MATH 1231 MATHEMATICS 1B 2010.
Calculus Section 5: Applications of integration.

S1: Motivation

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Lecture notes created by Chris Tisdell. All images are from "Thomas' Calculus" by Wier, Hass and Giordano, Pearson Inc., 2008.

1: Motivation

The concepts of average value and arc length appear frequently in applied mathematics and physics.

For example, average values of certain functions are used in the calculation of Fourier series.

Arc length is used in the calculation of total mass of wires and springs, where the (constant) density is known.

2: Average value of a function

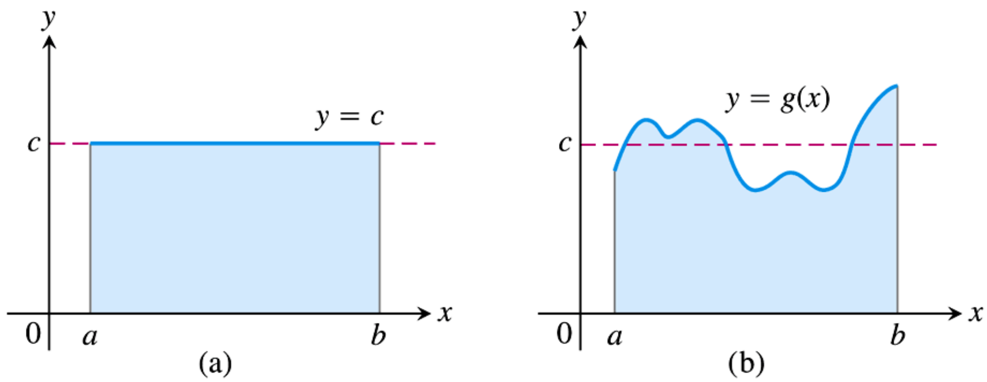


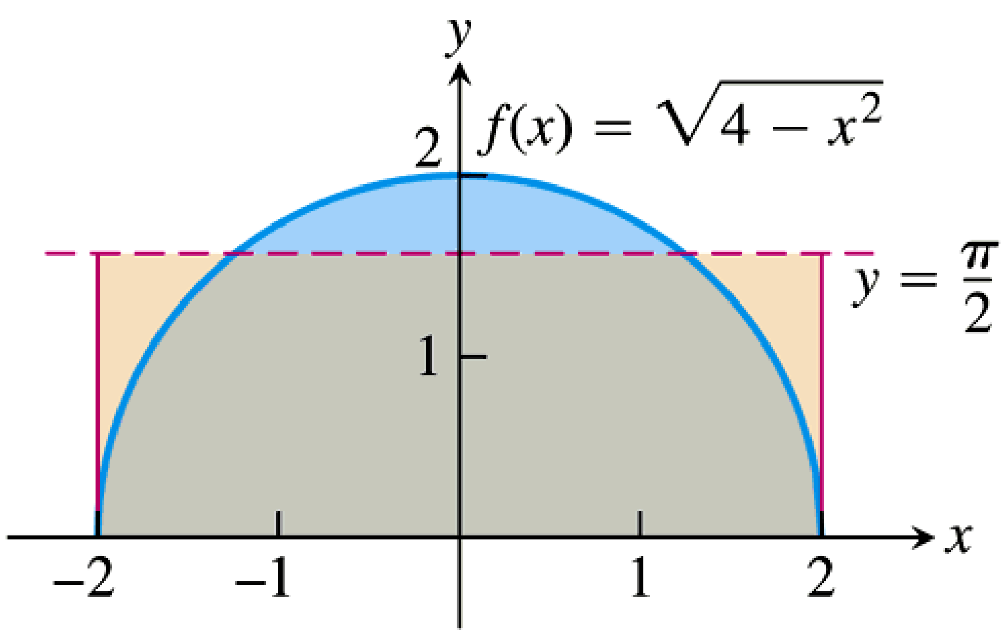
FIGURE (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

DEFINITION **The Average or Mean Value of a Function**

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Ex: Find the average value of $f(x) = \sqrt{4 - x^2}$ for $x \in [-2, 2]$.



The mean value theorem for integrals

THEOREM **The Mean Value Theorem for Definite Integrals**

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

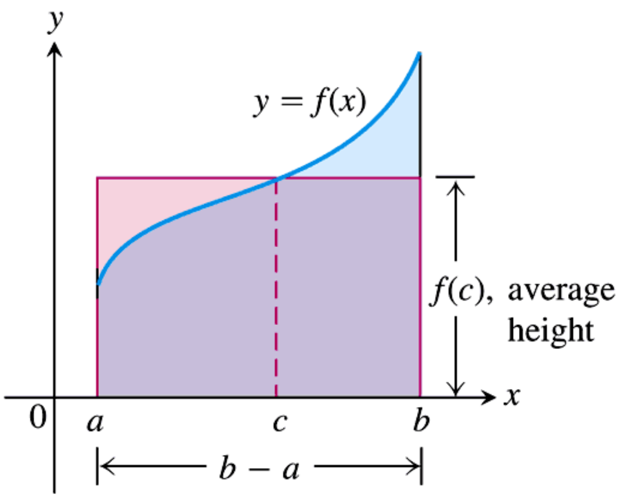


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

Independent learning ex: The previous result illustrates a significant distinction between the average of a list of numbers and the average value of a continuous function. What is it?

3. Arc length

Simple motivation: Consider the graph of $y = x^2$. How do we calculate the length of the curve between the points (0,0) and (1,1)?

Formula for the Length of $y = f(x), \quad a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

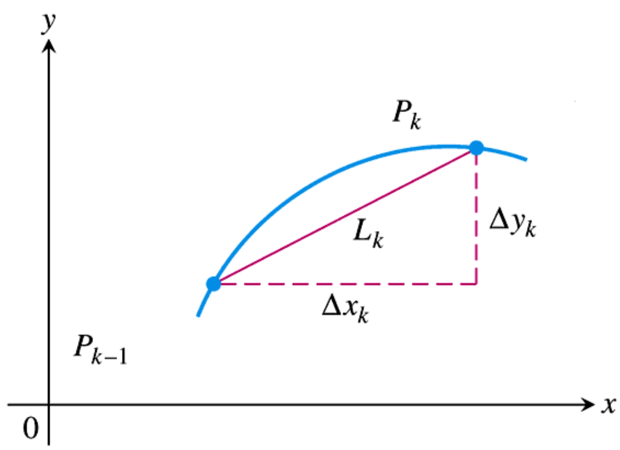


FIGURE The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Ex: Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \quad 0 \leq x \leq 2.$$

Applications Matter! Consider a thin wire of *constant* density δ (mass per unit length) that lies in the $XY-$ plane along the curve

$$y = f(x) := 4\sqrt{2}x^{3/2}/3 - 1, \quad 0 \le x \le 1.$$

If it can be shown that the total mass M of the wire is given by

$$M = \int_0^1 \delta\sqrt{1 + [f'(x)]^2}\, dx.$$

then calculate the total mass of the wire.

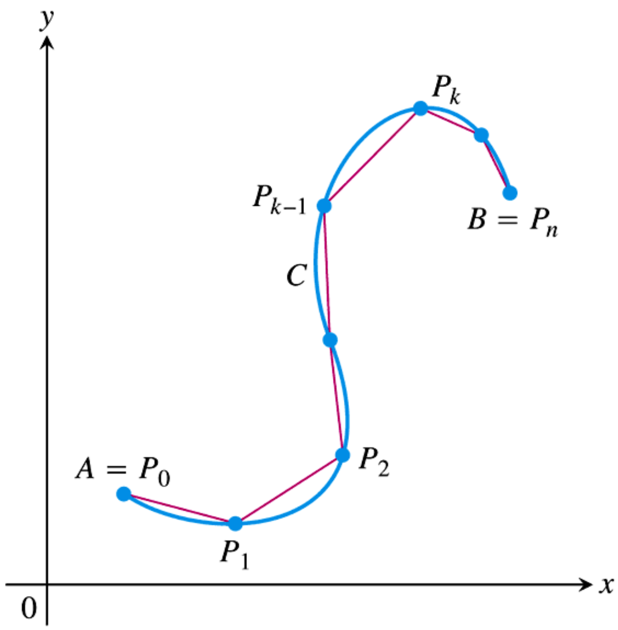


FIGURE The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

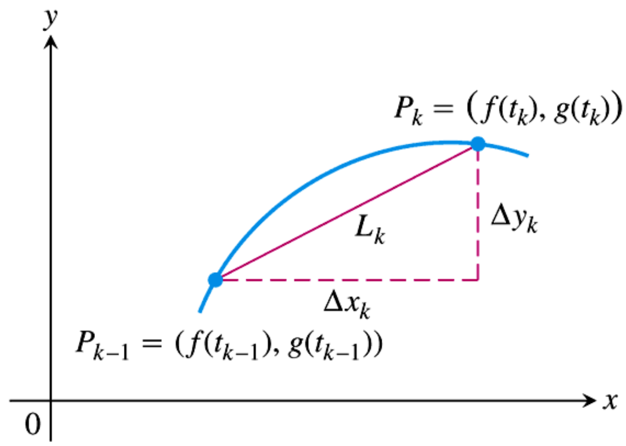


FIGURE The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Applications matter! Huygen's pendulum clock.

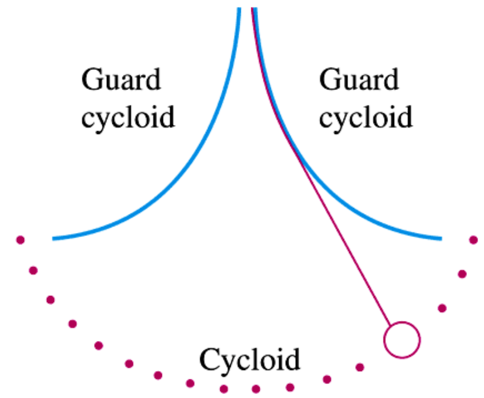


FIGURE 10.30 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

One challenge in creating accurate pendulum clocks is that a bob that swings in a circular arc has a frequency that depends on the amplitude of the swing. The wider the swing, the longer is take the bob to return to its lowest position.

To avoid this problem, Huygens designed a clock whose bob can be made to swing in a cycloid by hanging the bob from fine wire constrained by guards that caused it to draw up as it swung away from centre.

Ex: One arch of a "cycloid" generated by a circle of radius 2 has parametric equations

$$x = 2(t - \sin t), \quad y = 2(1 - \cos t), \quad 0 \le t \le 2\pi.$$

Calculate its length.

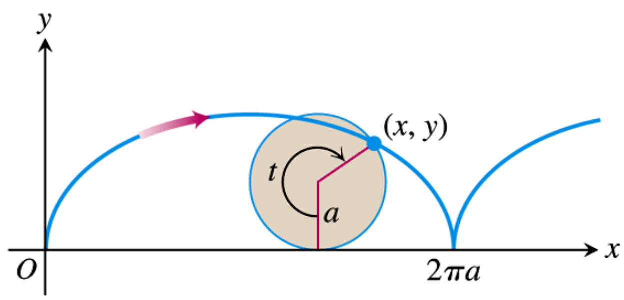


FIGURE The cycloid
$x = a(t - \sin t), y = a(1 - \cos t)$, for
$t \geq 0$.

4. Speed

The speed at time t of a particle with trajectory $c(t) = (x(t), y(t))$ is the derivative of the arc length integral

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du$$

and thus

$$\text{Speed } = s'(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

Proof:

Ex: A particle moves along the path $c(t) = (2t, 1 + t^{3/2})$, where t is in minutes, distance in metres. (a) Determine the speed at $t = 1$. (b) Determine the distance travelled s after 4 minutes.

5. Surface area

DEFINITION Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

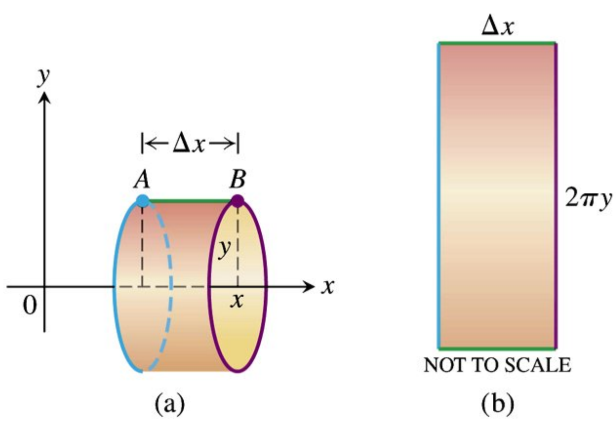


FIGURE (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled out cylindrical surface as a rectangle.

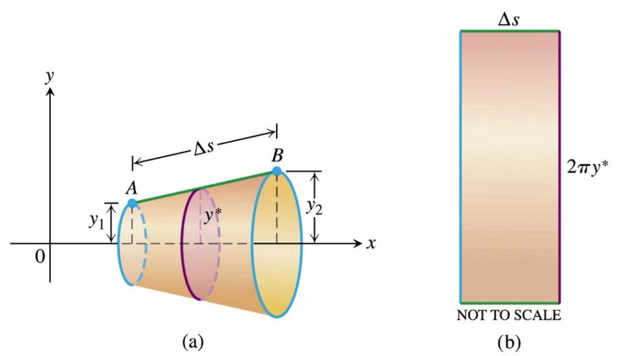


FIGURE (a) The frustum of a cone generated by rotating the slanted line segment AB of length Δs about the x-axis has area $2\pi y^* \Delta s$. (b) The area of the rectangle for $y^* = \frac{y_1 + y_2}{2}$, the average height of AB above the x-axis.

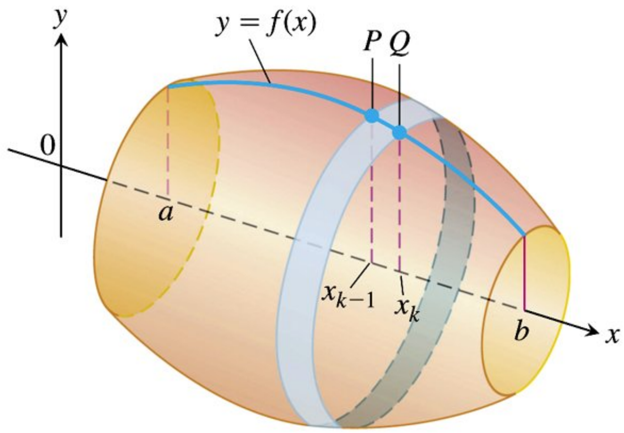


FIGURE The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

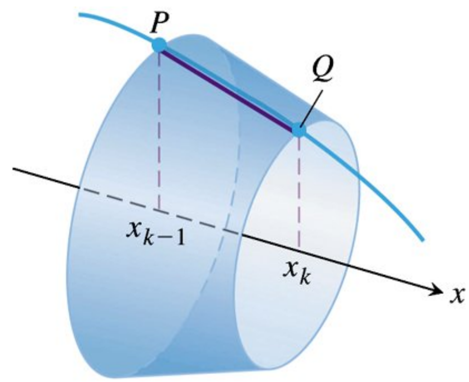


FIGURE The line segment joining P and Q sweeps out a frustum of a cone.

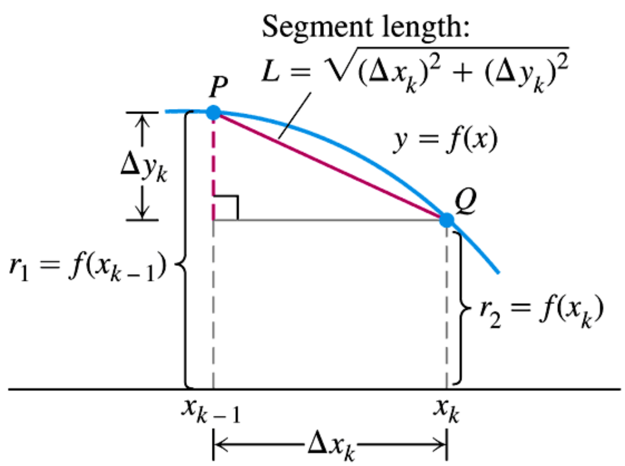


FIGURE Dimensions associated with the arc and line segment PQ.

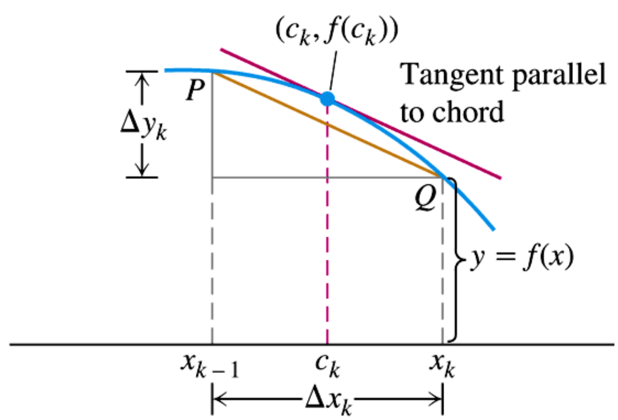


FIGURE If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. Revolution about the y-axis $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

Ex: Find the area of the surface obtained by rotating $y = x^{1/2} - \frac{1}{3}x^{3/2}$ about the x–axis for $1 \leq x \leq 3$.