

MATH 1231 MATHEMATICS 1B 2010.

Lecture notes by Chris Tisdell¹

Calculus Section 4.1: - Taylor Polynomials.

S1: Motivation

S2: Taylor polynomials

S3: Taylor's theorem

S4: Applications to max/min theory

S5: MAPLE.

¹Lecture notes created by Chris Tisdell. All images are from "Thomas' Calculus" by Wier, Hass and Giordano, Pearson, 2008.

1. Motivation

Our aim is develop a method for representing a (differentiable) function $f(x)$ as an (infinite) sum of powers of x .

For example, we will show

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Such a method will, for example, enable us to evaluate, differentiate and integrate a very wide class of functions.

For instance, we will show that

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

We will then be able to integrate this function, which we couldn't do previously.

The main thought-process behind our method is that powers of x are easy to evaluate, differentiate and integrate, so by rewriting complicated functions as sums of powers of x we can greatly simplify our analysis.

2. Taylor Polynomials

The “linearization” of a differentiable function f at a point a is the polynomial of degree one

$$P_1(x) := f(a) + f'(a)(x - a).$$

This linearization was used in MATH1131 to approximate $f(x)$ at values of x near a .

If f has derivatives of higher-order at a then it has higher-order approximations also, one for each available derivative. These polynomials are known as the “Taylor polynomials of f ”.

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots \\ + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$



Brook Taylor was an English mathematician who added to mathematics the ‘calculus of finite differences’, invented integration by parts, and discovered the celebrated formula known as Taylor’s expansion.

Note that in the definition of Taylor polynomials, we speak of a Taylor polynomial of order n , rather than of degree n . This is because $f^{(n)}(a)$ may be zero.

For example, the first two Taylor polynomials of

$$f(x) = \sin x$$

at $x = a = \pi/2$ are:

$$P_0(x) = 1; \quad \text{and} \quad P_1(x) = 1.$$

See that the first-order Taylor polynomial P_1 has degree zero, not one!

However, in your course pack, the term “Taylor polynomial of degree n ” is used...

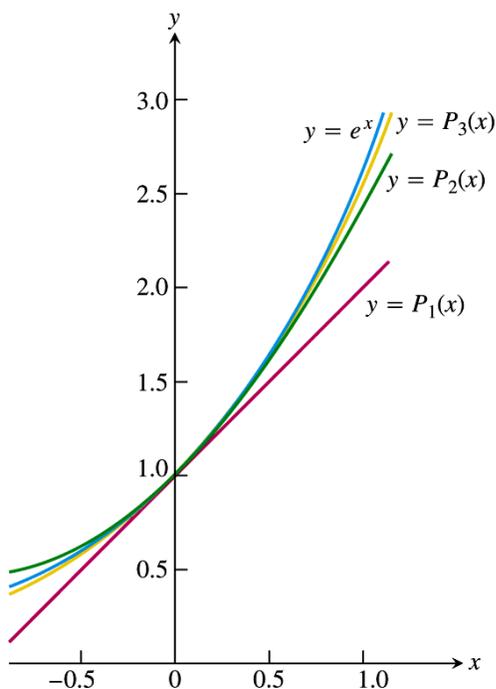
Ex: Compute the Taylor polynomial of order 3 generated by $f(x) = e^x$ at $x = 0 = a$.

We see:

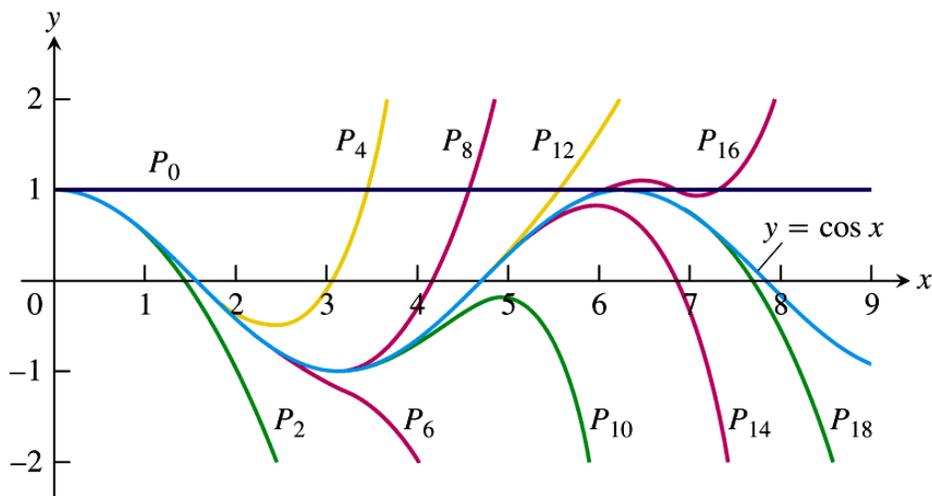
$$\begin{aligned}f(x) &= e^x, & f(0) &= 1; \\f'(x) &= e^x, & f'(0) &= 1; \\f''(x) &= e^x, & f''(0) &= 1; \\f'''(x) &= e^x, & f'''(0) &= 1.\end{aligned}$$

Our Taylor polynomial definition gives

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$



Ex: Compute the Taylor polynomial of order 4 generated by $f(x) = \cos x$ at $x = 0 = a$. What do you notice about $P_{2n}(x)$ and $P_{2n+1}(x)$?



3. Taylor's theorem

We can see from the previous graphs, that Taylor polynomials appear to approximate their generating functions if we are close to the point $x = a$. If we move away from $x = a$ then the approximations may not be so good.

In this section we answer the question of the accuracy of Taylor polynomials approximating their generating functions on a given interval.

THEOREM **Taylor's Theorem**

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b , and $f^{(n)}$ is differentiable on the open interval between a and b , then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \dots \\ + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's theorem is an extension of the mean value theorem.

When applying Taylor's theorem, we frequently wish to hold a fixed and consider b as an independent variable. If we change b to x in Taylor's formula, then it is easier to use in these cases. We obtain:

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I ,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

The above version of Taylor's theorem says that for all $x \in I$ we have

$$f(x) = P_n(x) + R_n(x).$$

Taylor's formula gives: a polynomial approximation to f to any order n ; and a formula for the error involved in employing that approximation over the interval I .

The function $R_n(x)$ is termed the *remainder of order n* , or the error term for the approximation of f by $P_n(x)$ over I . **Note:** in some texts the notation R_n is replaced with R_{n+1} with the right-hand side of (2) remaining unchanged.

It is common to be able to estimate $R_n(x)$ on the interval I . This estimation is contained in the following theorem.

THEOREM **The Remainder Estimation Theorem**

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a , inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

Proof: Taking absolute values in (2) of Taylor's formula we obtain, for all $x \in I$,

$$\begin{aligned} |R_n(x)| &= \left| \frac{f^{(n+1)}(c)}{(n + 1)!} (x - a)^{n+1} \right| \\ &\leq M \frac{|(x - a)^{n+1}|}{(n + 1)!}. \end{aligned}$$

Ex: Estimate the error of approximating $f(x) = e^x$ with $p_2(x) = 1 + x + x^2/2$ over the interval $x \in (-0.5, 0.5)$.

4. Applications to max/min theory.

At school you learnt the 2nd–derivative test for classifying stationary points of functions. We now can prove a “second” 2nd–derivative test by employing Taylor’s theorem!

Theorem. Let f have continuous 1st– and 2nd–order derivatives on an interval I and suppose that $f'(a) = 0$ for some $a \in I$.

(i) If $f'' \leq 0$ on I then f has a local max at a .

(ii) If $f'' \geq 0$ on I then f has a local min at a .

Proof. We prove (i) by using the equation

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c)}{2}(x - a)^2,$$

for all $x \in I$;

with c between a and x .

Note: (i) applies to functions such as $f(x) = -x^4$ at $a = 0$. The test also includes constant functions, which, of course have max and min values everywhere!

If the order of our Taylor “polynomial” becomes large (that is, $n \rightarrow \infty$) then we would like to determine if the error $R_n(x)$ goes to zero for each $x \in I$. This would mean that our “infinite” sum of powers of x would indeed express f accurately on I . This was our objective at the beginning of this section.

To deeply understand what we mean by $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ on I , we examine the limits of sequences.

5. MAPLE.

To obtain the Taylor polynomial of e^x of order 4 we use the following command:

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T1:=taylor(exp(x), x = 0, 5);
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$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)$$

We can remove the $O(x^5)$ and convert into a “proper” polynomial by using:

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convert(T1, polynom);
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$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$