

# **MATH 1231 MATHEMATICS 1B 2010.**

For use in Dr Chris Tisdell's lectures.

## **Calculus Section 3A: - First-order ODEs.**

*Created and compiled by Chris Tisdell*

S1: What is an ODE?

S2: Motivation

S3: Types and orders of ODEs

S4: What is a solution?

S5: First-order ODEs (Separable; Linear; Exact).

## 1. What is a differential equation?

A “differential equation” involves two things:

1. the derivative(s) of an unknown function
2. an equals sign.

The following are examples of differential equations:

$$\begin{aligned}y' &= y + x \\y'' &= y' + y \\ \frac{d^2y}{dx^2} + \frac{dy}{dx} &= 1 + y^2 - e^x.\end{aligned}$$

Like most equations, we want to “solve” the above equations. In this case we are looking for an unknown *function*  $y = y(x)$  that satisfies each differential equation for particular values of  $x$  in an interval.

## 2. Motivation.

When even the most foundational equations are applied to the modelling of phenomena, differential equations can arise. You may have seen Newton's 2nd law

$$F = ma \quad (1)$$

which means that when a body with (constant) mass  $m$  is acted on by a net force  $F$ , the body has acceleration  $a$  (with the body moving in one dimension).

From basic calculus we know that the velocity  $v$  and the acceleration  $a$  of the body are related by

$$\frac{dv}{dt} = a(t).$$

Also, if the net force  $F$  depends on  $v$  (which it does in many applications) then we may rewrite (1) as

$$F(v) = m \frac{dv}{dt}$$

or, equivalently,

$$\frac{dv}{dt} = \frac{F(v)}{m}$$

which we can then try to solve to find the velocity  $v = v(t)$  of the body at any time  $t$ .

An example of a particular net force  $F$  might be

$$F(v) := mg - kv$$

where  $m$  is the mass of the body,  $g$  is the gravitational constant and  $k$  is a constant. (This arises when we analyse a body falling under gravity subject to wind resistance.)

*How are differential equations useful?*

By solving differential equations we can gain a deeper understanding of the physical processes that the equations are describing.

The use of differential equations may empower us make precise predictions about the future behaviour of our models.

Even if we can't completely solve a differential equation, we may still be able to determine useful properties about its solution (qualitative information).

## 5. What is a solution?

When faced with an ODE, the really big question that we will want to answer is:

*How can we determine the solution to the ODE?*

However, first we need to understand what “a solution to an ODE” really means.

A *solution* to an ODE is a **function** which is differentiable and which satisfies the given equation.

More formally, consider the differential equation

$$\frac{dy}{dx} = f(x, y). \quad (2)$$

where  $f$  may be considered a real-valued function of two variables that has domain  $D$ . We call a function  $y = y(x)$  a solution to (2) on an interval  $I$  if, for all  $x \in I$ :  $y(x)$  satisfies (2); and  $(x, y(x)) \in D$ .

**Ex.** Consider

$$\frac{dy}{dx} = 2y + x. \quad (3)$$

We show that  $y(x) = -\frac{x}{2} - \frac{1}{4}$  is a solution to (3). The LHS of (3) is

$$\frac{d}{dx} \left[ -\frac{x}{2} - \frac{1}{4} \right] = -\frac{1}{2}$$

whereas the RHS of (3) is

$$\begin{aligned} 2y + x &= 2 \left[ -\frac{x}{2} - \frac{1}{4} \right] + x \\ &= -\frac{1}{2} \end{aligned}$$

Thus (3) holds for  $y(x) = -\frac{x}{2} - \frac{1}{4}$ .

Indep. learning ex: Can you find another solution to (3)? Show that  $y_1(x) := x^2$  is not a solution to (3).

## 4. Orders of ODEs

The **order** of a differential equation is the highest derivative that appears in the equation. Since only *ordinary* derivatives, rather than partial derivatives, are involved, the equations are called *ODEs*.

The order of an ODE is important as the order will play a role in choosing a particular technique to extract the solution.

Here are some types of ODEs:

$$\frac{dy}{dx} = y + x^2, \quad \text{first-order}$$
$$y'' + y' = 0, \quad \text{second-order}$$

$$\left(\frac{dy}{dx}\right)^2 + y = 0, \quad \text{first-order}$$
$$y''' = 1, \quad \text{third-order}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + y = 1 \quad \text{second-order.}$$

## 6. First order equations.

There is no universal method for solving ODEs of first (or any other) order.

We will develop and apply a collection of techniques for solving certain types of equations.

In this course we will be studying how to solve three basic classes of first order equations:

- separable equations
- linear equations
- exact equations.

Each type of equation requires elements from basic calculus to extract the solution.

# Separable Equations.

## Separable Equations

The equation  $y' = f(x, y)$  is **separable** if  $f$  can be expressed as a product of a function of  $x$  and a function of  $y$ . The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y)$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \quad H(y) = \frac{1}{h(y)}$$

The solution method is:

**Ex:** Solve for  $y = y(x)$

$$\frac{dy}{dx} = \frac{x}{e^y}.$$

Why does the solution method work?

Consider the separable ODE

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad (4)$$

and denote the solution by  $y = y(x)$ . let  $G$  and  $H$  be antiderivatives of  $g$  and  $h$ . Rearranging (4) we obtain

$$h(y) \frac{dy}{dx} = g(x)$$

and so integration yields

$$\int h(y) \frac{dy}{dx} dx = \int g(x) dx.$$

Now by the chain rule (substitution) we have

$$\int h(y) dy = \int g(x) dx$$

which gives us the solution (in implicit form):

$$H(y) = G(x) + C.$$

**Ex:** Solve for  $y = y(x)$  for  $x > 0$

$$\frac{dy}{dx} = xy^2.$$

## Initial value problems

If, in addition to an ODE, we are also given some extra piece(s) of information about the value of the solution at a particular point then we call this an “initial value problem”.

The extra piece(s) of information about the value of the solution at a point are known as “initial condition(s)”.

**Ex:** In the previous exercise, if we have an initial condition  $y(0) = 1$  then we can determine our constant of integration  $c$  (and exclude the general solution  $y \equiv 0$ ).

**Ex:** Solve the initial value problem

$$\frac{dy}{dx} = y(1 + x^2)$$

with initial value  $y(0) = 1$ .

**Applications matter!** How would you solve the problem

$$y \cdot (y')^3 = a, \quad y(\xi) = \nu$$

where  $a$ ,  $\xi$  and  $\nu$  are known constants? This IVP arises in the analysis of rotationally symmetric bodies of smallest wave drag at hypersonic flow.

The solution is

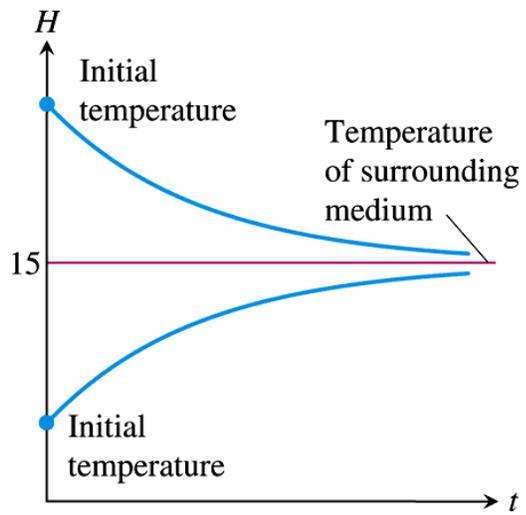
$$y(x) = \left[ (4/3) \cdot a^{1/3} \cdot (x - \xi) + \nu^{4/3} \right]^{3/4}.$$

See J. T. Heynatz, Application of the Newton method to the so-called integral equation method in transonic flow, Acta Mechanic 138 (1999), 123–127.

**Applications matter!** Heat transfer is the transition of thermal energy from a hotter object to a cooler object. For example, a hot aluminium ingot immersed in a large tub of water will cool down to the temperature of its surrounding water (ambient temperature). Such a process can be modelled by *Newton's law of cooling*

$$\frac{dT}{dt} = -k(T - A)$$

where:  $T = T(t)$  is the temperature of the ingot at any time;  $A$  is the ambient temperature; and  $k > 0$  is a constant. If  $A = 15^\circ\text{C}$  then solve the above ODE for any  $k$  and initial temperature  $T(0) = 50^\circ\text{C}$ .



Temperature versus time.

Regardless of initial temperature, the object's temperature  $H(t)$  tends toward  $15^{\circ}\text{C}$ , the temperature of the surrounding medium.

## Linear Equations.

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where  $P$  and  $Q$  are continuous functions of  $x$ .

To solve the linear equation  $y' + P(x)y = Q(x)$ , multiply both sides by the integrating factor  $v(x) = e^{\int P(x) dx}$  and integrate both sides.

The linear ODE

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (5)$$

has the solution

$$y(x) = \frac{1}{v(x)} \left[ \int v(x)Q(x) dx + C \right] \quad (6)$$

where

$$v(x) := e^{\int P(x)dx}.$$

The solution comes from an “integrating factor” and the product rule. If (5) holds then

$$\frac{dy}{dx} e^{\int P(x) dx} + P(x) e^{\int P(x) dx} y = e^{\int P(x) dx} Q(x)$$

and using the product rule we obtain

$$\frac{d}{dx} \left[ y e^{\int P(x) dx} \right] = e^{\int P(x) dx} Q(x).$$

Integrating both sides then gives

$$y e^{\int P(x) dx} = \int e^{\int P(x) dx} Q(x) dx + C.$$

with a rearrangement leading to our form of the solution (6). Note that we have slightly abused the notation in our justification!

**Ex:** Solve

$$\frac{dy}{dx} - 2y = e^{3x}.$$

**Ex:** Solve

$$\frac{dy}{dx} + \frac{y}{x} = x$$

subject to  $y(1) = 0$ .

**Ex:** Solve, for  $x > 0$ ,

$$x \frac{dy}{dx} + (x + 1)y = 2.$$

Sometimes an ODE already has an integrating factor present and there is not need for an exponential function.

**Ex:** Solve

$$t^2 \frac{dy}{dt} + 2ty = \sin t.$$

**Ex:** Solve

$$\frac{xy' - y}{x^2} = 0$$

Ex. An investor has a salary of \$60,000 per year and expects to get an annual increase of \$1000 per year. Suppose an initial deposit of \$1000 is invested in a program that pays 8% per annum and additional deposits are added yearly at a rate of 5% of the salary. Find the amount invested after  $t$  years.

We can approximate by assuming that the interest is calculated continuously and that deposits are made continuously.

Are the above assumptions really reasonable??

Note: in the previous example we were given the ODE to solve. In this example we are faced with the more difficult scenario of *constructing* the ODE and then solving it.

Let  $x(t)$  = amount of money invested at time  $t$  (in years),  
then

$$\frac{dx}{dt} = 0.08x + 0.05(60000 + 1000t)$$

rate of increase of investment

$$= 8\% \text{ of investment} + 5\% \text{ of salary}$$

Solving we have

$$x(t) = -625t - \frac{90625}{2} + Ce^{0.08t}.$$

Using  $t = 0, x = 1000$  we obtain

$$x(t) = \frac{92625}{2}e^{0.08t} - \left(625t + \frac{90625}{2}\right).$$

We can now predict, with accuracy, the amount invested at any future time.

Eg, when  $t = 3$  we have  $x = 11687.22$ ,

at  $t = 10$  we have  $x = 51507.86$

and when  $t = 30$  we have  $x = 446448$ .

## Exact equations.

Many first order ODEs can be written in the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0. \quad (7)$$

There are certain conditions under which the left hand side will be equal to the derivative (wrt  $x$ ) of a function  $F(x, y)$ , remembering that  $y = y(x)$ .

Suppose we have a function

$$F(x, y) = C, \quad \text{where } C \text{ is a constant.} \quad (8)$$

Applying the chain rule from Section 1, we obtain

$$\frac{d}{dx}[F(x, y)] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0.$$

Now **if**

$$M = \frac{\partial F}{\partial x} \text{ and } N = \frac{\partial F}{\partial y}, \quad (9)$$

then reversing the above steps, we could conclude that the solution to the differential equation (7) was

$$F(x, y) = C.$$

Thus, the problem is to **construct** an  $F = F(x, y)$  such that (9) hold for the functions  $M$  and  $N$  from the original problem. It turns out that such an  $F$  will exist if

$$M_y = N_x. \quad (10)$$

To see this, let  $(x_0, y_0)$  be a point such that  $y(x_0) = y_0$  and let  $F$  be defined by:

$$F(x, y) := \int_{x_0}^x M(s, y) ds + \int_{y_0}^y N(x_0, s) ds$$
$$F(x, y) := \int_{x_0}^x M(s, y_0) ds + \int_{y_0}^y N(x, s) ds$$

You can then show that

$$F_y = N \quad \text{and} \quad F_x = M.$$

Furthermore, if  $M_y = N_x$ , with each derivative continuous, then you can show that the two forms of  $F$  above are one and the same!

When solving problems, we usually construct our  $F$  from the equations

$$F_x = M \quad \text{and} \quad F_y = N.$$

Ex. Solve

$$(2x + y + 1)dx + (2y + x + 1)dy = 0.$$

Ex. Solve

$$2xy \, dx + (x^2 + 3y^2) \, dy = 0.$$

**Some additional applications.**

**Verhulst's Logistic Growth Model:** To model population growth or decay, we try

$$\frac{dP}{dt} = KP(P_m - P), \quad K, P_m = \text{const.}$$

## Mixing Problems:

At time  $t = 0$  a tank contains 1kg of salt dissolved in 100 litres of water. If salty water containing  $1/4$  kg per litre at a rate of 3 litres per minute and the (stirred) solution is draining from the tank at 3 litres per minute then predict how much salt will be in the tank at time  $t$ .

