function

$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$

has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2, |y| \leq 3\pi/2$.

Building the intuition: Geometrical interpretation.

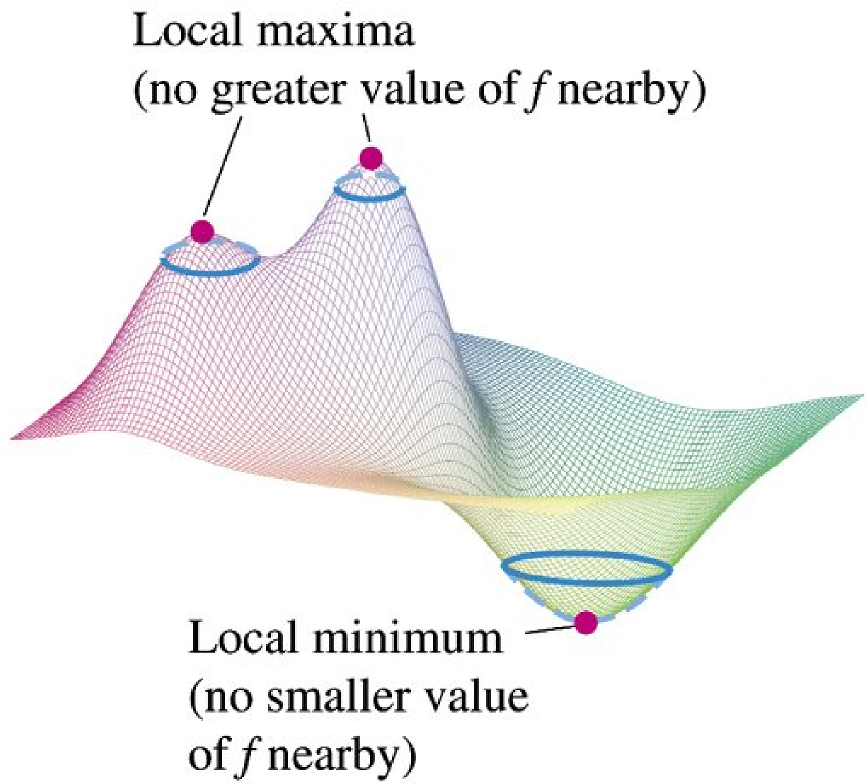


FIGURE A local maximum is a mountain peak and a local minimum is a valley low.

Local maxima correspond to mountain peaks of the surface $z = f(x, y)$.
Local minima correspond to valley bottoms of the surface $z = f(x, y)$.

First derivative test for local extreme values

THEOREM **First Derivative Test for Local Extreme Values**

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note that the First Derivative Test says that the surface has a horizontal tangent plane at local extremal points (provided the tangent plane actually exists).

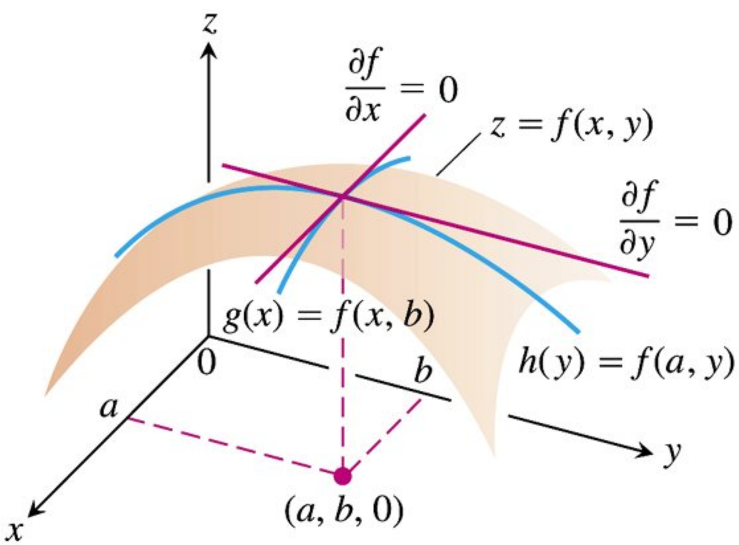


FIGURE If a local maximum of f occurs at $x = a, y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

S3. What is a critical point?

DEFINITION Critical Point

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

The First Derivative Test tells us that the only points where $f(x, y)$ can assume extreme values are: at critical points and at boundary points.

Ex: Determine all critical points of

$$f(x, y) := xy - x^2 - y^2 - 2x - 2y + 4.$$

Ex: Calculate the critical points of the function $f(x, y) := y^2 - x^2$.

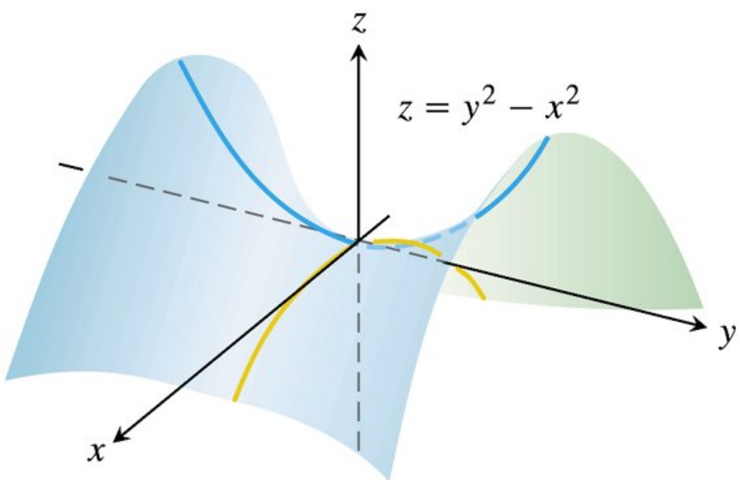


FIGURE The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values

Not every critical point gives rise to a local extremum, as the above example shows. Remember functions of one variable and points of inflection? For functions of two variables, we call this type of situation a "**saddle point**".

DEFINITION Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface

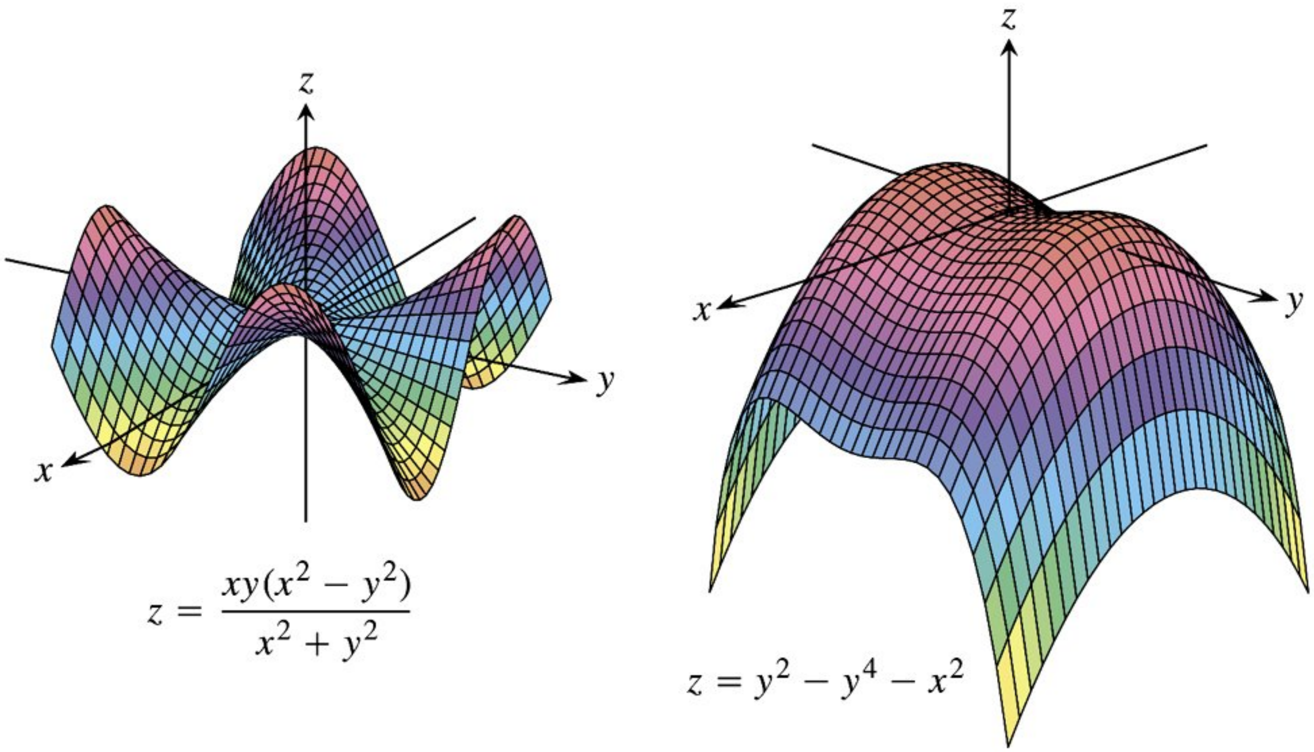


FIGURE Saddle points at the origin.

EXAMPLE Finding Local Extreme Values

Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum ■

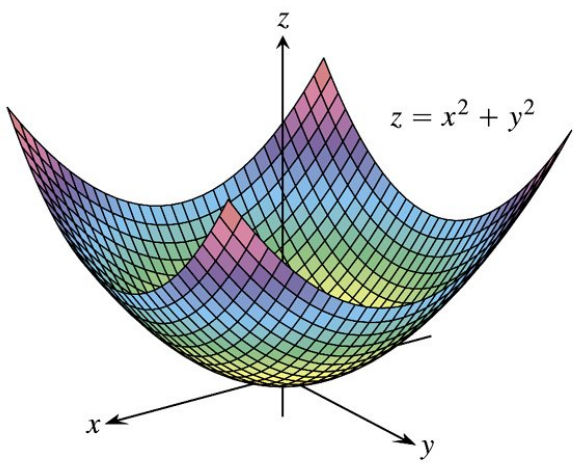


FIGURE The graph of the function $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$. The function has a local minimum value of 0 at the origin

S4. Second Derivative Test

Given a critical point (a, b) of $f(x, y)$, how can we determine its local nature?

THEOREM Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii. f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv. **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The proof relies on the use of Taylor polynomial techniques (to be discussed a little later).

Ex: Determine and classify the critical points of $f(x, y) := 2x^3 - 6xy + 3y^2$.

Why does the 2nd derivative test work? Case $(a, b) = (0, 0)$.

For functions of two variables, we can adapt and apply Taylor's formula to form

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}\left[x^2 f_{xx}(cx, cy) + 2xyf_{xy}(cx, cy) + y^2 f_{yy}(cx, cy)\right]$$

for some $c \in [0, 1]$. By assumption, $f_x(0, 0) = 0 = f_y(0, 0)$. Consider

$$Q(c) := x^2 f_{xx}(cx, cy) + 2xyf_{xy}(cx, cy) + y^2 f_{yy}(cx, cy).$$

If $Q(0) \neq 0$ then the sign of $Q(c)$ will be the same as the sign of $Q(0)$ for all sufficiently small values of x and y. Thus, we consider

$$f_{xx}(0, 0)Q(0) = f_{xx}(0, 0)[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$
$$= [xf_{xx}(0, 0) + yf_{xy}(0, 0)]^2 + (f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2)y^2$$

We see that the conditions of the 2nd derivative test imply that: $Q(0) < 0$ for all (x, y) close to $(0, 0)$ (case 1); $Q(0) > 0$ for all (x, y) close to $(0, 0)$ (case 2); $Q(0)$ changes sign (case 3).

S5. Maxima and Minima on closed, bounded regions.

The 2nd derivative test is powerful, but does have limitations:

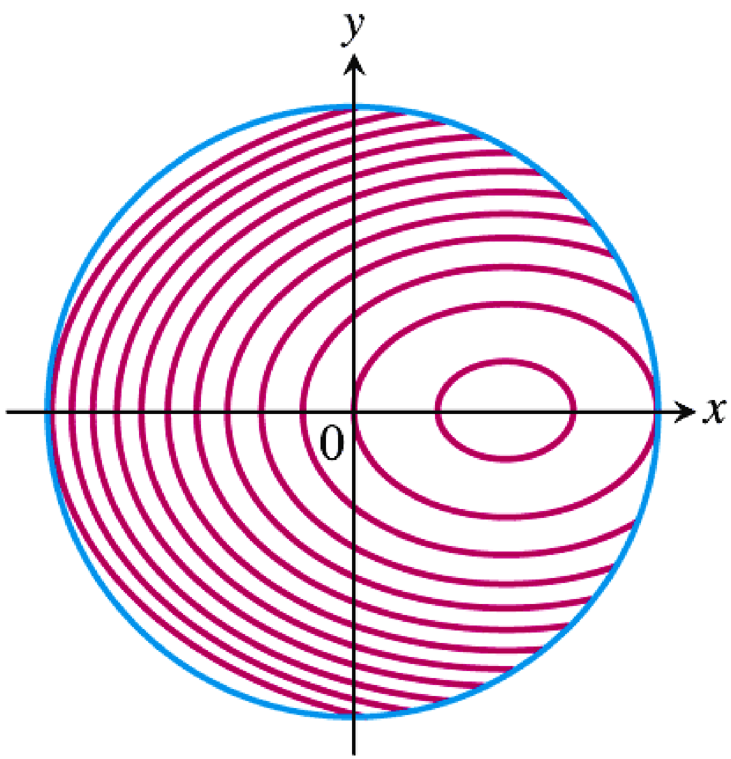
It does not apply to the boundary points of a function's domain, where it is possible for the function to have extreme values along with nonzero derivatives;

It does not cope with cases of critical points where first derivatives do not exist.

Ex: The function $f(x, y) := \sqrt{x^2 + y^2}$ does not satisfy the conditions of the 2nd derivative test.

Applications matter! A flat, circular plate occupies the region $x^2 + y^2 \leq 1$. The plate is heated so that the temperature at point (x, y) is $T(x, y) = x^2 + 2y^2 - x$. Where is the plate the hottest and coldest? What are the temperatures at these points?

FIGURE Curves of constant temperature are called isotherms. The figure shows isotherms of the temperature function $T(x, y) = x^2 + 2y^2 - x$ on the disk $x^2 + y^2 \leq 1$ in the xy-plane. Exercise asks you to locate the extreme temperatures.



S6. Summary of max–min tests

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive**.

S7. Lagrange Multipliers

Much of applied mathematics involves maximising or minimising a function where constraints occur.

The Method of Lagrange Multipliers
Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0.$$

For functions of two independent variables, the condition is similar, but without the variable z.

If we define the Lagrangian function L by

$$L(x, y, z, \lambda) := f(x, y, z) - \lambda g(x, y, z)$$

then see that the above method is just equivalent to determining critical points of L.

That is, we solve

$$
\begin{aligned}
0 &= \partial L/\partial x = \partial f/\partial x - \lambda \partial g/\partial x \\
0 &= \partial L/\partial y = \partial f/\partial y - \lambda \partial g/\partial y \\
0 &= \partial L/\partial z = \partial f/\partial z - \lambda \partial g/\partial z \\
0 &= g(x, y, z).
\end{aligned}
$$

Ex: Determine the maximum and minimum value of the function $f(x, y) := 3x + 4y$ that lie on the circle

$$x^2 + y^2 = 1.$$

(Cont):

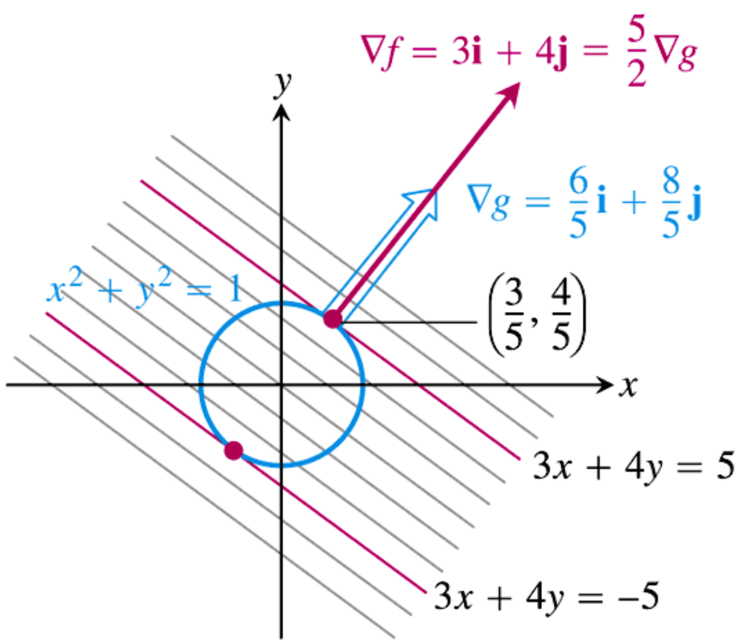


FIGURE The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$
At each of these points, ∇f is a scalar multiple of ∇g. The figure shows the gradients at the first point but not the second.

Applications matter! Surface heat loss in hot water systems is a problem of engineering design. A hot water storage tank is a vertical cylinder surmounted by a hemispherical top of the same diameter. Design such a hot water system that holds 400 m^3 while minimising surface heat loss.

Ex: Determine the point on the the surface $xyz = 1$ that is closest to the origin and satisfies $x > 0, y > 0, z > 0$.

Applications matter! A space probe in the shape of an ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters Earth's atmosphere. After 1 hour the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point(s) on the probe's surface.

S8. Why does the Lagrange method work?

THEOREM **The Orthogonal Gradient Theorem**

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

The above theorem is the key to the method of Lagrange multipliers. Suppose $f(x, y, z)$ and $g(x, y, z)$ are diff'able and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local max or min relative to its other values on the surface. Then f takes on a local max or min at P_0 relative to its value on every diff'able curve through P_0 on the surface $g(x, y, z) = 0$.

Therefore, ∇f is orthogonal to the velocity vector of every such diff'able curve through P_0. Also, ∇g is orthogonal to the velocity vector of every diff'able curve through P_0 (because ∇g is orthogonal to the level surface $g(x, y, z) = 0$). Therefore, at P_0, ∇f is some multiple λ of ∇g.

Joseph Lagrange was only 19 when he devised this method!

S9. Problems with two constraints

Sometimes we seek extreme values of a function f where two constraints $g_1 = 0$ and $g_2 = 0$ are now involved.

The method is similar as before. We solve the following for x, y, z, λ, μ:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$

Usually the surfaces $g_1 = 0$ and $g_2 = 0$ intersect to form a smooth curve C. Along this curve C we seek the points where f has local maximum and minimum values relative to its other values on the curve.

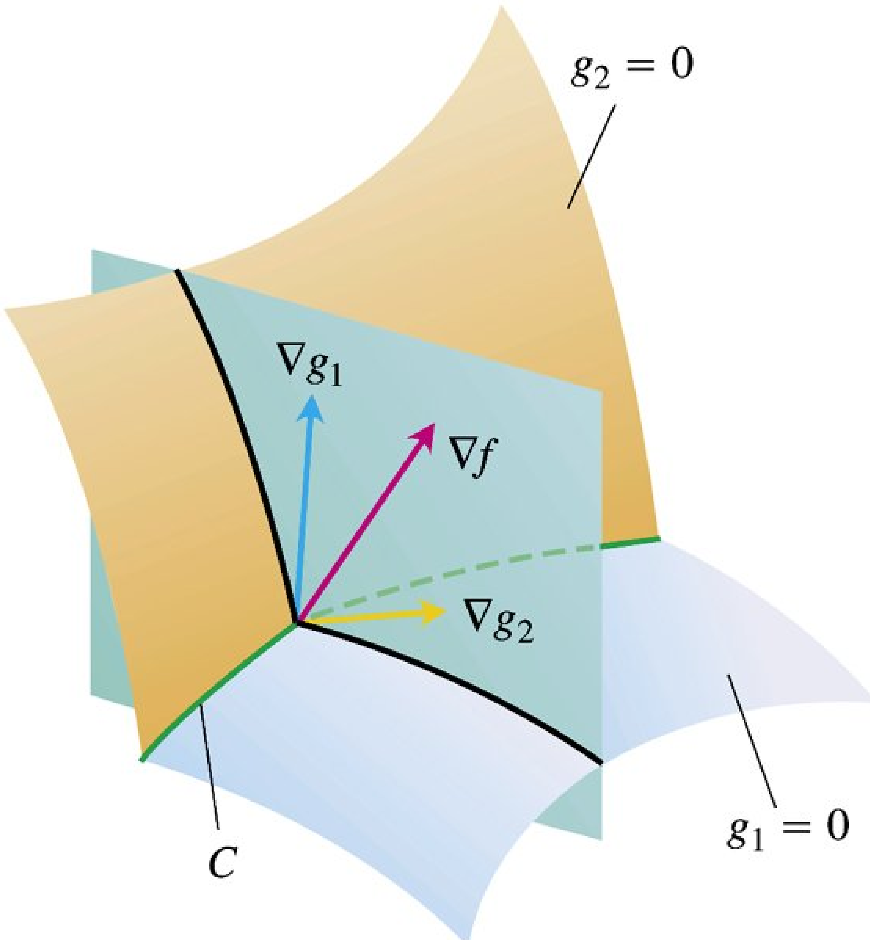


FIGURE The vectors ∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1 = 0$ and ∇g_2 is normal to the surface $g_2 = 0$.