MATH 1231 MATHEMATICS 1B 2009.
Calculus Section 4.2 - Sequences.

Lecture notes created by Chris Tisdell. All images are from "Thomas' Calculus" by Wier, Hass and Giordano, Pearson, 2008; and "Calculus" by Rogowski, W H Freeman & Co., 2008.

1: Motivation

Sequences are one of the most important parts of calculus and mathematical analysis.

Sequences occur in nature all around us and a good understanding of them enable us to accurately model many “discrete” phenomena. For example, “Fibonacci” sequences are seen in population models (breeding of rabbits; bee ancestry code).

Sequences are also a very useful tool in approximating solutions to complicated equations. For example, how can we approximate the roots of

$$10x - 1 - \cos x = 0?$$

2: What is a sequence?

DEFINITION **Infinite Sequence**

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by writing rules that specify their terms, such as:

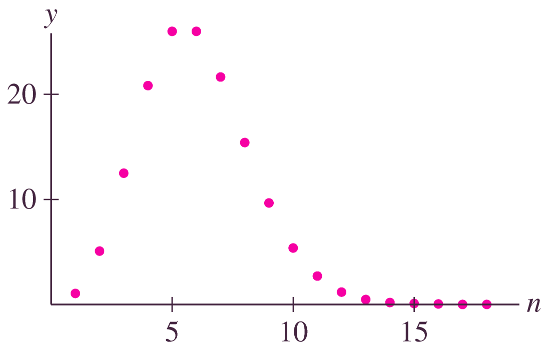
$$\begin{aligned} a_n &= n^2 \\ b_n &= (-1)^n \\ c_n &= 1/n \\ d_n &= e^{-n}. \end{aligned}$$

Or we can list the terms of the sequence:

$$\begin{aligned} \{a_n\} &= \{1, 4, 9, \ldots, n^2, \ldots\} \\ \{b_n\} &= \{-1, 1, -1, \ldots, (-1)^n, \ldots\} \\ \{c_n\} &= \{1, 1/2, 1/3, \ldots, 1/n, \ldots\} \\ \{d_n\} &= \{e^{-1}, e^{-2}, e^{-3}, \ldots, e^{-n}, \ldots\}. \end{aligned}$$

The order of the terms in a sequence is important. The sequence $1, 2, 3, 4, \ldots$ is not the same as the sequence $3, 1, 2, 4, \ldots$.

Graphs of sequences:



Graph of the sequence $a_n = \frac{5^n}{n!}$.

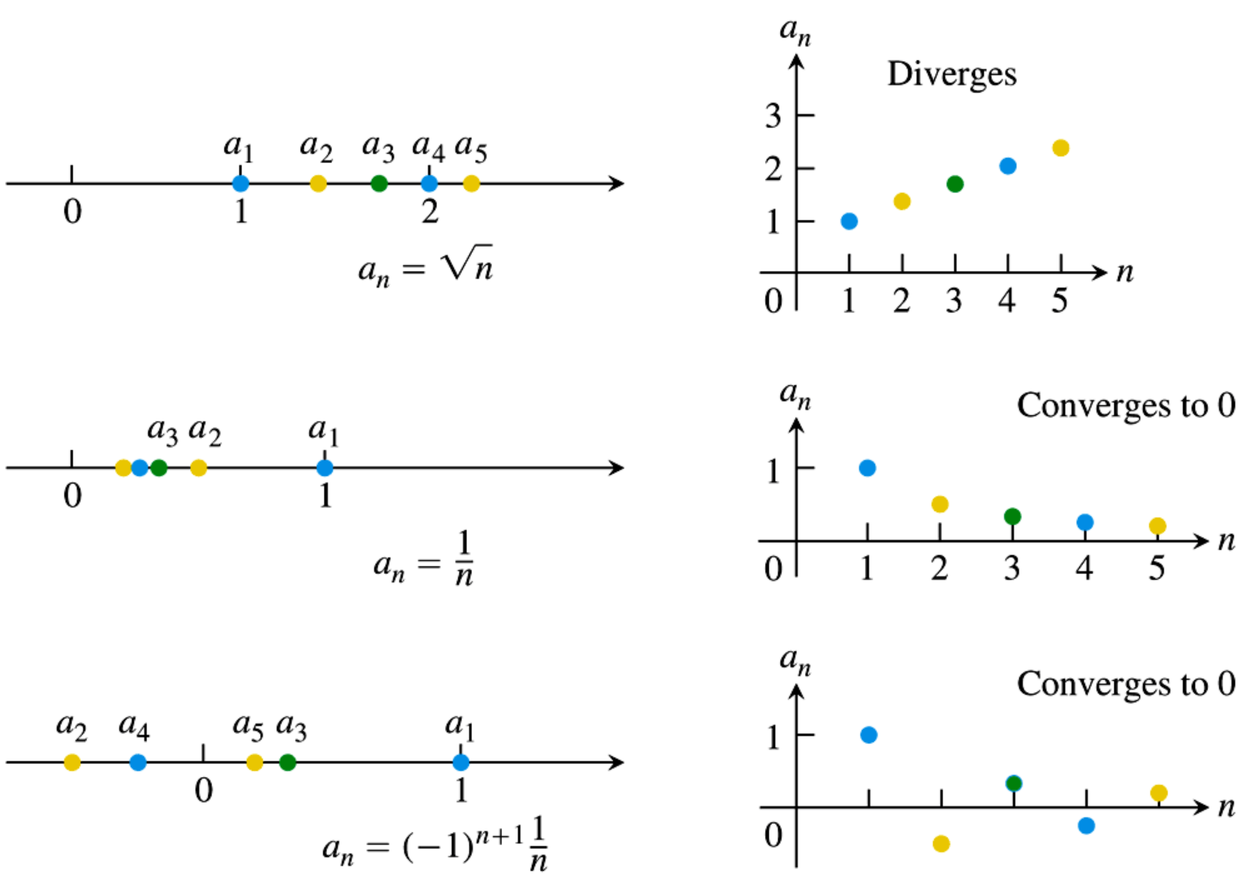


FIGURE Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

3: Limit of a sequence.

The big question that we wish to answer, is, *do the terms of our sequence get closer to some finite number as we go further and further along the sequence*?

That is, does

$$\lim_{n \to \infty} a_n$$

exist and, if so, then what is it?

Convergence

The formal definition of a convergent sequence is similar to what you learnt in MATH1131 for limits of functions $f(x)$ as $x \to \infty$.

DEFINITIONS **Converges, Diverges, Limit**

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence .

4: Geometrical interpretation.

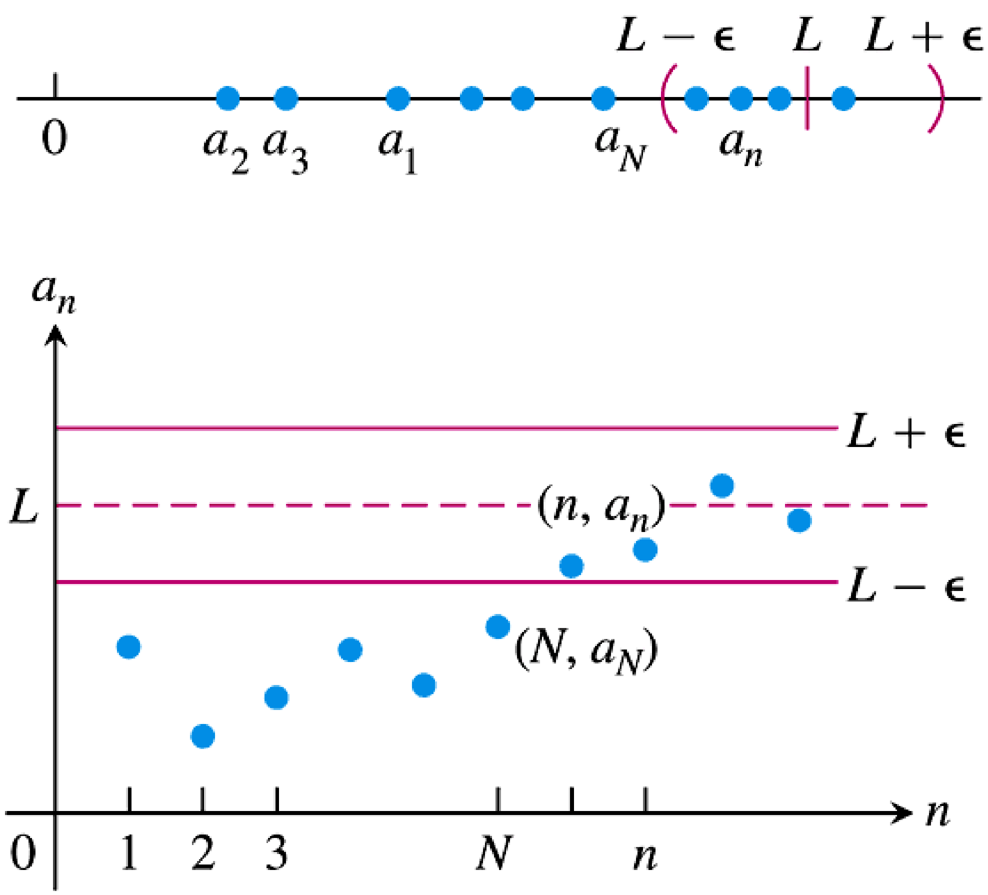


FIGURE $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

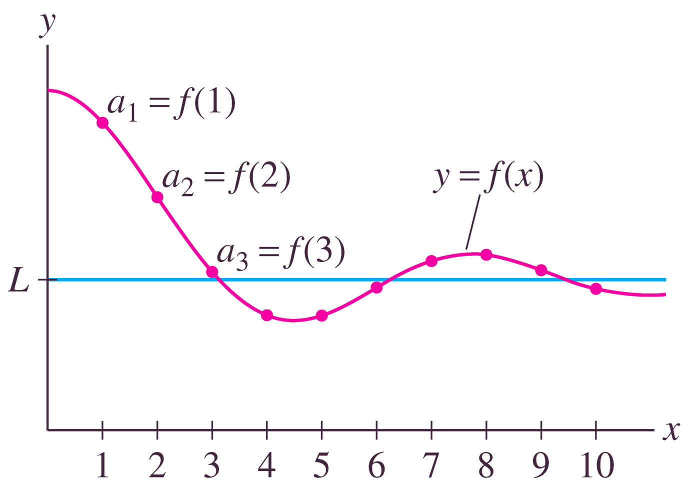
5: A function method for evaluating limits.

A very useful theorem relating limits of functions to limits of sequences is the following result:

THEOREM

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$



If $f(x)$ converges to L, then the sequence $a_n = f(n)$ also converges to L.

The above result enables us to use techniques for evaluating limits of functions, such as L'Hopital's theorem and squeeze theorem, and apply them to evaluate limits of sequences.

Ex: Discuss the behaviour of a_n as $n \to \infty$ where

$$a_n := \frac{n+4}{n+1}.$$

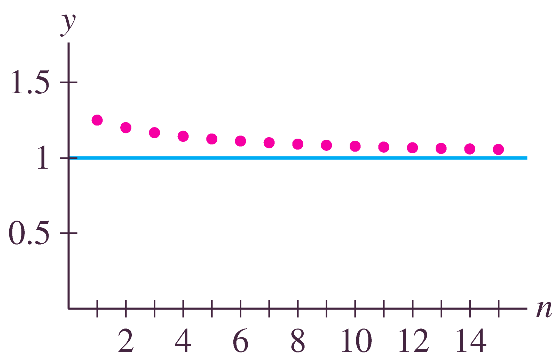


FIGURE The sequence $a_n = \frac{n+4}{n+1}$.

Ex: Evaluate $\lim_{n\to\infty} a_n$ where

$$a_n = \left(\frac{n+4}{n+1}\right)^n .$$

Ex: How large can n be for $1/n$ to be less than any given $\varepsilon > 0$? Prove formally that $\lim_{n\to\infty} \frac{1}{n} = 0$.

6: Divergence

There are two ways for a sequence a_n to diverge: (i) when $\lim_{n\to\infty} a_n = \pm\infty$ (ii) or when $\lim_{n\to\infty} a_n$ does not exist at all.

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty.$$

Ex: If $a_n = n^2$ then we see that $\lim_{n\to\infty} a_n = \infty$ and so n^2 diverges.

Ex: If $a_n = (-1)^n$ then we see "oscillation" and also that $\lim_{n\to\infty} a_n$ does not exist.

Ex: If $a_n = \cos n$ then $\lim_{n\to\infty} a_n$ does not exist.

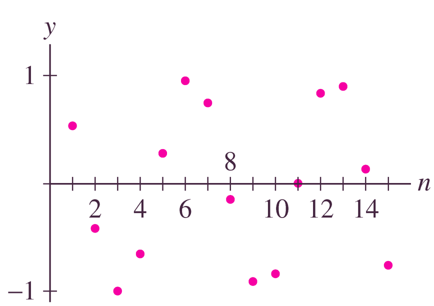


FIGURE The sequence $a_n = \cos n$ has no limit.

7: Basic limit laws.

THEOREM

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

THEOREM **The Continuous Function Theorem for Sequences**

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Ex: If $a_n = \sin(\frac{\pi}{2} - \frac{1}{n})$ then evaluate $\lim_{n\to\infty} a_n$.

THEOREM **The Sandwich Theorem for Sequences**

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Ex: Show that the sequence $\frac{(-1)^n}{n} \to 0$.

THEOREM

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

8: Bounded, monotonic sequences.

DEFINITION Bounded Sequences A sequence $\{a_n\}$ is:

- **Bounded from above** if there is a number M such that $a_n \leq M$ for all n. The number M is called an *upper bound*.
- **Bounded from below** if there exists m such that $a_n \geq m$ for all n. The number m is called a *lower bound*.

If $\{a_n\}$ is bounded from above and below, we say that $\{a_n\}$ is *bounded*. If $\{a_n\}$ is not bounded, we call $\{a_n\}$ an *unbounded sequence*.

THEOREM 5 Convergent Sequences Are Bounded If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

DEFINITION **Nondecreasing Sequence**

A sequence $\{a_n\}$ with the property that $a_n \le a_{n+1}$ for all n is called a **nondecreasing sequence**.

THEOREM 6 **Bounded Monotonic Sequences Converge**

- If $\{a_n\}$ is increasing and $a_n \le M$ for all n, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \le M$.
- If $\{a_n\}$ is decreasing and $a_n \ge m$ for all n, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \ge m$.

Note: we can replace "increasing" with "nondecreasing" and the conclusion of the theorem still holds. Similarly, we can replace "decreasing" with "nonincreasing".

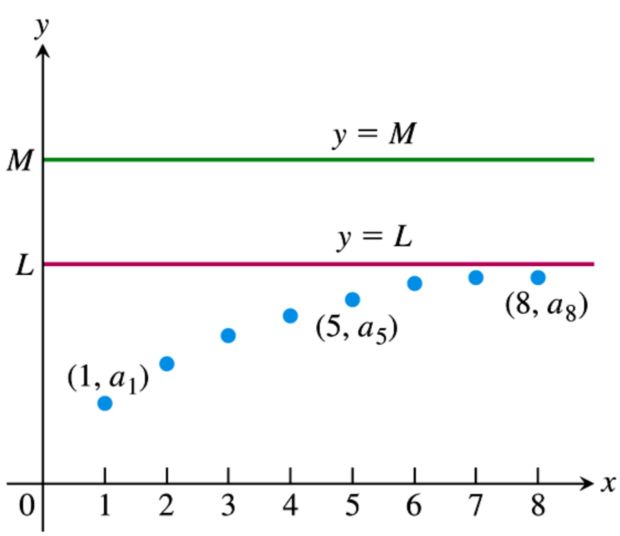


FIGURE If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \le M$.

9: Successive approximations.

The method of successive approximations is a very powerful tool that dates back to the works of Liouville and Picard. The method involves endeavouring to solve an equation of type

$$x = F(x); \tag{1}$$

where F is continuous, by starting from some a_0 and then defining a sequence a_n of approximations by

$$a_{n+1} := F(a_n), \quad n = 1, 2, \ldots$$

If a_n converges to some a^* then a^* will, in fact, be a solution to (1).


Emile Picard.

"A striking feature of Picard's scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known." (Hadamard). Between 1894 and 1937 Picard trained over 10000 engineers who were studying at the École Centrale des Arts et Manufactures.

Ex: Consider the equation $x^2 - x - 1 = 0$. Define a sequence of successive approximations a_n that converge to a root of our quadratic.

We recast our equation in the form $x = F(x)$, where $F(x) = x^2 - 1$. We now examine the recursively defined sequence

$$a_{n+1} = a_n^2 - 1$$

and choose $a_0 = -1/2$. If $\lim_{n\to\infty} a_n$ exists then it will be a root of our original equation.

We can show that $\lim_{n\to\infty} a_n$ exists by verifying by induction that: (i) $a_{n+1} \leq a_n$; (ii) $a_n \geq -1$.

Thus, the a_n is nonincreasing and bounded below. The previous theorem ensures that $\lim_{n\to\infty} a_n$ exists.

Independent learning exercise: There are two roots to our quadratic. Which one will our sequence a_n converge to?