MATH 1231 S2 2010: Calculus

For use in Dr Chris Tisdell's lectures

Section 2: Techniques of integration.

Created and compiled by Chris Tisdell

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Images from "Thomas' calculus" by Thomas, Wier, Hass & Giordano, 2008, Pearson Education, Inc.

1. Motivation.

Why study integration?

The theory of integration is a cornerstone of calculus. Integration finds a useful place in many disciplines, such as: engineering; physics; biology; economics and beyond.

Where are we going?

We will develop a number of important integration techniques that will be useful in the study of applied problems.

Throughout our discussions we will see HOW integration naturally arises in the analysis of applied problems and in modelling. This is critical to motivate the ideas and also to build the intuition.

2. What you should already know.

You studied integration at school and in MATH 1131. You should be comfortable with:

- using a table of integrals
- integrating by inspection
- integration by parts
- integration by simple substitution.

Just in case you've forgotten these techniques, I have included some examples to refresh your memory and to test your skills in the Appendix.

You will find this course *much easier* if you can integrate with ease. Your hard work will pay–off later in the session!!

TABLE Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

Procedures for Matching Integrals to Basic Formulas

Procedure	**Example**
Making a simplifying substitution	$\dfrac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x$ $= \sec^2 x + 2 \sec x \tan x + (\sec^2 x - 1)$ $= 2 \sec^2 x + 2 \sec x \tan x - 1$
Eliminating a square root	$\sqrt{1 + \cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2 - 7x}{3x + 2} = x - 3 + \dfrac{6}{3x + 2}$
Separating a fraction	$\dfrac{3x + 2}{\sqrt{1 - x^2}} = \dfrac{3x}{\sqrt{1 - x^2}} + \dfrac{2}{\sqrt{1 - x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

3. Integrals of trig functions.

You should be familiar with:

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2}\sin x \cos x + C$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2}\sin x \cos x + C$$

The above integrals are derived by using the "double angle formulas":

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

and for future reference

$$\sin 2x = 2 \sin x \cos x.$$

Integrals of products of sin and/or cos.

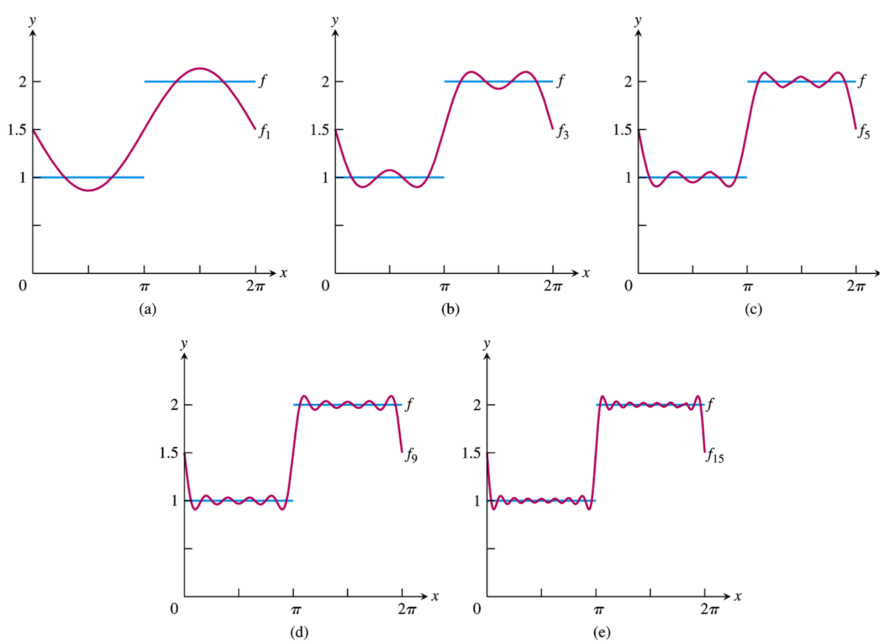
We first discuss how such integrals naturally arise.

Applications matter! So–called "Fourier series" play an important role in the study of **heat flow**.

A finite Fourier series of a function f is given by the sum

$$f(x) := \sum_{n=1}^{N} a_n \sin nx \tag{1}$$
$$= a_1 \sin x + a_2 \sin 2x + \cdots a_N \sin Nx$$

where the a_n are some numbers that are to be determined. That is, we aim to write $f(x)$ as a sum of sin functions.



The Fourier approximation functions f_1, f_3, f_5, f_9, and f_{15} of the function $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$

We now see how integration of trig functions is central to the method.

We claim that the mth coefficient a_m is given by

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \ dx. \tag{2}$$

If we multiply both sides of (1) by $\sin mx$ and then integrate over $[-\pi, \pi]$ we obtain

$$\begin{aligned}\int f(x) \sin mx \ dx &= \int_{-\pi}^{\pi} \sum_{n=1}^{N} a_n \sin nx \sin mx \ dx \\ &= \sum_{n=1}^{N} \left(\int_{-\pi}^{\pi} a_n \sin nx \sin mx \ dx \right).\end{aligned}$$

Thus we have obtained an integral that involves a product of $\sin$ functions.

How do we evaulate such an integral?

Integrals of products of sin and/or cos

These are done using the so–called *product to sum* formulae:

$$\begin{aligned} \sin A \cos B &= [\sin(A-B)+\sin(A+B)]/2 \\ \sin A \sin B &= [\cos(A-B)-\cos(A+B)]/2 \\ \cos A \cos B &= [\cos(A-B)+\cos(A+B)]/2 \end{aligned}$$

Ex: $\int \sin 4x \cos 2x \; dx =$

Independent learning ex: Now return to, and prove, (2).

Integrals of the form $\int \cos^m x \sin^n x \; dx$

If m or n are **odd** then our aim is to transform the integral into one of the types:

$$\begin{aligned}\int \sin^k x \cos x \; dx &= \frac{1}{k+1} \sin^{k+1} x + C \\ \int \cos^k x \sin x \; dx &= -\frac{1}{k+1} \cos^{k+1} x + C.\end{aligned}$$

At the heart of the approach is to factor out a $\sin x$ or $\cos x$ from the **odd** power term and then use the identity

$$\cos^2 x + \sin^2 x = 1$$

to transform the integral into one which can be directly evalutated.

Q: How can you **verify** the above integrals?

Ex. Evaluate

$$I := \int \cos^4 x \sin^3 x \; dx.$$

Ex: Evaluate

$$I := \int \cos^5 x \sin^5 x \; dx$$

If **both** m and n are **even**, then the integral is slightly harder. We will see shortly a recursive method for dealing with such an integral, but for smaller values of m and n one can use the identities

$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

Ex. Evaluate $I := \int \cos^4 x \sin^2 x \, dx$.

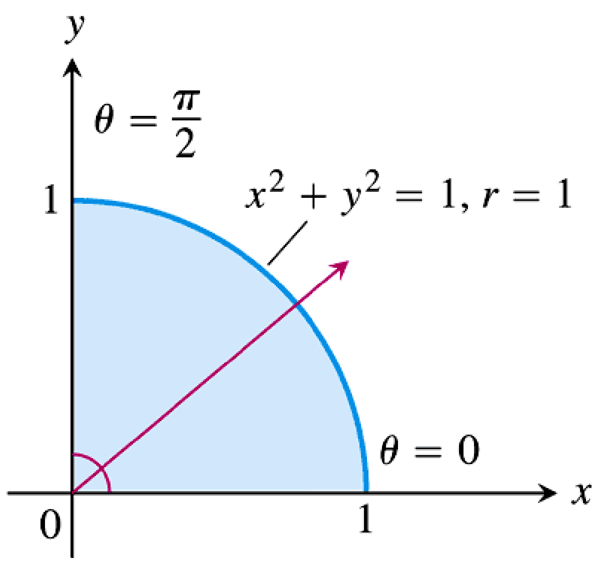
See that

$$\begin{aligned} I &= \int (\cos^2 x)^2 \sin^2 x \, dx \\ &= \int \left[\frac{1+\cos 2x}{2}\right]^2 \left[\frac{1-\cos 2x}{2}\right] \, dx \\ &= \frac{1}{8}\int 1+\cos 2x - \cos^2 2x - \cos^3 2x \, dx \end{aligned}$$

$$= \tfrac{1}{8}\int 1+\cos 2x - \cos^2 2x - (1-\sin^2 2x)\cos 2x \, dx$$

$$\begin{aligned} &= \tfrac{1}{8}\int 1 - \cos^2 2x + \sin^2 2x \cos 2x \, dx \\ &= \tfrac{1}{8}\left[\tfrac{x}{2} - \tfrac{1}{8}\sin 4x + \tfrac{1}{6}\sin^3 2x\right] + C. \end{aligned}$$

Applications matter! An important problem in applied math, engineering and physics is to calculate the total mass of an given object. Consider a thin plate occupying the unit quarter–disk in the first quadrant.



In polar coordinates, this region is described by simple inequalities:

$0 \le r \le 1$ and $0 \le \theta \le \pi/2$

If the density $\delta(x, y)$ of the plate at any point (x, y) is given by $\delta(x, y) = x^2y^4$ then the total mass of the plate is given by

$$M = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} x^2y^4 \; dy \right] \; dx.$$

To make this integral easier, we switch to polar co–ordinates: $x = r\cos\theta$; $y = r\sin\theta$; $dy\; dx = r\; dr\; d\theta$ with a substitution giving

$$\begin{aligned} M &= \int_0^{\pi/2} \int_0^1 (r^2\cos^2\theta)(r^4\sin^4\theta)\; r dr d\theta \\ &= \int_0^1 r^7\; dr \int_0^{\pi/2} (\cos^2\theta)(\sin^4\theta)\; d\theta. \end{aligned}$$

4. Reduction formulae

Repeated integration by parts can be a long process! We can shorten the process by applying so–called "reduction formulae".

Ex: The reduction formula

$$\int \tan^n x \; dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \; dx \qquad (*)$$

can be used in an iterative fashion to calcuate

$$\int \tan^5 x \; dx$$

Solution We apply Equation $(*)$ with $n = 5$ to get

$$\int \tan^5 x \, dx = \frac{1}{4}\tan^4 x - \int \tan^3 x \, dx.$$

We then apply Equation $(*)$ again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \int \tan x \, dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x \, dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$

If $I_n := \int \tan^n x \; dx$ then (*) may be compactly written as

$$I_n := \frac{1}{n-1}\tan^{n-1} x - I_{n-2}, \quad n \geq 2.$$

Ex. Let $I_n := \int_0^{\pi/2} \sin^n x \, dx$. Use the reduction formula

$$I_n := \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

to calculate I_7.

Ex: Construct a reduction formula for

$$I_n := \int x^n e^x \, dx.$$

Ex: Find the reduction formula for

$$I_n := \int_0^{\pi/2} \sin^n x \, dx.$$

The reduction formula for the integral of $\cos^n x$ is similar.

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Ex: Construct a reduction formulae from

$$I_n := \int_0^{\pi/4} \sec^n x \, dx.$$

A similar method is used to obtain the reduction formula for the integral $I_n := \int_0^{\pi/4} \tan^n x \, dx$.

Ex. [Q1, Class Test 1, 2002] Let

$$I_n := \int_0^{\pi/4} \tan^n \theta \sec \theta \, d\theta.$$

Show that

$$I_n := \frac{1}{n}\left(\sqrt{2} - (n-1)I_{n-2}\right), \quad \text{for } n \geq 2.$$

Note that

$$\frac{d}{d\theta} \sec \theta = \sec \theta \tan \theta.$$

A Two parameter recurrence:

Consider

$$\begin{aligned} I_{m,n} &= \int_0^{\pi/2} \cos^m x \sin^n x \, dx \\ &= \int_0^{\pi/2} [\cos^{m-1} x][\sin^n x \cos x] \, dx \end{aligned}$$

and use integration by parts. Choose $u = \cos^{m-1} x$ and $v' = \sin^n x \cos x$. Thus,

$$u' = -(m-1) \sin x \cos^{m-2} x, \; v = \frac{1}{n+1} \sin^{n+1} x.$$

Integration by parts then leads to

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}, \quad m \geq 2.$$

In a similar way, one could also obtain the recurrence

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}. \quad n \geq 2.$$

In applying the above formulae we must reach one of $I_{1,1} = \frac{1}{2}$, $I_{1,0} = I_{0,1} = 1$ or $I_{0,0} = \frac{\pi}{2}$.

You are not expected to memorise this formula.

5. Trig & Hyperbolic Substitutions.

You will have already seen integration by simple substitution.

THEOREM Substitution in Definite Integrals

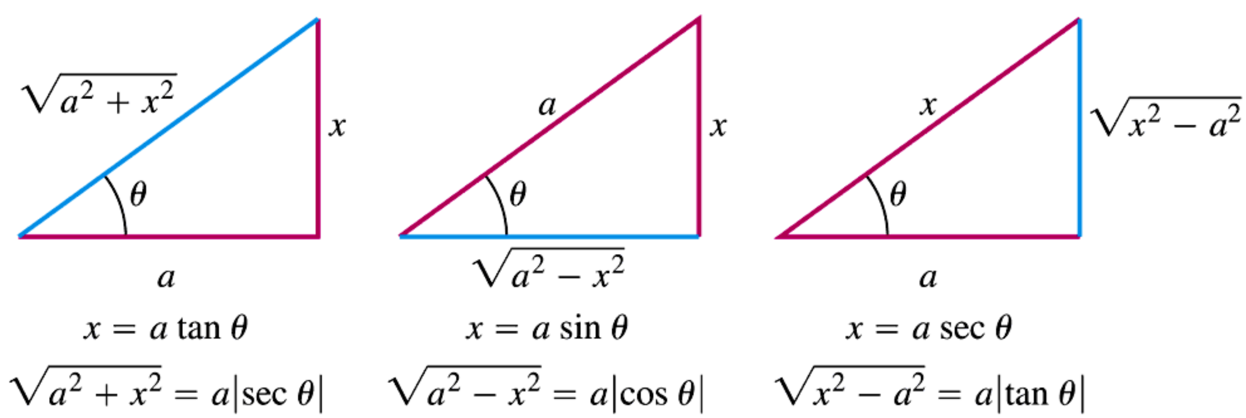
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

With integrals involving square roots of quadratics, the idea is to make a suitable trigonometric or hyperbolic substitution that greatly simplifies the integral.

Integrals involving square roots of quadratics can be worked out using the following trigonometric or hyperbolic substitutions.

$\sqrt{a^2 - x^2}$	try $x = a \sin\theta$
$\sqrt{a^2 + x^2}$	try $x = a \tan\theta$ or $x = a \sinh\theta$
$\sqrt{x^2 - a^2}$	try $x = a \sec\theta$ or $x = a \cosh\theta$



Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Ex: Evaluate

$$I = \int_0^4 x^2 \sqrt{16 - x^2}\, dx.$$

Ex: Evaluate

$$I := \int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

Ex: Evaluate

$$I := \int \frac{dx}{x^2\sqrt{x^2-1}}$$

via the substitution $x = \sec\theta$

(This last integral can also be done using a $\cosh\theta$ substitution but it is not easy.)

Applications matter! An important problem in applied math, engineering and physics is to calculate the total mass of an given object.

Consider a thin wire of *constant* density δ (mass per unit length) that lies in the XY–plane along the curve

$$y = f(x) := x^2/2, \quad a \leq x \leq b.$$

It can be shown that the total mass M of the wire is

$$\begin{aligned} M &= \int_a^b \delta\sqrt{1 + [f'(x)]^2}\, dx \\ &= \delta \int_a^b \sqrt{1 + x^2}\, dx. \end{aligned}$$

See how an integral of $\sqrt{1 + x^2}$ naturally arises?

6. Partial Fractions.

A *rational function* is a function of the form $\dfrac{f(x)}{g(x)}$ where f and g are polynomials.

Ex: Which of the following are rational functions?

(a) $\dfrac{2x+1}{3x^2-4}$

(b) $\dfrac{\cos x}{x^2+1}$

(c) $\dfrac{x^2}{x-1}$

(d) $\dfrac{e^x}{x+1}$.

For the purpose of integration, we ensure that the degree of the numerator is less than the degree of the denominator. If not then we may need to divide.

Building the intuition.

An integral like

$$\int \frac{dx}{x(1-x)}$$

cannot be easily computed directly, however, we know that the integrals

$$\int \frac{dx}{x}, \quad \int \frac{dx}{1-x}$$

can both be directly evaluated.

Thus, can we write $\frac{1}{x(1-x)}$ as some linear combination of $\frac{1}{x}$ and $\frac{1}{1-x}$?

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Heaviside Method

1. *Write the quotient with g(x) factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Heaviside proved important results in electromagnetism and vector calculus. He reduced Maxwell's 20 equations in 20 variables to 4 equations in 2 variables.

"Mathematics is an experimental science, and definitions do not come first, but later on."

"Why should I refuse a good dinner simply because I don't understand the digestive processes involved." [reply when criticised for his daring use of operators before they could be justified formally.]

Factors are Linear:

Ex. Evaluate

$$I := \int \frac{2x-1}{x^2+5x+6}dx = \int \frac{2x-1}{(x+3)(x+2)}dx.$$

Ex. Evaluate

$$\int \frac{2x^2 + 2x + 6}{(x-1)(x+1)(x-2)} dx.$$

Repeated Linear Factors:

Ex: Evaluate

$$I := \int \frac{x+3}{(x+2)^3} dx.$$

In general, each factor in the denominator of the form $(x - a)^k$ gives rise to an expression of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots\ldots + \frac{A_k}{(x-a)^k}.$$

Ex: Evaluate

$$I := \int \frac{dx}{x(x+1)^3}.$$

Ex: [Q2, Class Test 1, 2003] Calculate

$$\int \frac{8x+9}{(x-2)(x+3)^2}dx.$$

Irreducible Quadratic Factors:

Suppose the denominator of our rational function is a quadratic which does not factor (over the real numbers). For example,

$$f(x) = \frac{2x + 1}{x^2 + 4x + 5}.$$

To find the integral of such a function, we manipulate the integrand so that

the numerator equals the derivative of the denominator.

This leads to a logarithm and complete the square on the denominator of the remaining term leading to an inverse tangent function.

Ex: Evaluate

$$I := \int \frac{2x+1}{x^2+4x+5}\,dx.$$

(N.B. If the quadratic does have real but irrational roots, one can use this same method to avoid nasty partial fractions. The second factor will then lead to an inverse tanh.

e.g.

$$\begin{aligned} & \int \frac{2x+1}{x^2+4x-6}dx \\ = & \int \frac{2x+4}{x^2+4x-6} + \frac{3}{10-(x+2)^2}dx \\ = & \ln|x^2+4x-6| + \frac{3}{\sqrt{10}}\tanh^{-1}\left(\frac{x+2}{\sqrt{10}}\right) + C. \end{aligned}$$

Ex; [Q1, Class Test 1, 2002] Calculate

$$\int \frac{x}{x^2 + 2x + 10} dx.$$

You are given that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$$

Combinations of the Rules:

These rules can be combined to cover a variety of situations.

Ex: Calculate

$$I := \int \frac{x+6}{x(x^2+2x+3)}\, dx.$$

In general, an irreducible quadratic factor

$$x^2 + ax + b$$

in the denominator will give rise to a term of the form

$$\frac{Ax + B}{x^2 + ax + b}$$

and more generally again, a factor

$$(x^2 + ax + b)^k$$

in the denominator will give rise to terms of the form

$$\frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \ldots + \frac{A_kx + B_k}{(x^2 + ax + b)^k}.$$

Ex. Evaluate

$$I := \int \frac{x}{(x+1)^2(x^2+1)}\, dx.$$

Applications matter! Where does an integral involving partial fractions arise? Let $P = P(t)$ denote the population size of a species at time t. It can be shown that P and dP/dt satisfy the "differential equation"

$$\frac{dP}{dt} = P\left(1 - \frac{P}{M}\right) \tag{3}$$

where $M > 0$ is a constant.

The challenge is to determine the unknown function P from (3) and hence *predict* what will happen to the population over time.

If we rearrange (3) and integrate both sides w.r.t t then we obtain

$$\frac{1}{M}\int \frac{1}{P\,(M-P)}\frac{dP}{dt}dt = \int 1dt$$

which becomes

$$\frac{1}{M}\int \frac{dP}{P\,(M-P)} = t + K.$$

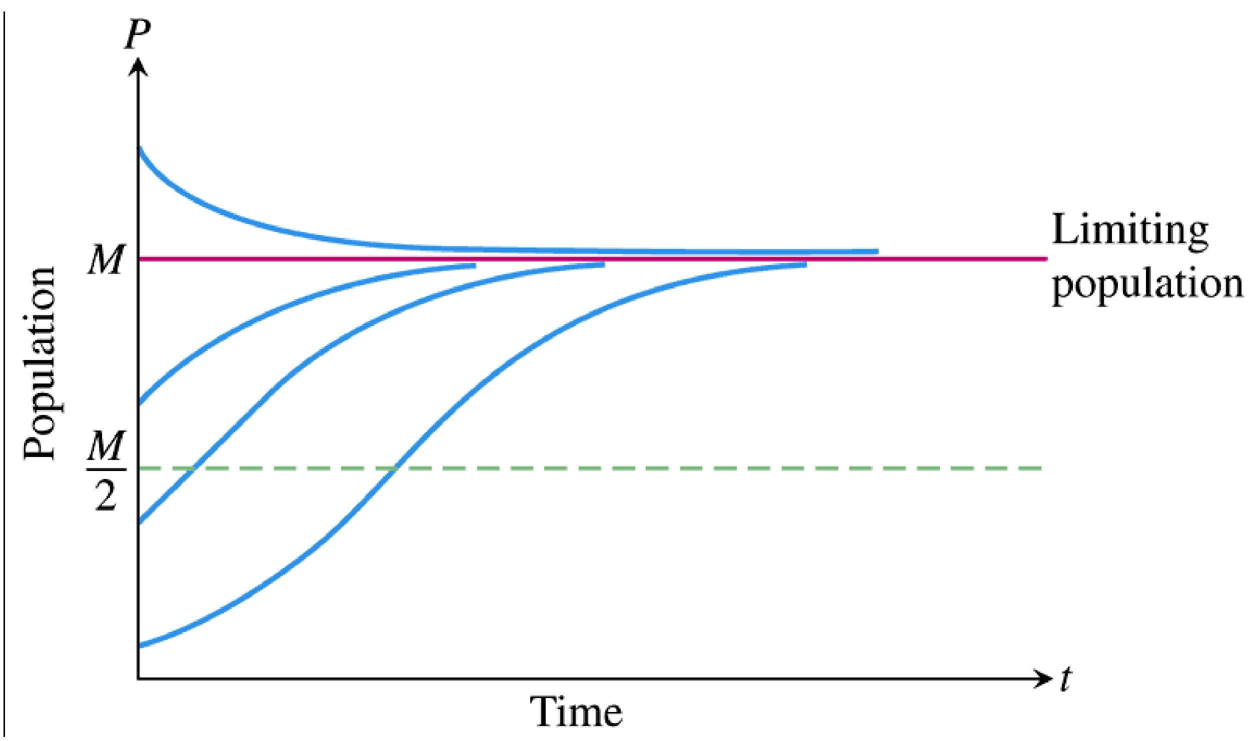
Applying partial fractions we obtain

$$\frac{1}{M}\int\frac{dP}{P\,(M-P)}=\frac{1}{M}\left[\int\frac{dP}{P}+\int\frac{dP}{M-P}\right]$$

which can be integrated to form

$$\frac{P}{|M-P|}=Ce^{t}.$$

By considering two separate cases $M-P>0$ and $M-P<0$ we can rearrange, solve for P and plot the following graph.



7. Rationalising substitutions

The above methods using partial fractions allow us to integrate (in principle) any rational function, so we often make a change of variable in an integral that will lead us to some rational function.

Ex: Evaluate

$$I := \int \frac{dx}{1 + x^{1/4}}.$$

Ex: Calculate

$$I := \int \frac{dx}{\sqrt{e^{2x} - 1}}.$$

8. Weierstrass' substitution.

Weierstrass' substitution is very useful for integrals involving a rational expression in $\sin x$ and/or $\cos x$.

Weierstrass' substitution is

$$t = \tan \frac{x}{2}.$$

Recall, that $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, and that $dx = \dfrac{2\,dt}{1+t^2}$.

You should know how to derive these formulae. For example, if $t = \tan x/2$ then

$$\begin{aligned} \frac{2t}{1+t^2} &= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \\ &= 2\frac{(\sin\frac{x}{2})/(\cos\frac{x}{2})}{1+(\sin^2\frac{x}{2})/(\cos^2\frac{x}{2})} \\ &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ &= \sin x. \end{aligned}$$

Who was Weierstrass??

Ex: Calculate

$$I := \int \frac{dx}{1 + \cos x + \sin x}.$$

.

Ex: Show that $I = \int_0^{\pi/2} \frac{dx}{2+\cos x} = \frac{\pi}{3\sqrt{3}}$.

We let $t = \tan(x/2)$, so that

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2\ dt}{1+t^2}$$

in a similar way to the previous example. Substitution then leads to

$$\begin{aligned} I &= \int_0^1 \frac{2}{3+t^2} dt \\ &= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}}\right]_0^1 \\ &= \frac{\pi}{3\sqrt{3}}. \end{aligned}$$

Applications matter! Where does an integral like

$$\int \frac{d\theta}{1+\cos\theta+\sin\theta}$$

arise? Consider a thin plate in the XY–plane that occupies the region (in polars)

$$1 \le r \le 2, \quad 0 \le \theta \le \pi/2.$$

Suppose at each point $(x, y) \neq (0, 0)$ the plate has density function

$$\delta(x,y) = \frac{1}{\sqrt{x^2+y^2}+x+y}.$$

To evaluate the total mass M of the plate we consider

$$M = \int_0^1 \left[\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \delta(x,y)\ dx \right] dy.$$

Now, making substitutions: $x = r\cos\theta$; $y = r\sin\theta$; $dx\ dy = r\ dr\ d\theta$ we obtain

$$M = \int_1^2 dr \int_0^{\pi/2} \frac{d\theta}{1+\cos\theta+\sin\theta}.$$

9. APPENDIX

MAPLE

The following examples show how to use MAPLE to perform partial fraction decompositions.

```
> convert(x^2/(x+2), parfrac, x);
```

$$x - 2 + \frac{4}{x+2}$$

```
> convert(x/(x-b)^2, parfrac, x);
```

$$\frac{b}{(x-b)^2} + \frac{1}{x-b}$$

Integration by Parts.

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

The analysis requires you to:

choose functions f and g';

then calculate f' and g from your above choices;

and then apply the above formula.

Ex: Evaluate $I = \int xe^x\,dx$.

We choose $f = x$ and $g' = e^x$. Thus, $f' = 1$ and $g = e^x$. Using our IBP formula

$$\begin{aligned} I = \int xe^x\,dx &= xe^x - \int 1e^x\,dx \\ &= xe^x - e^x + C. \end{aligned}$$

Integration by inspection.

When confronted by an integral, you should always see if the integral can be 'guessed' and then fine–tuned by multiplying or dividing by a constant.

Alternatively, multiply and divide the integral by a particular constant to put it into a useful form.

Ex: Evaluate $\int x(x^2+1)^{10}\,dx$.

See that x is "almost" the derivative of (x^2+1).

We make the guess $(x^2+1)^{11}$ and then check our guess by evaluating

$$\frac{d}{dx}[(x^2+1)^{11}] = 11{\cdot}2x(x^2+1)^{10} = 22x(x^2+1)^{10}.$$

See that we have an unwanted 22 in the above. Thus, we divide our initial guess by 22 to obtain our answer

$$\int x(x^2+1)^{10}\,dx = \frac{1}{22}(x^2+1)^{11} + C.$$

Observe that this method ONLY works for integrals of the form

$$\int f(g(x))g'(x)\,dx$$

when the derivative $g'(x)$ is present in the integrand up to a constant.

$$\begin{aligned}\int \sin t \cos^5 t \; dt &= \int \frac{d}{dt}\left[\frac{-\cos^6 t}{6}\right] dt \\ &= \frac{-\cos^6 t}{6}.\end{aligned}$$

The above methods should remind you of the chain rule!

$$\int x^4(x^3+1)^{10}\,dx$$

cannot be 'guessed' and fine tuned since the derivative of x^3+1 is not in the integrand.

Other useful integrals.

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

These four should be learnt carefully.