MATH 1231 MATHEMATICS 1B 2010.

For use in Dr Chris Tisdell's lectures

Calculus Section 3B: - Second order ODEs.

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S1: The homogeneous case

S2: The nonhomogoeneous case

S3: MAPLE.

Images from "Thomas' calculus" by Thomas, Wier, Hass & Giordano, 2008, Pearson Education, Inc.

S1: Second order ODEs

The second order linear ODE with constant coefficients, has the form

$$(NH) \quad ay''+by'+cy = f(x), \quad a \neq 0, b, c = \text{consts}$$

where y is a function of x which we have to determine. If $f(x) = 0$, then we say that the equation

$$ay'' + by' + cy = 0 \qquad (H)$$

is $homogeneous$.

How do we solve the above ODE and what is the form of the solution?

Ex: Solve

$$y'' - y' - 6y = 0. \qquad (D)$$

In the previous example, the coefficient of y' was "split" into two parts to form

$$-1y' = -(3-2)y'$$

Also note that the coefficient of y in (D) is obtained by multiplying our 3 and -2. Furthermore, 3 and -2 occur in our final answer for y. This is no accident and both 3 and -2 will be solve the "characteristic equation" associated with (D), namely

$$\lambda^2 - \lambda - 6 = 0.$$

We normally go straight to the characteristic equation, solve it and simply write down the general solution.

In general the characteristic equation for

$$ay'' + by' + cy = 0 \text{ is } a\lambda^2 + b\lambda + c = 0.$$

Real and distinct roots.

If the characteristic equation has real and distinct roots λ_1 and λ_2 then the general solution to our ODE is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \qquad A, B = \text{ constants.}$$

Ex: Solve $y'' - 4y' - 5y = 0,\ y(0) = 1,\ y'(0) = 0.$

Repeated roots.

Ex: Solve

$$y'' - 4y' + 4y = 0.$$

Note that we can rearrange our ODE to form

$$[y'' - 2y'] - 2[y' - 2y] = 0$$

or, equivalently,

$$\frac{d}{dx}[y' - 2y] - 2[y' - 2y] = 0.$$

We can now reduce our eqn to a first order ODE via the substution $u = y' - 2y$, obtaining

$$u' - 2u = 0$$

which may be solved to obtain $u = Ke^{2x}$. Backsubsitution for u yields

$$Ke^{2x} = y' - 2y$$

which can be solved for y to form

$$y = Ae^{2x} + Bxe^{2x}.$$

Note that the coefficent of x in the exponents our solution (ie, 2) is just the repeated root of the characteristic eqn

$$\lambda^2 - 4\lambda - 4 = 0.$$

The technique we used above works in general and so if the characteristic equation has a single repeated root, λ, then the general solution is

$$y = Ae^{\lambda x} + Bxe^{\lambda x}.$$

Ex: Solve

$$y'' + 6y' + 9y = 0.$$

Complex roots.

Finally, the third possibility is that the roots are complex. They occur in conjugate pairs and so are of the form

$$\lambda = \alpha \pm i\beta.$$

The general solution will be

$$y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

This solution can be justified from our earlier method with some work and the use of Euler's formula for e^{ix}.

Ex: Solve

$$y'' - 2y' + 5y = 0.$$

To summarise, we solve the characteristic equation of

$$ay'' + by' + cy = 0$$

which is

$$a\lambda^2 + b\lambda + c = 0.$$

If

• roots are real and distinct, λ_1, λ_2, then the general solution is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x};$$

• roots are equal, λ, then the general solution is

$$y(x) = Ae^{\lambda_1 x} + Bxe^{\lambda_2 x};$$

• roots are complex, $\lambda = \alpha \pm i\beta$, then the general solution is

$$y(x) = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right].$$

S2: The nonhomogeneous case.

If $y_1(x)$ solves (H) and $y_2(x)$ solves (NH) then the linear combo $Y := y_1 + y_2$ also satisfies (NH) as

$$\begin{aligned} & Y'' + aY' + bY \\ = \;& a[y_1'' + y_2''] + b[y_1' + y_2'] + c[y_1 + y_2] \\ = \;& [ay_1'' + by_1' + cy_1] + [ay_2'' + by_2' + cy_2] \\ = \;& [0] + f(x) \\ = \;& f(x). \end{aligned}$$

The idea above is will be used frequently to solve (NH).

We first solve (H) and then construct a particular solution to (NH). We then add the two solutions to obtain the general solution to (NH).

Ex: Solve

$$y'' - 5y' + 6y = 2x + 3.$$

Ex: Solve

$$y'' - 5y' + 6y = 12e^{5x}.$$

We can construct a table of what to try as a particular solution for given $f(x)$.

$f(x)$
$P(x)$, a nth deg. polyn.
$P(x)e^{ax}$
$P(x)\cos ax$
$P(x)\sin ax$
$P(x)e^{ax}\sin bx$ or $P(x)e^{ax}\cos bx$

Respective construction for y_p.
$Q(x)$, nth deg. polyn.
$Q(x)e^{ax}$
$Q_1(x)\cos ax + Q_2(x)\sin ax$
$Q_1(x)\cos ax + Q_2(x)\sin ax$
$Q_1(x)e^{ax}\cos bx + Q_2(x)e^{ax}\sin bx$

***Care must be taken however, when using the above table as the following examples will show. ***

Ex: Solve $y'' - 5y' + 6y = 12e^{2x}$.

An even more unpleasant example is:

Ex: Solve $y'' - 4y' + 4y = 2e^{2x}$.

Thus, as a general rule, if the right hand side of the equation has a function which is already in the kernel (i.e. one of the homogeneous solutions), we multiply by x until the resulting function is no longer a solution to the homogeneous equation.

Ex: Solve $y'' + y = \cos x$, $y(0) = 3$, $y'(0) = 0$.

3. Appendix: MAPLE

The following command is used in MAPLE to solve ODE's (if possible).

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>dsolve(deqn, y(x));
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For example

```
>dsolve(diff(y(x),x\$2)-y(x) = 1, y(x));
```

$$y(x) = -1 + \_C1 \exp(x) + \_C2 \exp(-x)$$

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>dsolve({diff(v(t),t) + 2*t = 0, v(1) = 5}, v(t));
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$$v(t) = -t^2 + 6$$