SEMESTER OVERVIEW

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Contents

Introduction

This document is an overview of the math that I have learned during Fall 2020 Semester regarding Kac-Moody Lie algebras. In this paper, I will outline the main concepts, definitions, and examples I have learned throughout the semester. The goal of this write-up is to organize and solidify what I have learned and to have this as a future reference for myself.

1. Basic definitions

We began the semester by going through the first chapter of Kac's book [2].

1.1. Generalized Cartan matrices and realizations.

We will follow Section 1.1 in [2].

Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with rank ℓ.

Definition 1.1.1. The matrix A is called a *generalized Cartan matrix* if it satisfies the following conditions:

(1) $a_{ii} = 2$ for $i = 1, \ldots, n$,

(2) a_{ij} are non-positive integers for $i \neq j$, and
(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Example 1.1.1.

$$A = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 2 & -1 \\ -5 & -19 & 2 \end{pmatrix}$$

Now let A be any $n \times n$ matrix. In particular, A does not have to be a generalized Cartan matrix.

Definition 1.1.2. A *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets, satisfying the following conditions:

(1) both sets Π and $\Pi^\vee$ are linearly independent,
(2) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $i, j = 1, \ldots, n$, and
(3) $n - \ell = \dim \mathfrak{h} - n$.

Remark 1.1.1. The notation here is slightly unintuitive as $\Pi \subset \mathfrak{h}^*$ and $\Pi^\vee \subset \mathfrak{h}$, but this is Kac's notation, so I will stick to it.

Definition 1.1.3. If $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}', \Pi', \Pi'^\vee)$ are two realizations of the matrix A, then an *isomorphism of realizations* is an invertible linear map $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi(\alpha_i^\vee) = \alpha_i'^\vee$ and $\phi^*(\alpha_i') = \alpha_i$.

Proposition 1.1.1. *There exists a realization for every $n \times n$ matrix A. Furthermore this realization is unique up to isomorphism. (The isomorphism is not unique if* $\det A = 0$*.)*

Proof. First, by shuffling around the indices through row and column operations, we can assume that

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1 is a non-degenerate $\ell \times \ell$ submatrix. Notice that if the matrix is full rank, i.e., the $\det A \neq 0$, then $A_1 = A$. Now build the following $(2n - \ell) \times (2n - \ell)$ matrix:

$$C = \begin{pmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}$$

One can check that $\det C = \pm \det A_1$, thus we have built a non-degenerate matrix C. Now let $\mathfrak{h} = \mathbb{C}^{2n-\ell}$, pick $\alpha_1, \ldots, \alpha_n$ be the first n standard basis $e_1, \ldots, e_n$, and choose $\alpha_1^\vee, \ldots, \alpha_n^\vee$ to be the first n rows of the matrix C. Then this is a realization of the matrix A.

□

Example 1.1.2. Let

$$A = \begin{pmatrix} 2 & -1 & -8 \\ -3 & 2 & 0 \\ -10 & 0 & 2 \end{pmatrix}$$

First notice that this matrix is full rank, so $A = C$. Let $\mathfrak{h} = \mathbb{C}^3$, then choose $\alpha_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. Finally let $\alpha_1^\vee = \begin{pmatrix} 2 & -1 & -8 \end{pmatrix}^T$, $\alpha_2^\vee = \begin{pmatrix} -3 & 2 & 0 \end{pmatrix}^T$, and $\alpha_3^\vee = \begin{pmatrix} -10 & 0 & 2 \end{pmatrix}^T$. Notice, $\langle \alpha_i^\vee, \alpha_j \rangle = \alpha_j(\alpha_i^\vee) = a_{ij}$. Therefore this is a realization of A.

Example 1.1.3. Let

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

Now, the rank of A is 2, so we construct the matix

$$C = \begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\mathfrak{h} = \mathbb{C}^4$, $\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$, $\alpha_1^\vee = \begin{pmatrix} 2 & 0 & -2 & 0 \end{pmatrix}^T$, $\alpha_2^\vee = \begin{pmatrix} 0 & 2 & 0 & 0 \end{pmatrix}^T$, and $\alpha_3^\vee = \begin{pmatrix} -2 & 0 & 2 & 1 \end{pmatrix}^T$.

Remark 1.1.2. Example 1.6 illustrates several important concepts about matrices with determinant 0.

(1) First, notice the importance of adding the identity block matrices to C. Without them, $\Pi^\vee$ would not be a linearly independent set.
(2) Second, Notice the set $\{\alpha_i^\vee\}$ is linearly independent but does not span $\mathfrak{h}$. Therefore we can conclude that $\Pi^\vee$ **is a basis for $\mathfrak{h}$ if and only if the determinant of A is nonzero.**
(3) Finally, it is a simple exercise to see if $\det A \neq 0$ and we are given two different realizations of A, then there is exactly one isomorphism between the two realizations. Conversely, if $\det A = 0$, one can construct infinitely many different isomorphism between two separate realizations of A.

Definition 1.1.4. The set Π is called the *root basis* and $\Pi^\vee$ the *coroot basis*. Elements from Π (resp. $\Pi^\vee$) are called *simple roots* (resp. *simple coroots*). Set

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i \text{ and } Q_+ = \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i.$$

The lattice Q is called the *root lattice*.

Definition 1.1.5. For $\alpha = \sum_i k_i\alpha_i \in Q$ then the number $\mathrm{ht}\alpha := \sum_i k_i$ is called the *height of α*.

We can put a partial order $\geq$ on $\mathfrak{h}^*$ as follows. For $\lambda, \mu \in \mathfrak{h}^*$, let $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.

1.2. **The auxiliary Lie algebra $\mathfrak{g}$ and the PBW theorem.** Following section 1.2 in [2], we get closer to defining a Kac-Moody Lie algebra. Kac does this in two steps by first defining an auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$.

Definition 1.2.1. Let A be an $n \times n$ matrix over $\mathbb{C}$, and $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Then we define the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ to be the Lie algebra with the generators e_i, f_i ($i = 1, \ldots, n$) and $\mathfrak{h}$, and the following defining relations:

(1) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$, $i, j = 1, \ldots, n$,
(2) $[h, h'] = 0$, $h, h' \in \mathfrak{h}$,
(3) $[h, e_i] = \langle \alpha_i, h \rangle e_i$, and
(4) $[h, f_i] = -\langle \alpha_i, h \rangle f_i$.

Let $\tilde{\mathfrak{n}}_-$ be the subalgebra generated by $e_1, \ldots, e_n$ and $\tilde{n}_+$ the subalgebra generated by $f_1, \ldots, f_n$. The first big theorem is the following.

Theorem 1.2.1. *(1) The auxiliary Lie algebra can be decomposed as $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.*
(2) The subalgebra $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) is freely generated by $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_n$).
(3) The map $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, $h \mapsto -h$ can be uniquely extended to an involution $\tilde{\omega}$ of the Lie algebra $\tilde{g}(A)$.
(4) With respect to $\mathfrak{h}$ one has the root space decomposition

$$\tilde{\mathfrak{g}}(A) = \left(\bigoplus_{\alpha \in Q_+, \alpha \neq 0} \tilde{\mathfrak{g}}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in Q_+, \alpha \neq 0} \tilde{\mathfrak{g}}_{\alpha} \right)$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) | [h,x] = \alpha(h)x\}$, $\dim \tilde{\mathfrak{g}}(A)_\alpha < \infty$, *and* $\tilde{\mathfrak{g}}(A)_\alpha \subset \tilde{\mathfrak{n}}_\pm$ *for* $\pm\alpha \in Q_+$.

(5) *There exists a unique maximal ideal* $\mathfrak{r}$ *in* $\tilde{\mathfrak{g}}(A)$ *among the ideals intersecting* $\mathfrak{h}$ *trivially. Furthermore,*

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+).$$

We never went carefully through the proof of this theorem; however, the proof uses universal enveloping algebras and the PBW theorem so we will define these here as we will used them in future sections as well.

Definition 1.2.2. If V is a vector space over a field $\mathbb{F}$ and $k \in \mathbb{Z}_{\geq 2}$ then the k^{th} *tensor power* of V is

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k\text{-times}}$$

Define $V^{\otimes 0} = \mathbb{F}$ and $V^{\otimes 1} = V$. Then the *tensor algebra of* V is defined to be

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

Definition 1.2.3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. Then the *universal enveloping algebra of* $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y])$$

where $x, y \in \mathfrak{g}$.

Definition 1.2.4. Let R be a fixed commutative ring. An *associative R-algebra* is both an associative ring and an R-module, such that if $x, y \in A$ and $r \in R$, then

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$$

Remark 1.2.2. Notice that an associative algebra always has the structure of a Lie algebra as we can define a bracket operation to be $[x,y] = xy - yx$. However, a Lie algebra is not necessarily an associative algebra as the Lie bracket satisfies the Jacobi Identity and is not associative.

These definitions lead to an important proposition:

Proposition 1.2.1. *Let* σ *be a Lie algebra homomorphism of* $\mathfrak{g}$ *into an associative algebra* L. *Then there exists a unique homomorphism* ψ *of* $U(\mathfrak{g})$ *into* L *such that* $\sigma = \psi \circ \sigma_0$, *where* $\sigma_0 : \mathfrak{g} \to U(\mathfrak{g})$ *is the canonical mapping.*

Remark 1.2.3. This proposition tells us that representations of Lie algebras are in 1-1 correspondence with representations of the associative algebra $U(\mathfrak{g})$.

We will now explore the structure of the universal enveloping algebra through the Poincaré-Birkhoff-Witt (PBW) Theorem.

Theorem 1.2.4 (Poincaré-Birkhoff-Witt)**.** *Let* $\{v_1, \ldots, v_n\}$ *be a basis for* $\mathfrak{g}$. *Choose a total order such that* $v_1 \leq v_2 \leq \cdots \leq v_n$. *Then a basis for* $U(\mathfrak{g})$ *is*

$$\{v_{i_1}^{\otimes d_i} \otimes \cdots \otimes v_{i_k}^{\otimes d_k} | 1 \leq i_1 \leq \cdots \leq n, d_i \geq 0, k \leq n\}$$

This is the **Poincaré-Birkhoff-Witt** *(or PBW) basis of* $U(\mathfrak{g})$.

So what the PBW theorem states is given an ordered basis of a Lie algebra $\mathfrak{g}$, a basis of $U(\mathfrak{g})$ can be constructed from all monomials in the basis elements of $\mathfrak{g}$ where you keep the fixed order.

Example 1.2.1. To see what this is saying we will look at $\mathfrak{sl}_2(\mathbb{C})$, the Lie algebra of 2×2 complex matrices with trace 0 and Lie bracket given by matrix commutation. Recall a basis of $\mathfrak{sl}_2(\mathbb{C})$ is

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad F=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

For convenience of notation, express $x\otimes y=xy$ and $x^{\otimes d}=x^d$. Then $U(\mathfrak{sl}_2(\mathbb{C}))$ is generated by E,F, and H with the relations,

$$HE-EH=[H,E]=2E,\quad HF-FH=[H,F]=-2F,\quad EF-FE=[E,F]=H$$

Order the basis elements such that $F\leq H\leq E$. The PBW theorem states that a basis of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is

$$\{F^iH^jE^k|i,j,k\geq 0\}.$$

We will now demonstrate with a few examples how to write any element of the universal enveloping algebra as a linear combination of these monomials.

To do this we will be using that the universal enveloping algebra is associative and the relations above. For example $EF=H+FE$. Now a more complicated example is below.

$$\begin{aligned}EHF&=E(HF)\\&=E(-2F+FH)\\&=-2(EF)+(EF)H\\&=-2(H+FE)+(H+FE)H\\&=-2H-2FE+H^2+F(EH)\\&=-2H-2FE+H^2+F(HE-2E)\\&=-2H-2FE+H^2+FHE-2FE\\&=-2H-4FE+H^2+FHE.\end{aligned}$$

Now we will return to defining a Kac-Moody Lie algebra. We will do this is two different but equivalent ways. One following Kac's book [2] and one following Kumar's book [3].

1.3. **Construction of a Kac-Moody algebra.** We will first construct a Kac-Moody algebra following section 1.3 in [2].

Let $(\mathfrak{h},\Pi,\Pi^\vee)$ be a realization of A, and let $\tilde{\mathfrak{g}}(A)$ be the auxiliary Lie algebra constructed in the previous section. We have a natural imbedding $\mathfrak{h}\to\tilde{\mathfrak{g}}(A)$. Let $\mathfrak{r}$ be the unique maximal ideal that intersects $\mathfrak{h}$ trivially (Theorem 1.2.1). Set

$$\mathfrak{g}(A)=\tilde{\mathfrak{g}}(A)/\mathfrak{r}.$$

This is a Lie algebra. The matrix A is called the *Cartan matrix* of the Lie algebra $\mathfrak{g}(A)$, and n is called the *rank* of $\mathfrak{g}(A)$. Kac calls the quadruple $(\mathfrak{g}(A),\mathfrak{h},\Pi,\Pi^\vee)$ the $(\mathfrak{g},\mathfrak{h})$*-pair associated to the matrix* A. It is important to notice that the construction of this Lie algebra does not currently have any restrictions on the matrix A. So for any square matrix, we can construct a Lie algebra corresponding to that matrix.

Definition 1.3.1. Let A be a generalized Cartan matrix. The Lie algebra $\mathfrak{g}(A)$ is the *Kac-Moody algebra* associated to A.

If we assume that A is a generalized Cartan matrix, then there is an equivalent construction of a Kac-Moody algebra which comes from [3]. This construction lends more direct information to the structure of a Kac-Moody algebra.

Definition 1.3.2. Let A be a generalized Cartan matrix with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$. The *Kac-Moody algebra* associated to A is the Lie algebra $\mathfrak{g}(A)$ over $\mathbb{C}$ generated by e_i, f_i, $(i = 1, \dots, n)$, and $\mathfrak{h}$ with relations

1. $[h, h'] = 0, \qquad (h, h' \in \mathfrak{h})$,
2. $[h, e_i] = \alpha_i(h)e_i, \qquad [h, f_i] = -\alpha_i(h)f_i$,
3. $[e_i, f_j] = \delta_{i,j}\alpha_i^\vee$,
4. $(\mathrm{ad}\, e_i)^{1-a_{i,j}}(e_j) = 0, \qquad i \neq j$, and
5. $(\mathrm{ad}\, f_i)^{1-a_{i,j}}(f_j) = 0, \qquad i \neq j$.

Relations (4) and (5) are called the *Serre relations*. We will now go through an example which demonstrates how these two constructions are equivalent.

Example 1.3.1. Consider the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

. Both the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ from Section 1.1 and the Kac-Moody algebra from Definition 1.3.2 are generated by e_1, e_2, f_1, f_2 and $\mathfrak{h} = \mathbb{C}^2$. The Serre relations for this matrix are

$$[e_1[e_1, e_2]] = 0, \quad [e_2, [e_2, e_1]] = 0, \quad [f_1[f_1, f_2]] = 0, \quad [f_2, [f_2, f_1]] = 0.$$

We claim the unique maximal ideal $\mathfrak{r}$ which intersects $\mathfrak{h}$ trivially is

$$\langle [e_1[e_1, e_2]], [e_2, [e_2, e_1]], [f_1[f_1, f_2]], [f_2, [f_2, f_1]] \rangle.$$

Which is to say for a generalized Cartan matrix, quotienting by the ideal $\mathfrak{r}$ is equivalent to imposing the Serre relations.

Proof. Call the ideal above $\mathfrak{r}'$. To show that this is actually the unique maximal ideal $\mathfrak{r}$, we will first show that $\mathfrak{r}'$ intersects $\mathfrak{h}$ trivially. Elements of $\mathfrak{r}'$ will be of the form

$$[a, [e_1[e_1, e_2]]] + [b, [e_2, [e_2, e_1]]] + [c, [f_1[f_1, f_2]]] + [d, [f_2, [f_2, f_1]]]$$

where $a, b, c, d \in \tilde{\mathfrak{g}}(A)$. Recall that $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$, so to show $\mathfrak{r}'$ intersects $\mathfrak{h}$ trivially, we need to show that for $h \in \mathfrak{h}$, $\ell_- \in \tilde{\mathfrak{n}}_-$ and $\ell_+ \in \tilde{\mathfrak{n}}_+$ that

$$[h, \bullet] \notin \mathfrak{h}, \quad [\ell_-, \bullet] \notin \mathfrak{h}, \quad [\ell_+, \bullet] \notin \mathfrak{h}$$

where $\bullet$ represents any of the generators of the ideal $\mathfrak{r}'$.

We will only show $[h, [e_1, [e_1, e_2]]] \notin \mathfrak{h}$, $[\ell_-, [e_1, [e_1, e_2]]] \notin \mathfrak{h}$, and $[\ell_+, [e_1, [e_1, e_2]]] \notin \mathfrak{h}$ for all $h \in \mathfrak{h}$, $\ell_- \in \tilde{\mathfrak{n}}_-$, and $\ell_+ \in \tilde{\mathfrak{n}}_+$. Note that arguments for the other generators of $\mathfrak{r}'$ will be equivalent. For simplicity, we will denote the element $[e_1, [e_1, e_2]]$ as $[1, [1, 2]]$. First consider any $h \in \mathfrak{h}$,

$$\begin{aligned}
[h, [1, [1, 2]]] &= -[[1, 2], [h, 1]] - [1, [[1, 2], h]] \\
&= -[[1, 2], \alpha_1(h)1] + [1, [h, [1, 2]]] \\
&= -\alpha_1(h)[[1, 2], 1] - [1, [2, [h, 1]]] - [1, [1, [2, h]]] \\
&= \alpha_1(h)[1, [1, 2]] - [1, [2, \alpha_1(h)1]] - [1, [1, -\alpha_2(h)2]] \\
&= (\alpha_1(h) + \alpha_1(h) + \alpha_2(h))[1, [1, 2]] \notin \mathfrak{h}.
\end{aligned}$$

Now notice for any $\ell_+ \in \tilde{\mathfrak{n}}_+$ we have that

$$[\ell_+, [1, [1, 2]]] \in \tilde{\mathfrak{n}}_+.$$

Therefore $[\ell_+, [1, [1, 2]]]$ cannot be in $\mathfrak{h}$. So $\ell_- \in \mathfrak{n}_-$ is the last type of element in $\tilde{\mathfrak{g}}(A)$ which could possibly yield an element of $\mathfrak{h}$ when bracketed with $[1, [1, 2]]$. Although there are infinitely many elements $\ell_- \in \tilde{\mathfrak{n}}_-$ that we would need to check, we will do a few computations with some of the most simple elements in $\tilde{\mathfrak{n}}_-$ and notice a trend. So begin with $\ell_- = f_1$, $\ell_- = f_2$, and $\ell_- = [f_1, f_2]$.

$$[f_1, [1, [1, 2]]]] = \gamma_1[1, 2]$$

$$[f_2,[1,[1,2]]]] = \gamma_2[1,1] = 0$$

$$[[f_1,f_2],[1,[1,2]]]] = \gamma_3 e_1$$

where γ_i are constants. We notice that the computation of each bracket above yielded an element of $\tilde{\mathfrak{n}}_+$, thus none of these are in $\mathfrak{h}$. The computations above illustrate an important observation concerning the bracket of elements in $\tilde{\mathfrak{n}}_-$ with elements in $\tilde{\mathfrak{n}}_+$.

Observation. *Let $n < m$. Then*

$$[[f_{i_1},[f_{i_2},[\dots,[f_{i_{n-1}},f_{i_n}]]],[e_{j_1},[e_{j_2},[\dots,[e_{j_{m-1}},e_{j_m}]]]]]] \in \tilde{\mathfrak{n}}_+.$$

Similarly if $n > m$,

$$[[f_{i_1},[f_{i_2},[\dots,[f_{i_{n-1}},f_{i_n}]]],[e_{j_1},[e_{j_2},[\dots,[e_{j_{m-1}},e_{j_m}]]]]]] \in \tilde{\mathfrak{n}}_-.$$

Finally if $n = m$,

$$[[f_{i_1},[f_{i_2},[\dots,[f_{i_{n-1}},f_{i_n}]]],[e_{j_1},[e_{j_2},[\dots,[e_{j_{m-1}},e_{j_m}]]]]]] \in \mathfrak{h}.$$

From this observation, we note that the only possible elements $\ell_- \in \tilde{\mathfrak{n}}_-$ which will be in $\mathfrak{h}$ after bracketing with $[1,[1,2]]$, are $\ell_- = [f_1,[f_1,f_2]]$ or $\ell_- = [f_2,[f_2,f_1]]$. Therefore, in order for $\mathfrak{r}'$ to intersect $\mathfrak{h}$ trivially, we would need $[[f_1,[f_1,f_2]],[1,[1,2]]] = 0$ and $[[f_2,[f_2,f_1]],[1,[1,2]]] = 0$. Both of these computations do indeed yield 0, and thus we can conclude that for every element $g \in \tilde{\mathfrak{g}}(A)$, $[g,[1,[1,2]]] \notin \mathfrak{h}$.

Remark 1.3.1. The computations to prove $[[f_1,[f_1,f_2]],[1,[1,2]]] = 0$ and $[[f_2,[f_2,f_1]],[1,[1,2]]] = 0$ were very long and tedious. We will include one example of this type of computation below in which we show $[[f_1,[f_1,f_2]],[1,[1,2]]] = 0$.

Proof. The computation of $[[f_1,[f_1,f_2]],[e_1,[e_1,e_2]]]$ is large, so we will break it up into pieces.

First,

$$[[f_1,[f_1,f_2]],[e_1,[e_1,e_2]]] = -[[e_1,e_2],[[f_1,[f_1,f_2]],e_1]] - [e_1,[[e_1,e_2],[f_1,[f_1,f_2]]]].$$

We will focus on the purple section.

$$\begin{aligned} -[[e_1,e_2],[[f_1,[f_1,f_2]],e_1]] &= [[e_1,e_2],[e_1,[f_1,[f_1,f_2]]]] \\ &= -[[e_1,e_2],[[f_1,f_2],[e_1,f_1]]] - [[e_1,e_2],[f_1,[[f_1,f_2],e_1]]]. \end{aligned}$$

Now we further focus on the blue section within the purple section.

$$\begin{aligned} -[[e_1,e_2],[[f_1,f_2],[e_1,f_1]]] &= -[[e_1,e_2],[[f_1,f_2],\alpha_1^\vee]] \\ &= [[e_1,e_2],[\alpha_1^\vee,[f_1,f_2]]] \\ &= -[[e_1,e_2],[f_2,[\alpha_1^\vee,f_1]]] - [[e_1,e_2],[f_1,[f_2,\alpha_1^\vee]]] \\ &= -[[e_1,e_2],[f_2,-\alpha_1(\alpha_1^\vee)f_1]] - [[e_1,e_2],[f_1,\alpha_2(\alpha_1^\vee)f_2)]] \\ &= \alpha_1(\alpha_1^\vee)[[e_1,e_2],[f_2,f_1]] - \alpha_2(\alpha_1^\vee)[[e_1,e_2],[f_1,f_2]] \\ &= -\alpha_1(\alpha_1^\vee)[[e_1,e_2],[f_1,f_2]] - \alpha_2(\alpha_1^\vee)[[e_1,e_2],[f_1,f_2]]. \end{aligned}$$

The orange section becomes,

$$\begin{aligned}
[[e_1,e_2],[f_1,f_2]] &= -[f_2,[[e_1,e_2],f_1]] - [f_1,[f_2,[e_1,e_2]]] \\
&= [f_2,[f_1,[e_1,e_2]]] - [f_1,[f_2,[e_1,e_2]]] \\
&= -[f_2,[e_2,[f_1,e_1]]] - [f_2,[e-1,[e_2,f_1]]] + [f_1,[e_2,[f_2,e_1]]] + [f_1,[e_1,[e_2,f_2]] \\
&= -[f_2,[e_2,-\alpha_1^\vee]] + [f_1,[e_1,\alpha_2^\vee]] \\
&= -[f_2,[\alpha_1^\vee,e_2]] - [f_1,[\alpha_2^\vee,e_1]] \\
&= -[f_2,\alpha_2(\alpha_1^\vee)e_2] - [f_1,\alpha_1(\alpha_2^\vee)e_1] \\
&= -\alpha_2(\alpha_1^\vee)[f_2,e_2] - \alpha_1(\alpha_2^\vee)[f_1,e_1] \\
&= -\alpha_2(\alpha_1^\vee)(-\alpha_2^\vee) - \alpha_1(\alpha_2^\vee)(-\alpha_1^\vee) \\
&= \alpha_2(\alpha_1^\vee)\alpha_2^\vee + \alpha_1(\alpha_2^\vee)\alpha_1^\vee \\
&= a_{12}\alpha_2^\vee + a_{21}\alpha_1^\vee.
\end{aligned}$$

Thus,

$$-[[e_1,e_2],[[f_1,f_2],[e_1,f_1]]] = -a_{11}(a_{12}\alpha_2^\vee + a_{21}\alpha_1^\vee) - a_{12}(a_{12}\alpha_2^\vee + a_{21}\alpha_1^\vee).$$

Focusing on the red section we have,

$$\begin{aligned}
-[[e_1,e_2],[f_1,[[f_1,f_2],e_1]]] &= [[e_1,e_2],[f_1,[e_1,[f_1,f_2]]]] \\
&= -[[e_1,e_2],[f_1,[f_2,[e_1,f_1]]]] - [[e_1,e_2],[f_1[f_1,[f_2,e_1]]]] \\
&= -[[e_1,e_2],[f_1,[f_2,\alpha_1^\vee]]] \\
&= -[[e_1,e_2],[f_1,\alpha_2(\alpha_1^\vee)f_2]] \\
&= -\alpha_2(\alpha_1^\vee)[[e_1,e_2],[f_1,f_2]] \\
&= -a_{12}(a_{12}\alpha_2^\vee) - a_{12}(a_{21}\alpha_1^\vee).
\end{aligned}$$

Thus overall,

$$-[[e_1,e_2],[[f_1,[f_1,f_2]],e_1]] = -a_{11}a_{12}\alpha_2^\vee - a_{11}a_{21}\alpha_1^\vee - a_{12}a_{12}\alpha_2^\vee - a_{12}a_{21}\alpha_1^\vee - a_{12}a_{12}\alpha_2^\vee - a_{12}a_{21}\alpha_1^\vee$$

Now we will focus on the green section.

$$-[e_1,[[e_1,e_2],[f_1,[f_1,f_2]]]] = [e_1,[[f_1,f_2],[[e_1,e_2],f_1]]] + [e_1,[f_1,[[f_1,f_2],[e_1,e_2]]]]$$

We will first focus on the pink section.

$$\begin{aligned}
[e_1,[[f_1,f_2],[[e_1,e_2],f_1]]] &= -[e_1,[[f_1,f_2],[f_1,[e_1,e_2]]]] \\
&= [e_1,[[f_1,f_2],[e_2,[f_1,e_1]]]] + [e_1,[[f_1,f_2],[e_1,[e_2,f_1]]]] \\
&= [e_1,[[f_1,f_2],[e_2,-\alpha_1^\vee]]] \\
&= [e_1,[[f_1,f_2],\alpha_2(\alpha_1^\vee)e_2]] \\
&= -\alpha_2(\alpha_1^\vee)[e_1,[e_2,[f_1,f_2]]] \\
&= a_{12}[e_1,[f_2,[e_2,f_1]]] + a_{12}[e_1,[f_1,[f_2,e_2]]] \\
&= a_{12}[e_1,[f_1,-\alpha_2^\vee]] \\
&= -a_{12}[e_1,\alpha_1(\alpha_2^\vee)f_1] \\
&= -a_{12}a_{21}[e_1,f_1] \\
&= -a_{12}a_{21}\alpha_1^\vee
\end{aligned}$$

Finally, the light blue section becomes,

$$\begin{aligned}[e_1,[f_1,[[f_1,f_2],[e_1,e_2]]]] &= -[e_1,[f_1,[[e_1,e_2],[f_1,f_2]]]]\\ &= [e_1,[f_1,-[[e_1,e_2],[f_1,f_2]]]]\\ &= [e_1,[f_1,-a_{12}\alpha_2^\vee - a_{21}\alpha_1^\vee]]\\ &= -a_{12}[e_1,[f_1,\alpha_2^\vee]] - a_{21}[e_1,[f_1,\alpha_1^\vee]]\\ &= -a_{21}[e_1,\alpha_1(\alpha_2^\vee)f_1] - a_{21}[e_1,\alpha_1(\alpha_1^\vee)f_1]\\ &= -a_{21}a_{21}\alpha_1^\vee - a_{21}a_{11}\alpha_1^\vee.\end{aligned}$$

Therefore, the green section is,

$$-[e_1,[[e_1,e_2],[f_1,[f_1,f_2]]]] = -a_{12}a_{21}\alpha_1^\vee - a_{21}a_{21}\alpha_1^\vee - a_{21}a_{11}\alpha_1^\vee.$$

Thus finally we get,

$$\begin{aligned}[[f_1,[f_1,f_2]],[e_1,[e_1,e_2]]] = &- a_{11}a_{12}\alpha_2^\vee - a_{11}a_{21}\alpha_1^\vee - a_{12}a_{12}\alpha_2^\vee - a_{12}a_{21}\alpha_1^\vee - a_{12}a_{12}\alpha_2^\vee\\ &- a_{12}a_{21}\alpha_1^\vee - a_{12}a_{21}\alpha_1^\vee - a_{21}a_{21}\alpha_1^\vee - a_{21}a_{11}\alpha_1^\vee.\end{aligned}$$

The entries of the matrix are $a_{11}=2$, $a_{12}=-1$, $a_{21}=-1$, and $a_{22}=2$. Thus,

$$\begin{aligned}[[f_1,[f_1,f_2]],[e_1,[e_1,e_2]]] &= -(2)(-1)\alpha_2^\vee - (2)(-1)\alpha_1^\vee - (-1)(-1)\alpha_2^\vee - (-1)(-1)\alpha_1^\vee - (-1)(-1)\alpha_2^\vee\\ &\quad - (-1)(-1)\alpha_1^\vee - (-1)(-1)\alpha_1^\vee - (-1)(-1)\alpha_1^\vee - (-1)(2)\alpha_1^\vee\\ &= 0\alpha_2^\vee + 0\alpha_1^\vee\\ &= 0\end{aligned}$$

□

We have shown above that $\mathfrak{r}'$ intersects $\mathfrak{h}$ trivially. Since $\mathfrak{r}$ is unique, to show $\mathfrak{r}'=\mathfrak{r}$, we need to show $\mathfrak{r}'$ is maximal. So assume there was some ideal $\mathfrak{q}$ which intersects $\mathfrak{h}$ trivially such that

$$\mathfrak{r}' \subsetneq \mathfrak{q} \subsetneq \tilde{\mathfrak{g}}(A)$$

First note the only possibilities for the ideal $\mathfrak{q}$ are following:

(1) $\langle [e_1,e_2],[f_1,f_2]\rangle$,
(2) $\langle [e_1,e_2],f_1\rangle$,
(3) $\langle [e_1,e_2],f_2\rangle$,
(4) $\langle [f_1,f_2],e_1\rangle$, or
(5) $\langle [f_1,f_2],e_2\rangle$.

This is due to the fact that $\mathfrak{r}'$ is strictly contained in $\mathfrak{q}$, but not equal to $\mathfrak{q}$. Now if $\mathfrak{q}$ is any of the options (2) – (5), we have that either $[e_1,f_1]=\alpha_1^\vee\in\mathfrak{q}$, or $[e_2,f_2]=\alpha_2^\vee\in\mathfrak{q}$. Thus $\mathfrak{q}$ does not intersect $\mathfrak{h}$ trivially. Now if $\mathfrak{q} = (1)$ then,

$$\begin{aligned}[][e_1, e_2], [f_1, f_2]] &= -[f_2, [[e_1, e_2], f_1]] - [f_1, [f_2, [e_1, e_2]]] \\ &= [f_2, [f_1, [e_1, e_2]]] - [f_1, [f_2, [e_1, e_2]]] \\ &= -[f_2, [e_2, [f_1, e_1]]] - [f_2, [e_1, [e_2, f_1]]] + [f_1, [e_2, [f_2, e_1]]] + [f_1, [e_1, [e_2, f_2]]] \\ &= -[f_2, [e_2, \alpha_1^\vee]] + [f_1, [e_1, \alpha_2^\vee] \\ &= [f_2, [\alpha_1^\vee, e_2]] - [f_1, [\alpha_2^\vee, e_1]] \\ &= [f_2, \alpha_2(\alpha_1^\vee)e_2] - [f_1, [\alpha_1(\alpha_2^\vee)e_1]] \\ &= -a_{12}[e_2, f_2] + a_{21}[e_1, f_1] \\ &= -a_{12}\alpha_2^\vee + a_{21}\alpha_1^\vee \in \mathfrak{q}.\end{aligned}$$

Thus $\mathfrak{q}$ does not intersect $\mathfrak{h}$ trivially in any case. Therefore we can conclude $\mathfrak{r}'$ is maximal, and thus $\mathfrak{r}' = \mathfrak{r}$. □

Through all of these computations we have shown that the two different constructions of a Kac-Moody algebra are actually the same in this example. This remains true for any generalized Cartan Matrix. We will now show that $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{sl}_3(\mathbb{C})$. Now from the Serre relations, a basis of $\mathfrak{g}(A)$ is

$$\{\alpha_1^\vee, \alpha_2^\vee, e_1, e_2, f_1, f_2, [e_1, e_2], [f_1, f_2]\}$$

We can construct an isomorphism of $\mathfrak{g}(A) \to \mathfrak{sl}_3(\mathbb{C})$ by sending

$$\alpha_1^\vee \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_2^\vee \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[e_1, e_2] \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[f_1, f_2] \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

An important observation is that this matrix produced a finite dimensional Lie algebra. Now consider the matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The only difference between this matrix and the previous one is we changed the -1's to -2's. The Serre relations for this matrix are

$$[e_1,[e_1,[e_1,e_2]]]]=0, \quad [e_2,[e_2,[e_2,e_1]]]]=0, \quad [f_1,[f_1,[f_1,f_2]]]]=0, \quad [f_2,[f_2,[f_2,f_1]]]]=0$$

This will NOT yield a finite dimensional Lie algebra as we can alternate e_1's and e_2's to construct infinitely many linearly independent elements of the Lie algebra. For example,

$$[e_1,[e_2,[e_1,e_2]]], \quad [e_1,[e_2,[e_1,[e_2,e_1]]]], \quad [e_1,[e_2,[e_1,[\ldots]]]]$$

are all linearly independent elements of the Lie algebra. This leads us to the following classification.

Classification: Let A be a generalized Cartan matrix then A is of

(1) **finite type** if $\det A \neq 0$ and all of its proper principal minors are positive,
(2) **affine type** if $\det A = 0$ and all of its proper principal minors are positive,
(3) **indefinite type** otherwise.

If A is finite type then $\mathfrak{g}(A)$ will be a finite dimensional Lie algebra.

Definition 1.3.3. Let A be of affine type. The Lie algebra $\mathfrak{g}(A)$ is called the *affine Lie algebra associated to A.*

We will continue with some facts about the Lie algebra $\mathfrak{g}(A)$. First we call the subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ the *Cartan subalgebra* and we call the elements e_i and f_i the *Chevalley generators*. We will now explore some of the properties of $\mathfrak{g}(A)$.

Proposition 1.3.1. *The Chevalley generators generate the commutator subalgebra*[1] $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$. *Furthermore,* $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$, *and* $\mathfrak{g}(A) = \mathfrak{g}'(A)$ *if and only if* $\det A = 0$.

Proof. We will only prove the third claim. Assume $\det A \neq 0$. If $\mathfrak{g}(A) = \mathfrak{g}'(A)$ then for all $h \in \mathfrak{h}$ we must have that $h \in \mathfrak{g}'(A)$. Notice that since $\det A \neq 0$, $\Pi^\vee$ is a basis for $\mathfrak{h}$; thus for all $h \in \mathfrak{h}$, there will exist $a_1, \ldots, a_n$ such that

$$h = a_1\alpha_1^\vee + \cdots + a_n\alpha_n^\vee = a_1[e_1, f_1] + \cdots + a_n[e_n, f_n] \in \mathfrak{g}'(A).$$

If $\det A = 0$, then $\Pi^\vee$ is not a basis of $\mathfrak{h}$, thus there must exist some $h \in \mathfrak{h}$ such that h is not a linear combination of the $\alpha_i^\vee$. Now using the fact that $\mathfrak{g}'(A)$ is generated by the Chevalley generators and using the relations on $\mathfrak{g}(A)$, we can conclude that the only elements of h which are in $\mathfrak{g}'(A)$ are linear combinations of the elements in $\Pi^\vee$. Thus there will exist an $h \in \mathfrak{g}$ such that $h \notin \mathfrak{g}'(A)$. Thus $\mathfrak{g}(A) \neq \mathfrak{g}'(A)$. □

Let $\mathfrak{h}' = \mathbb{C}\alpha_i^\vee$. Then we have the following facts:

$$\mathfrak{g}'(A) \cap \mathfrak{h} = \mathfrak{h}' \text{ and } \mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha \text{ if } \alpha \neq 0.$$

We have the following *root space decomposition:*

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

where as before, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) | [h,x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$.

Definition 1.3.4. The number $\text{mult}\alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity of α*.

[1] $[\mathfrak{g}(A), \mathfrak{g}(A)]$ is the Lie algebra whose elements are linear combinations of the form $[g, g']$, where $g, g' \in \mathfrak{g}(A)$

We have the following estimation:

$$\text{mult}\alpha \le n^{ht\alpha}$$

where $\alpha = \sum_i k_i\alpha_i \in Q$. We call $\alpha \in Q$ a *root* if $\alpha \neq 0$ and $\text{mult}\alpha \neq 0$. Using the partial order defined in Section 1.1, a root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). Here $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$. Any root is either positive or negative. We denote Δ, Δ_+, and Δ_- to be the set of all roots, positive roots, and negative roots respectively. We have

$$\Delta = \Delta_+ \sqcup \Delta_-.$$

We know that $\mathfrak{g}_\alpha \subset \mathfrak{n}_+$ if $\alpha > 0$ and $\mathfrak{g}_\alpha \subset \mathfrak{n}_-$ if $\alpha < 0$. Therefore we have the following important observation.

Observation. *For $\alpha > 0$ (resp. $\alpha < 0$), $\mathfrak{g}_\alpha$ is the linear span of elements of the form $[e_{i_1}, [e_{i_2}, [e_{i_3}, [\cdots, [e_{i_{s-1}}, e_{i_s}]\cdots]$, (resp. $[f_{i_1}, [f_{i_2}, [f_{i_3}, [\cdots, [f_{i_{s-1}}, f_{i_s}]\cdots]$), such that $\alpha_{i_1} + \cdots + \alpha_{i_s} = \alpha$ (resp. $-\alpha$).*

If follows that

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i, \quad \mathfrak{g}_{s\alpha_i} = 0 \text{ if } |s| > 1$$

We have the following lemma.

Lemma 1.3.1. *If $\beta \in \Delta_+ \backslash \{\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i)] \cap \Delta \subset \Delta_+$.*

Example 1.3.2. In this example we will look at the root spaces of the Lie algebra associated to the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We let $\mathfrak{h} = \mathbb{C}^2$, $\alpha_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $\alpha_1^\vee \begin{pmatrix} 2 & -1 \end{pmatrix}^T$, $\alpha_2^\vee = \begin{pmatrix} -1 & 2 \end{pmatrix}^T$.

The Serre relations for this Lie algebra are

$$[e_1, [e_1, e_2]] = 0, \quad [e_2, [e_2, e_1]] = 0, \quad [f_1, [f_1, f_2]] = 0, \quad [f_2, [f_2, f_1]] = 0.$$

From the observation above, we have

$$\mathfrak{g}_{\alpha_1} = \mathbb{C}e_1, \quad \mathfrak{g}_{\alpha_2} = \mathbb{C}e_2, \quad \mathfrak{g}_{-\alpha_1} = \mathbb{C}f_1, \quad \mathfrak{g}_{-\alpha_2} = \mathbb{C}f_2.$$

We also have that $\mathfrak{g}_{\alpha_1+\alpha_2} = \mathbb{C}[e_1, e_2]$ and $\mathfrak{g}_{-\alpha_1-\alpha_2} = \mathbb{C}[f_1, f_2]$.

We claim that these are all of the root spaces. To see why this is true, consider $\alpha = a\alpha_1 + b\alpha_2$, where $a, b > 0$. The root space $\mathfrak{g}_\alpha$ is the linear span of elements of the form

$$[e_{i_1}, [e_{i_2}, [\cdots, [e_{i_{s-1}}, e_{i_s}]]]],$$

where $\alpha_{i_1} + \cdots + \alpha_{i_s} = a\alpha_1 + b\alpha_2$. Notice the only possible way the above bracket will be nonzero is if we alternate between α_1 and α_2. Thus the root space will be spanned by

$$[e_1, [e_2, [e_1, [\cdots, e_2]]]].$$

However,

$$[e_1, [e_2, [e_1, [\cdots, [e_2, [e_1, e_2]]]]]] = -[e_1, [e_2, [e_1, [\cdots, [e_2, [e_2, e_1]]]]]] = -[e_1, [e_2, [e_1, [\cdots, 0]]]] = 0$$

A similar argument is used if $a, b < 0$. Thus we can conclude there are six root spaces, so we have

$$\mathfrak{g}(A) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

Recall that $\mathfrak{r} \subset \tilde{\mathfrak{g}}(A)$ is $\tilde{\omega}$-invariant, where $\tilde{\omega}$ is the involution introduced in Theorem 1.2.1. Thus we get an induced involutive automorphism ω of the Lie algebra $\mathfrak{g}(A)$ called the *Cartan involution* of $\mathfrak{g}(A)$ determined by

$$\omega(e_i) = -f_i \quad \omega(f_i) = -e_i, \quad \omega(h) = -h.$$

Thus $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. This implies that $\text{mult}\alpha = \text{mult}(-\alpha)$. Therefore

$$\Delta_- = -\Delta_+.$$

1.4. **Uniqueness of $\mathfrak{g}(A)$.** Section 1.4 in [2] is comprised of statements of the uniqueness of the Lie algebras $\mathfrak{g}(A)$. I did not find these to be particularly important so I will not include the statements here.

1.5. **The principal gradation of $\mathfrak{g}(A)$.** We follow section 1.5 in [2] to create tools to further break up the Lie algebra $\mathfrak{g}(A)$.

Definition 1.5.1. Given an abelian group M, a decomposition $V = \bigoplus_{\alpha\in M} V_\alpha$ of a vector space V into a direct sum of its subspaces is called an *M-gradation of V*. We call a subspace $U \subset V$ *graded* if $U = \bigoplus_{\alpha\in M}(U \cap V_\alpha)$. The elements from V_α are called *homogeneous of degree α*.

This leads to an important proposition

Proposition 1.5.1. *Let $\mathfrak{h}$ be a commutative Lie algebra, V a diagonalizable $\mathfrak{h}$-module, i.e.*

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V | \lambda(h)v \text{ for all } h \in \mathfrak{h}\} \tag{1.5.1}$$

Then any submodule U of V is graded with respect to the gradation (1.5.1).

The proof of this proposition is included in [2].

Definition 1.5.2. Given a Lie algebra $\mathfrak{g}$, we define an *M-gradation of* $\mathfrak{g}$ as its gradation as a vector space such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Example 1.5.1. One example of an M gradation of $\mathfrak{g}(A)$ is the root space decomposition.

To introduce an M-gradation in a Lie algebra $\mathfrak{g}$, one chooses a system of generators $a_1, \dots, a_n$ of $\mathfrak{g}$ and elements $\lambda_1, \dots, \lambda_k$ of M and assigns a degree to each a_i. Let $\deg a_i = \lambda_i$. An M-gradation of $\mathfrak{g}$ with $\deg a_i = \lambda_i$ does not always exist, but if it does it is unique. If $a_1, \dots, a_n$ is a free system of generators of $\mathfrak{g}$, then such a gradation exists.

We will now describe how to find a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{g}(A)$. Let $s = (s_1, \dots, s_n)$ be an n-tuple of integers. Set

$$\deg e_i = -\deg f_i = s_i, \quad \deg \mathfrak{h} = 0$$

Then the *gradation of type s* is

$$\mathfrak{g}(A) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j(s),$$

where

$$\mathfrak{g}_j(s) = \bigoplus_{\alpha} \mathfrak{g}_\alpha,$$

and the sum is taken over $\alpha = \sum_i k_i\alpha_i \in Q$ such that $\sum_i k_i s_i = j$.

Notice that if $s_i > 0$ for every i, then $\mathfrak{g}_0(\mathfrak{h}) = \mathfrak{h}$. Also note for each $j \in \mathbb{Z}$ we have $\dim \mathfrak{g}_j(s) < \infty$.

The gradation we will be focusing on for the remainder of this section is called the *principal gradation*, which is what we obtain by setting $s = \mathbf{1} = (1, \dots, 1)$ in the construction above. Notice in this case,

$$\mathfrak{g}_j(\mathbf{1}) = \bigoplus_{\alpha:\mathrm{ht}\alpha=j} \mathfrak{g}_\alpha.$$

We can immediately conclude the following:

$$\mathfrak{g}_0(\mathbf{1}) = \mathfrak{h}, \quad \mathfrak{g}_{-1}(\mathbf{1}) = \sum_i \mathbb{C}f_i, \quad \mathfrak{g}_1(\mathbf{1}) = \sum_i \mathbb{C}e_i.$$

We also have that $\mathfrak{n}_\pm = \bigoplus_{j\geq 1} \mathfrak{g}_{\pm j}(\mathbf{1})$.

Example 1.5.2. Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Consider the Lie algebra $\mathfrak{g}(A)$ associated to this matrix. We know from previous sections that we already have two ways to decompose this; namely, the triangular decomposition and the root space decomposition. Thus we can express $\mathfrak{g}(A)$ as

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

or

$$\mathfrak{g}(A) = \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$$

Equipped with the principal gradation, we can decompose $\mathfrak{g}(A)$ in yet another way. Notice $\mathfrak{g}_0(\mathbf{1}) = \mathfrak{h}$, $\mathfrak{g}_{-2}(\mathbf{1}) = \mathfrak{g}_{-\alpha_1-\alpha_2}$, $\mathfrak{g}_2(\mathbf{1}) = \mathfrak{g}_{\alpha_1+\alpha_2}$, $\mathfrak{g}_{-1}(\mathbf{1}) = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$, $\mathfrak{g}_1(\mathbf{1}) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$, $\mathfrak{g}_{n\geq|3|}(\mathbf{1}) = 0$. So,

$$\mathfrak{g}(A) = \mathfrak{g}_{-2}(\mathbf{1}) \oplus \mathfrak{g}_{-1}(\mathbf{1}) \oplus \mathfrak{g}_0(\mathbf{1}) \oplus \mathfrak{g}_1(\mathbf{1}) \oplus \mathfrak{g}_2(\mathbf{1}).$$

To see exactly how each decomposition is partitioning $\mathfrak{g}(A)$, we will highlight the triangular decomposition in pink, the root space decomposition in green, and the principal gradation in yellow.

$$\mathfrak{g}(A) = \underbrace{\mathbb{C}[f_1, f_2] \oplus \mathbb{C}f_1 \oplus \mathbb{C}f_2}_{\mathfrak{n}_1} \oplus \underbrace{\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}\alpha_2^\vee}_{\mathfrak{h}} \oplus \underbrace{\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}[e_1, e_2]}_{\mathfrak{n}_+}$$

$$\mathfrak{g}(A) = \underbrace{\mathbb{C}[f_1, f_2]}_{\mathfrak{g}_{-\alpha_1-\alpha_2}} \oplus \underbrace{\mathbb{C}f_1}_{\mathfrak{g}_{-\alpha_1}} \oplus \underbrace{\mathbb{C}f_2}_{\mathfrak{g}_{-\alpha_2}} \oplus \underbrace{\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}\alpha_2^\vee}_{\mathfrak{g}_0} \oplus \underbrace{\mathbb{C}e_1}_{\mathfrak{g}_{\alpha_1}} \oplus \underbrace{\mathbb{C}e_2}_{\mathfrak{g}_{\alpha_2}} \oplus \underbrace{\mathbb{C}[e_1, e_2]}_{\mathfrak{g}_{\alpha_1+\alpha_2}}$$

$$\mathfrak{g}(A) = \underbrace{\mathbb{C}[f_1, f_2]}_{\mathfrak{g}_{-2}(\mathbf{1})} \oplus \underbrace{\mathbb{C}f_1 \oplus \mathbb{C}f_2}_{\mathfrak{g}_{-1}(\mathbf{1})} \oplus \underbrace{\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}\alpha_2^\vee}_{\mathfrak{g}_0(\mathbf{1})} \oplus \underbrace{\mathbb{C}e_1 \oplus \mathbb{C}e_2}_{\mathfrak{g}_1(\mathbf{1})} \oplus \underbrace{\mathbb{C}[e_1, e_2]}_{\mathfrak{g}_2(\mathbf{1})}$$

Example 1.5.3. For a slightly more difficult example we will consider the Lie algebra $\widehat{\mathfrak{sl}}_2$, which is associated to the matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The Serre relations for $\mathfrak{g}(A)$ are

$$[e_1, [e_1, [e_1, e_2]]] = 0, \quad [e_2, [e_2, [e_2, e_1]]] = 0, \quad [f_1, [f_1, [f_1, f_2]]] = 0, \quad [f_2, [f_2, [f_2, f_1]]] = 0.$$

The Lie algebra $\mathfrak{g}(A)$ is infinite dimensional, but it does have a nice basis. A basis for $\mathfrak{g}(A)$ is a basis for $\mathfrak{h}$, along with the sets

$$\{e_1, e_2, [e_1, e_2], [e_1, [e_2, e_1]], [e_2, [e_1, e_2]], [e_1, [e_2, [e_1, e_2]]], [e_1, [e_2, [e_1, [e_2, e_1]]]], [e_2, [e_1, [e_2, [e_1, e_2]]]], \dots, \}$$

and

$$\{f_1, f_2, [f_1, f_2], [f_1, [f_2, f_1]], [f_2, [f_1, f_2]], [f_1, [f_2, [f_1, f_2]]], [f_1, [f_2, [f_1, [f_2, f_1]]]], [f_2, [f_1, [f_2, [f_1, f_2]]]], \dots, \}$$

Observation. *In this basis we note brackets containing $n+1$ e_1's and n e_2's are linearly independent from the bracket containing n e_1's and $n+1$ e_2's; i.e. $[e_1, [e_2, [\dots, [e_2, e_1] \cdots]]]$ is linearly independent to $[e_2, [e_1, [\dots, [e_1, e_2] \cdots]]]$. However, if there are the same number of e_1's and e_2's in a bracket, then $[e_1, [e_2, [\dots, [e_1, e_2] \cdots]]] = -[e_2, [e_1, [\dots, [e_2, e_1] \cdots]]]$. We will prove this for $n = 1, 2, 3$.*

Proof. For $n = 1$ and $n = 2$, this claim is not difficult to see.

$n = 1$:

$$[e_1, e_2] = -[e_2, e_1]$$

$n = 2$:

$$[e_1, [e_2, [e_1, e_2]]] = -[e_1, [[e_1, e_2], e_2]] = [e_2, [e_1, [e_1, e_2]]] + [[e_1, e_2], [e_2, e_1]] = -[e_2, [e_1, [e_2, e_1]]]$$

The computations become more intense from this point forward. We will include one last computation.

$n = 3$:

We first need the following calculations:

$$\begin{aligned}
[[e_2, [e_1, e_2]], [e_1, e_2]] &= -[e_2, [[e_2, [e_1, e_2]], e_1]] - [e_1, [e_2, [[e_2, [e_1, e_2]]]]] \\
&= -[e_2, [[e_2, [e_1, e_2]], e_1]] + 0 \\
&= [e_2, [e_1, [e_2, [e_1, e_2]]]],
\end{aligned}$$

and

$$[[e_1, [e_2, e_1]], [e_2, e_1]] = [e_1, [e_2, [e_1, [e_2, e_1]]]].$$

Use the Jacobi Identity to obtain

$$\begin{aligned}
[e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]] &= -[[e_2, [e_1, [e_2, e_1]]], [e_2, e_1]] - [e_1, [[e_2, [e_1, [e_2, e_1]]], e_2]] \\
&= -[[e_2, [e_1, [e_2, e_1]]], [e_2, e_1]] + [e_1, [e_2, [e_2, [e_1, [e_2, e_1]]]]] \\
&= -[[e_2, [e_1, [e_2, e_1]]], [e_2, e_1]] - [e_1, [e_2, [e_1, [e_2, [e_1, e_2]]]]].
\end{aligned}$$

Using the first computations we completed, we have

$$\begin{aligned}
-[[e_2, [e_1, [e_2, e_1]]], [e_2, e_1]] &= [[e_2, e_1], [e_2, [e_1, [e_2, e_1]]]] \\
&= -[[e_1, [e_2, e_1]], [[e_2, e_1], e_2]] - [e_2, [[e_1, [e_2, e_1]], [e_2, e_1]]] \\
&= -[[e_1, [e_2, e_1]], [e_2, [e_1, e_2]]] - [e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]].
\end{aligned}$$

Therefore,

$$[e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]] = -[[e_1, [e_2, e_1]], [e_2, [e_1, e_2]]] - [e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]] - [e_1, [e_2, [e_1, [e_2, [e_1, e_2]]]]].$$

Simplifying,

$$2[e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]] = [[e_2, [e_1, e_2]], [e_1, [e_2, e_1]]] - [e_1, [e_2, [e_1, [e_2, [e_1, e_2]]]]].$$

We can use the exact same process as above to obtain

$$2[e_1, [e_2, [e_1, [e_2, [e_1, e_2]]]]] = -[[e_2, [e_1, e_2]], [e_1, [e_2, e_1]]] - [e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]].$$

Therefore,

$$\begin{aligned}
4[e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]] &= 2[[e_2, [e_1, e_2]], [e_1, [e_2, e_1]]] - 2[e_1, [e_2, [e_1, [e_2, [e_1, e_2]]]]] \\
&= 3[[e_2, [e_1, e_2]], [e_1, [e_2, e_1]]] + [e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]]
\end{aligned}$$

So we get the equality

$$[e_2,[e_1,[e_2,[e_1,[e_2,e_1]]]]] = [[e_2,[e_1,e_2]],[e_1,[e_2,e_1]]].$$

Similarly,

$$[e_1,[e_2,[e_1,[e_2,[e_1,e_2]]]]] = -[[e_2,[e_1,e_2]],[e_1,[e_2,e_1]]].$$

Finally get the desired result of

$$-[e_2,[e_1,[e_2,[e_1,[e_2,e_1]]]]] = [e_1,[e_2,[e_1,[e_2,[e_1,e_2]]]]].$$

□

Due to this alternating basis we have three possibilities for what the roots of $\mathfrak{g}(A)$ can be; namely

$$\alpha = n\alpha_1 + n\alpha_2, \quad \alpha = n\alpha_1 + (n-1)\alpha_2, \quad \alpha = (n-1)\alpha_1 + n\alpha_2$$

for $n \in \mathbb{Z}$.

Combining all the information above we can decompose $\mathfrak{g}(A)$ into its root spaces which are all one dimensional.

$$\begin{aligned}\mathfrak{g}(A) = \cdots &\oplus \mathfrak{g}_{-2\alpha_1-2\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-2\alpha_2} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \\ &\oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+2\alpha_2} \oplus \cdots\end{aligned}$$

Now we will decompose $\mathfrak{g}(A)$ into its principal gradation decomposition.

$$\mathfrak{g}(A) = \cdots \oplus \mathfrak{g}_{-2}(\mathbf{1}) \oplus \mathfrak{g}_{-1}(\mathbf{1}) \oplus \mathfrak{g}_0(\mathbf{1}) \oplus \mathfrak{g}_1(\mathbf{1}) \oplus \mathfrak{g}_2(\mathbf{1}) \oplus \cdots$$

Here

$$\mathfrak{g}_0(\mathbf{1}) = \mathfrak{h}, \;\; \mathfrak{g}_1(\mathbf{1}) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}, \;\; \mathfrak{g}_2(\mathbf{1}) = \mathfrak{g}_{\alpha_1+\alpha_2}, \;\; \mathfrak{g}_3(\mathbf{1}) = \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}, \;\; \mathfrak{g}_4(\mathbf{1}) = \mathfrak{g}_{2\alpha_1+2\alpha}.$$

Thus we observe that for j even, we have $\dim \mathfrak{g}_j(\mathbf{1}) = 1$ and for j odd we have $\dim \mathfrak{g}_j(\mathbf{1}) = 2$.

We will again include graphics to illustrate how each decomposition partitions $\mathfrak{g}(A)$. We will again highlight the triangular decomposition in pink, the root space decomposition in green, and the principal gradation in yellow.

$$\begin{aligned}\mathfrak{g}(A) = &\underbrace{\cdots \oplus \mathbb{C}[f_1,[f_2,[f_1,f_2]]] \oplus \mathbb{C}[f_1,[f_2,f_1]] \oplus \mathbb{C}[f_2,[f_1,f_2]] \oplus \mathbb{C}[f_1,f_2] \oplus \mathbb{C}f_1 \oplus \mathbb{C}f_2}_{\mathfrak{n}_-} \\ &\oplus \underbrace{\mathfrak{h}}_{\mathfrak{h}} \\ &\oplus \underbrace{\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}[e_1,e_2] \oplus \mathbb{C}[e_1,[e_2,e_1]] \oplus \mathbb{C}[e_2,[e_1,e_2]] \oplus \mathbb{C}[e_1,[e_2,[e_1,e_2]]] \oplus \cdots}_{\mathfrak{n}_+}\end{aligned}$$

$$\begin{aligned}\mathfrak{g}(A) = \cdots &\oplus \underbrace{\mathbb{C}[f_1,[f_2,[f_1,f_2]]]}_{\mathfrak{g}_{-2\alpha_1-2\alpha_2}} \oplus \underbrace{\mathbb{C}[f_1,[f_2,f_1]]}_{\mathfrak{g}_{-2\alpha_1-\alpha_2}} \oplus \underbrace{\mathbb{C}[f_2,[f_1,f_2]]}_{\mathfrak{g}_{-\alpha_1-2\alpha_2}} \oplus \underbrace{\mathbb{C}[f_1,f_2]}_{\mathfrak{g}_{-\alpha_1-\alpha_2}} \oplus \underbrace{\mathbb{C}f_1}_{\mathfrak{g}_{-\alpha_1}} \oplus \underbrace{\mathbb{C}f_2}_{\mathfrak{g}_{-\alpha_2}} \\ &\oplus \underbrace{\mathfrak{h}}_{\mathfrak{g}_0} \\ &\oplus \underbrace{\mathbb{C}e_1}_{\mathfrak{g}_{\alpha_1}} \oplus \underbrace{\mathbb{C}e_2}_{\mathfrak{g}_{\alpha_2}} \oplus \underbrace{\mathbb{C}[e_1,e_2]}_{\mathfrak{g}_{\alpha_1+\alpha_2}} \oplus \underbrace{\mathbb{C}[e_1,[e_2,e_1]]}_{\mathfrak{g}_{2\alpha_1+\alpha_2}} \oplus \underbrace{\mathbb{C}[e_2,[e_1,e_2]]}_{\mathfrak{g}_{\alpha_1+2\alpha_2}} \oplus \underbrace{\mathbb{C}[e_1,[e_2,[e_1,e_2]]]}_{\mathfrak{g}_{2\alpha_1+2\alpha_2}} \oplus \cdots\end{aligned}$$

$$\mathfrak{g}(A) = \cdots \oplus \underbrace{\mathbb{C}[f_1,[f_2,[f_1,f_2]]]}_{\mathfrak{g}_{-4}(\mathbf{1})} \oplus \underbrace{\mathbb{C}[f_1,[f_2,f_1]] \oplus \mathbb{C}[f_2,[f_1,f_2]]}_{\mathfrak{g}_{-3}(\mathbf{1})} \oplus \underbrace{\mathbb{C}[f_1,f_2]}_{\mathfrak{g}_{-2}(\mathbf{1})} \oplus \underbrace{\mathbb{C}f_1 \oplus \mathbb{C}f_2}_{\mathfrak{g}_{-1}(\mathbf{1})}$$
$$\oplus \underbrace{\mathfrak{h}}_{\mathfrak{g}_0(\mathbf{1})}$$
$$\oplus \underbrace{\mathbb{C}e_1 \oplus \mathbb{C}e_2}_{\mathfrak{g}_1(\mathbf{1})} \oplus \underbrace{\mathbb{C}[e_1,e_2]}_{\mathfrak{g}_2(\mathbf{1})} \oplus \underbrace{\mathbb{C}[e_1,[e_2,e_1]] \oplus \mathbb{C}[e_2,[e_1,e_2]]}_{\mathfrak{g}_3(\mathbf{1})} \oplus \underbrace{\mathbb{C}[e_1,[e_2,[e_1,e_2]]]}_{\mathfrak{g}_4(\mathbf{1})} \oplus \cdots$$

The last thing we will include from this section is a lemma which is useful in future proofs.

Lemma 1.5.1. *Let $a \in \mathfrak{n}_+$ be such that $[a, f_i] = 0$ for all $i = 1, \ldots, n$, then $a = 0$. Similarly if $a \in \mathfrak{n}_-$ and $[a, e_i] = 0$ for all $i = 1, \ldots, n$, then $a = 0$.*

1.6. **The center of $\mathfrak{g}(A)$.** The following section, which follows section 1.6 in [2] is comprised of two facts about the Lie algebra $\mathfrak{g}(A)$.

Proposition 1.6.1. *The center of the Lie algebra $\mathfrak{g}(A)$ or $\mathfrak{g}'(A)$ is*

$$\mathfrak{c} = \{h \in \mathfrak{h} | \langle \alpha_i, h\rangle = 0 \, for \, all \, i = 1, \ldots, n\}.$$

Proof. This proof originates from [1]. We will first prove $\mathfrak{c}$ is the center of $\mathfrak{g}(A)$. Let c be in the center of $\mathfrak{g}(A)$. Then $[h, c] = 0$ for all $h \in \mathfrak{h}$ thus $c \in \mathfrak{g}_0 = \mathfrak{h}$. We also have $[c, e_i] = \alpha_i(c)e_i = 0$, for all i, thus $\alpha_i(c) = \langle \alpha_i, c\rangle = 0$ for all $i = 1, \ldots, n$. Thus $c \in \mathfrak{c}$. Take some $c \in \mathfrak{c}$. To show it is in the center we need to show it commutes with the Chevalley generators. Since $\alpha_i(c) = 0$ for all i, we must have that $[c, e_i] = 0$ and $[c, f_i] = 0$ for all i. Thus c is in the center.

To show that $\mathfrak{c}$ is also the center of $\mathfrak{g}'(A)$ we need to show $\mathfrak{c} \subseteq \mathfrak{h}'$. First notice that since $\alpha_i(\alpha_j^\vee) = A_{ji}$, each $\alpha_j^\vee$ picks out the j^{th} row of A over $i = 1, \ldots, n$. Thus any linear combination of the rows induces a linear combination of the $\alpha_j^\vee$'s. Therefore linear combination of rows which sum to zero are in correspondence with the maximal number of linearly independent elements of $\mathfrak{h}'$ which are contained in center. Thus,

$$n - \ell = \text{corank}(A) = \dim(\mathfrak{c} \cap \mathfrak{h}') \leq \dim \mathfrak{c}$$

However, since for all $j = 1, \ldots, n$ we have $\alpha_j(\alpha_j^\vee) = 2 \neq 0$, we must have that all simple coroots must be contained in the vector space complement of $\mathfrak{c}$ in $\mathfrak{h}$. Recall all the $\alpha_j^\vee$'s are linearly independent so we must have

$$\dim \mathfrak{c} \leq \dim \mathfrak{h} - n = n - \ell = \text{corank}(A)$$

Thus,

$$\dim(\mathfrak{c} \cap \mathfrak{h}') = \dim \mathfrak{c}$$

Thus $\mathfrak{c} \subseteq \mathfrak{h}'$ □

Lemma 1.6.1. *Let $I_1, I_2 \subset \{1, \ldots, n\}$ be disjoint subsets such that $a_{ij} = 0 = a_{ji}$ whenever $i \in I_1$ and $j \in I_2$. Let $\beta_s = \sum_{i \in I_s} k_i^s \alpha_i$ $(s = 1, 2)$. Suppose $\alpha = \beta_1 + \beta_2$ is a root of the Lie algebra $\mathfrak{g}(A)$. Then either β_1 or β_2 is zero.*

Proof. Let $i \in I_1$ and $j \in I_2$. We can see that $[\alpha_i^\vee, e_j] = 0$, $[\alpha_j^\vee, e_i] = 0$, $[e_i, f_j] = 0$, $[e_j^\vee, f_i] = 0$ by using the conditions $a_{ij} = a_{ji} = 0$ and I_1 and I_2 are disjoint. Notice that

$$[[e_i, e_j], f_i] = -[f_i, [e_i, e_j]] = [e_j, [f_i, e_i]] + [e_i, [e_j, f_i]] = 0,$$

so using Lemma 1.5.1, we have that $[e_i, e_j] = 0$. We can similarly show that $[f_i, f_j] = 0$. Let $\mathfrak{g}^{(s)}$ be the subalgebra generated by e_i, f_i for $i \in I_s$. From all of the computations above we know that

$\mathfrak{g}^{(1)}$ commutes with $\mathfrak{g}^{(2)}$. Since $\mathfrak{g}_\alpha$ lies in the subalgebra generated by $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ we must have that $\mathfrak{g}_\alpha$ lies either in $\mathfrak{g}^{(1)}$ or $\mathfrak{g}^{(2)}$. Since $\alpha = \beta_1 + \beta_2$, either β_1 or β_2 is zero.

□

1.7. **Ideals of $\mathfrak{g}(A)$.** The focus of section 1.7 in [2] is giving a description of the structure of the ideals of $\mathfrak{g}(a)$.

Proposition 1.7.1. *(1) The Lie algebra $\mathfrak{g}(A)$ is simple if and only if $\det A \neq 0$ and for each pair of indices i, j the following condition holds;*

(1.7.1) *there exists indices $i_1, \ldots, i_s$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_s j} \neq 0$.*

(2) Provided that the condition above holds, every ideal of $\mathfrak{g}(A)$ either contains $\mathfrak{g}'(A)$ or is contained in the center.

Proof. First assume $\det A = 0$, then we know the center of $\mathfrak{g}(A)$ is a nontrivial proper nonzero ideal. Thus $\mathfrak{g}(A)$ is not simple.

Now suppose $\det A \neq 0$ and condition (1.7.1) is satisfied. Let $\mathfrak{i} \subset \mathfrak{g}(A)$ be a nonzero ideal. Then i contains a nonzero element of $h \in \mathfrak{h}$ as $\mathfrak{r}$ is the only ideal of $\mathfrak{g}(\tilde{A})$ which intersects $\mathfrak{h}$ trivially. Now since $\det A \neq 0$ by Proposition 1.6.1 we know $\mathfrak{c} = 0$. Thus $[h, e_i] = ae_i \neq 0$ for some i. Therefore we know $e_i \in \mathfrak{i}$ and $\alpha_i = [e_i, f_i] \in \mathfrak{i}$. From condition (1.7.1), we have for every $j = 1, \ldots n$ there exist indices $i_1, \ldots, i_s$ such that

$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_s j} \neq 0.$$

Therefore, $a_{ii_1} \neq 0, \cdots, a_{i_s j} \neq 0$. Therefore,

$$[\alpha_i^\vee, e_{i_1}] = a_{ii_1} e_{i_1} \neq 0, \ \cdots, \ [\alpha_{i_s}^\vee, e_j] = a_{i_s j} e_j \neq 0.$$

Thus we get for every $j = 1, \ldots n$, $e_j \in \mathfrak{i}$ for all j, which directly implies $e_j, f_j, \alpha_j^\vee \in \mathfrak{i}$ for all $j = 1 \ldots n$. Since $\det A \neq 0$, $\mathfrak{h}$ is the linear span of the set $\{\alpha_j^\vee\}$ thus $i = \mathfrak{g}(A)$ which is to say $\mathfrak{g}(A)$ is simple.

Now assume that condition (1.7.1) holds and consider some ideal $\mathfrak{i} \subset \mathfrak{g}(A)$. If $\det A \neq 0$, then $\mathfrak{g}(A)$ is simple so the only ideals are 0 and $\mathfrak{g}(A)$. Notice $0 \in \mathfrak{c}$, and $\mathfrak{g}'(A) \subset \mathfrak{g}(A)$ so (2) is true. Now if $\det A = 0$, then the center is non trivial. Assume that $\mathfrak{i}$ is not contained in the center. Then take some $h \in \mathfrak{h}$ such that $h \in \mathfrak{i}$ and $h \notin \mathfrak{c}$. Then

$$[h, e_i] = \alpha_i(h) e_i \neq 0$$

for some $i = 1, \ldots n$. Thus $e_i \in \mathfrak{i}$. Similarly we have $f_i \in \mathfrak{i}$ and $\alpha_i^\vee \in \mathfrak{i}$. Using the same process as above we can utilize (1.7.1) to get $e_i, f_i, \alpha_i^\vee \in i$ for every $i = 1, \cdots, n$. Thus $\mathfrak{g}'(A) \subset \mathfrak{i}$.

□

2. The Casimir operator for finite dimensional Lie algebras

2.1. **Construction of the Casimir operator.** The focus of this section is how to construct the Casimir operator for finite dimensional Lie algebras. We will begin with some pertinent definitions.

Definition 2.1.1. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras with Lie bracket operations $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$. A linear transformation $T : \mathfrak{g}_1 \to \mathfrak{g}_2$, is a *Lie algebra homomorphism* if

$$T([A, B]_1) = [T(A), T(B)]_2$$

for all $A, B \in \mathfrak{g}_1$. Furthermore T is a *Lie algebra isomorphism* if T is bijective.

Definition 2.1.2. Let $\mathfrak{g}$ be a Lie algebra and V a vector space. Then a *representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to L(V)$$

where $L(V)$ are linear maps of V to itself. An equivalent definition of representation of $\mathfrak{g}$ on V is saying V is a $\mathfrak{g}$-module. We will denote a representation as $(\mathfrak{g}, V, \rho)$.

Definition 2.1.3. Suppose $(\mathfrak{g}, V, \rho)$ and $(\mathfrak{g}, W, \psi)$ are two representations of the Lie algebra $\mathfrak{g}$. A linear transformation $T : V \to W$ is called a *homomorphism of representations* if for all $g \in \mathfrak{g}$,

$$T \circ \rho(g) = \psi(g) \circ T.$$

Furthermore T is an *isomorphism of representations* if T is invertible.

Definition 2.1.4. A *subrepresentation* of a representation $(\mathfrak{g}, V, \rho)$ is a subspace $W \subseteq V$ satisfying for all $x \in \mathfrak{g}$

$$(\rho(x))(W) \subseteq W.$$

A representation that contains no non trivial proper subrepresentations is called *irreducible.*

Recall from Remark 1.2.3 there is a $1 : 1$ correspondence between $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules. Thus when working with $\mathfrak{g}$-modules it is often more useful to consider them as $U(\mathfrak{g})$-modules. This leads us to the construction of the Casimir operator.

Definition 2.1.5. Let $\mathfrak{g}$ be a semi simple Lie algebra over a field K, $U(\mathfrak{g})$ its universal enveloping algebra, $\mathfrak{h}$ a finite dimensional ideal of $\mathfrak{g}$, and B an invariant bilinear form on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is nondegenerate. Let $(e_i)_{1\leq i\leq n}$, $(e'_j)_{1\leq j\leq n}$ be two bases of $\mathfrak{h}$ such that $B(e_i, e'_j) = \delta_{ij}$. Then the element $c = \sum_{i=1}^n e_i e'_i$ belongs to the center of $U(\mathfrak{g})$ and is independent of choice of bases. The element c as an element of $U(\mathfrak{g})$ is called the *Casimir element* and c as an operator on a $U(\mathfrak{g})$-module is called the *Casimir operator.*

Remark 2.1.1. When $\mathfrak{g}$ is finite dimensional, the center of $U(\mathfrak{g})$ is large and can be difficult to construct. So the Casimir element is useful as we can always find one element in the center of $U(\mathfrak{g})$.

We now arrive at an important theorem.

Theorem 2.1.2. *Let V be a $\mathfrak{g}$-module for some finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let c be the Casimir operator. Then c will act on V as a scalar multiple of the identity, i.e.*

$$\begin{aligned} V &\to V \\ v &\mapsto c.v = \chi(c)v \end{aligned}$$

where $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is called an infinitesimal character, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.

In order to prove this we will need to utilize Schur's lemma.

Lemma 2.1.1 (Schur)**.** *Suppose $(\mathfrak{g}, V, \rho)$ and $(\mathfrak{g}, W, \psi)$ are irreducible representations of a Lie algebra $\mathfrak{g}$. Let $\phi : V \to W$ be a homomorphism of representations. Then ϕ is either zero or an isomorphsim.*

Proof. Assume that ϕ is not zero. We will first show $\ker\phi$ and $\mathrm{Im}\phi$ are subrepresentations of V and W respectively. $\ker\phi \subseteq V$, so take any $x \in \mathfrak{g}$ and $v \in \ker\phi$. Then,

$$\phi(\rho(x)(v)) = \psi(x) \circ \phi(v) = \psi(x)(0) = 0.$$

Thus $\rho(x)(v) \in \ker\phi$ so the kernel is a subrepresentation of V. Now for the image, take any $x \in \mathfrak{g}$ and $w \in \mathrm{Im}\phi$. In order for $\psi(x)(w) \in \mathrm{Im}\phi$, there must be some $v \in V$ such that $\phi(v) = \psi(x)(w)$. Now since $w \in \mathrm{Im}\phi$, there is some $v_1 \in V$ such that $\phi(v_1) = w$. Let $v = \rho(x)(v_1)$, then

$$\phi(v) = \phi(\rho(x)(v_1)) = \psi(x) \circ \phi(v_1) = \psi(x)(w)$$

Therefore, $\mathrm{Im}\phi$ is a subrepresentation of W. Since W and V are irreducible, this implies $\ker\phi = 0$ or $\ker\phi = V$ and $\mathrm{Im}\phi = W$ or $\mathrm{Im}\phi = 0$. By assumption, ϕ is not 0, thus $\ker\phi = 0$ and $\mathrm{Im}\phi = W$, so ϕ is an isomorphism. □

From Schur's lemma we get the following important corollary for when V is finite dimensional.

Corollary 2.1.1. *Let V be a finite dimensional irreducible representation of a Lie algebra over an algebraically closed field. Let $\phi : V \to V$ be a homomorphism of representations. Then ϕ is a scalar multiple of the identity.*

Proof. Since V is finite dimensional over an algebraically closed field, there will exist an eigenvalue λ of ϕ. Let $\phi' = \phi - \lambda_I$. Let x be the eigenvector associated with λ. Then,

$$\phi'(x) = (\phi - \lambda I)x = \phi(x) - \lambda x = \lambda x - \lambda x = 0.$$

So $\ker \phi'$ is nontrivial thus by Schur's lemma we know $\ker \phi' = V$, which is to say $\phi' = 0 = \phi - \lambda I$. □

Remark 2.1.3. It turns out that V can also be infinite dimensional for the corollary above to remain true, but we will not prove that case here. This generalization is often called Diximier's Lemma.

This Corollary directly proves Theorem 2.1.2. This leads us to an important remark.

Remark 2.1.4. If two representations have different infinitesimal characters, then they are NOT isomorphic. Thus we see how the Casimir operator can be extremely useful in the question of characterizing representations of a given Lie algebra.

2.2. **The Casimir operator for $\mathfrak{sl}_2(\mathbb{C})$.** In this section we will give an explicit example of how to construct the Casimir element and operator for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

To construct the Casimir element, we first need an invariant bilinear form on $\mathfrak{sl}_2(\mathbb{C})$ that is nondegenerate. First recall the *adjoint representation*

$$ad : \mathfrak{sl}_2(\mathbb{C}) \to L(\mathfrak{sl}_2(\mathbb{C}))$$
$$x \mapsto ad_x$$

where $ad_x : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ is defined by $ad_x(y) = [x, y]$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is a three dimensional Lie algebra with basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \text{and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we can represent ad_x as a 3×3 matrix for every $x \in \mathfrak{sl}_2(\mathbb{C})$. We define the *Killing form* on $\mathfrak{sl}_2(\mathbb{C})$ as the nondegenerate invariant bilinear form

$$\kappa : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$$
$$(x, y) \mapsto \frac{1}{2}\text{tr}(ad_x ad_y)$$

where $\text{tr}(ad_x ad_y)$ is the trace of the matrix $ad_x ad_y$. To define the Casimir operator we need to find dual bases $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', e_3'\}$ of $\mathfrak{sl}_2(\mathbb{C})$. Let $\{e_1, e_2, e_3\} = \{E, F, H\}$, then we need to find $\{X, Y, Z\} = \{e_1', e_2', e_3'\}$. We include a computation on how to find the first basis element X and note that the other computations are similar. First we find the matrices for ad_E, ad_F, and ad_H are as follows:

$$ad_E = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ ad_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},\ ad_H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $X = aE + bF + cH$ for $a, b, c \in \mathfrak{sl}_2(\mathbb{C})$, then since $\{X, Y, Z\}$ is dual to $\{E, F, H\}$ we should have

$$\kappa(E, X) = a\kappa(E, E) + b\kappa(E, F) + c\kappa(E, H) = 1,$$
$$\kappa(F, X) = a\kappa(F, E) + b\kappa(F, F) + c\kappa(F, H) = 0,$$
$$\kappa(H, X) = a\kappa(H, E) + b\kappa(H, F) + c\kappa(H, H) = 0.$$

So we obtain the following system of equations:

$$\frac{1}{2}a\text{tr}(ad_E ad_E) + \frac{1}{2}b\text{tr}(ad_E ad_F) + \frac{1}{2}c\text{tr}(ad_E ad_H) = 1,$$

$$\frac{1}{2}a\mathrm{tr}(ad_F ad_E) + \frac{1}{2}b\mathrm{tr}(ad_F ad_F) + \frac{1}{2}c\mathrm{tr}(ad_F ad_H) = 0,$$

$$\frac{1}{2}a\mathrm{tr}(ad_H ad_E) + \frac{1}{2}b\mathrm{tr}(ad_H ad_F) + \frac{1}{2}c\mathrm{tr}(ad_H ad_H) = 0.$$

Solving this system of equations yields $X = \frac{1}{2}F$. Doing the same for Y and Z, we obtain the basis $\{e'_1, e'_2, e'_3\} = \{\frac{1}{2}F, \frac{1}{2}E, \frac{1}{4}H\}$. Thus we get the Casimir element for $\mathfrak{sl}_2(\mathbb{C})$ is

$$c = \frac{1}{2}EF + \frac{1}{2}FE + \frac{1}{4}H^2.$$

We will verify c is in the center of $U(\mathfrak{sl}_2(\mathbb{C}))$ by showing it commutes with the basis elements of $\mathfrak{sl}_2(\mathbb{C})$. To do this we first will prove the following useful fact.

Proposition 2.2.1. *Let $a, b, c \in U(\mathfrak{g})$, then $[a, bc] = [a, b]c + b[a, c]$.*

Proof.

$$[a, b]c + b[a, c] = abc - bac + bac - bca = abc - bac = [a, bc]$$

□

Now,

$$\begin{aligned} [H, c] &= \frac{1}{2}[H, EF] + \frac{1}{2}[H, FE] + \frac{1}{4}[H, H^2] \\ &= \frac{1}{2}([H, E]F + E[H, F]) + \frac{1}{2}([H, F]E + F[H, E]) \\ &= \frac{1}{2}(2EF - 2EF) + \frac{1}{2}(-2FE + 2FE) \\ &= 0. \end{aligned}$$

Similar computations show that $[E, c] = 0$ and $[F, c] = 0$, so we have found that c is indeed in the center of $U(\mathfrak{sl}_2(\mathbb{C}))$.

Remark 2.2.1. Notice that any scalar multiple of c is also in the center of $U(\mathfrak{g})$. Thus the Casimir operator for $\mathfrak{sl}_2(\mathbb{C})$ is often written as $c = EF + FE + \frac{1}{2}H^2$.

We will now give an example of an infinitesimal character of a representation of $\mathfrak{sl}_2(\mathbb{C})$. An irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is $\mathcal{P}^2$ which is the vector space consisting of homogeneous polynomial of degree two in x and y. The representation is described by

$$\begin{aligned} \rho : \mathfrak{sl}_2(C) &\to L(\mathbb{C}[x, y]) \\ E &\mapsto -y\frac{d}{dx} \\ F &\mapsto -x\frac{d}{dy} \\ H &\mapsto y\frac{d}{dy} - x\frac{d}{dx} \end{aligned}$$

A basis of $\mathcal{P}^2$ is $\{x^2, xy, y^2\}$. One can check the action of the Casimir operator on $\mathcal{P}^2$ is

$$\begin{aligned} \mathcal{P}^2 &\to \mathcal{P}^2 \\ v &\mapsto 2v \end{aligned}$$

for all $v \in \mathcal{P}^2$. Thus $\chi(c) = 2$.

3. The invariant bilinear form and the generalized Casimir operator

The goal of Chapter 2 in [2] is to the introduce the generalized Casimir operator Ω which is a generalization of the Casimir element in the finite-dimensional theory. We follow sections 2.1-2.8 in [2].

3.1. **A symmetric bilinear form on the Cartan subalgebra.** This section follows 2.1 in [2] which focuses on building a symmetric bilinear form on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$.

We begin this section with an observation.

Observation. *Let*

$$D = \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

where ε_i are nonzero numbers. Then we obtain an isomorphism $\mathfrak{g}(A) \to \mathfrak{g}(DA)$ by rescaling the Chevalley generators $e_i \mapsto e_i$ and $f_i \mapsto \varepsilon_i f_i$.

Definition 3.1.1. An $n\times n$ matrix $A = (a_{ij})$ is called *symmetrizable* if there exists a non-degenerate diagonal matrix $D = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ and a symmetric matrix $B = (b_{ij})$, such that

$$A = DB. \tag{3.1.1}$$

Let A be a symmetrizable matrix with a fixed decomposition as (3.1.1) and choose $(\mathfrak{h}, \Pi, \Pi^\vee)$ to be a realization of A. Let $\mathfrak{h}''$ be a complementary subspace to $\mathfrak{h}' = \sum \mathbb{C}\alpha_i^\vee$ in $\mathfrak{h}$. Then we can define a symmetric bilinear $\mathbb{C}$-valued form $(\cdot|\cdot)$ on $\mathfrak{h}$ via the following equations:

$$(\alpha_i^\vee|h) = \langle \alpha_i, h\rangle \varepsilon_i \text{ for } h \in \mathfrak{h}, i = 1, \ldots, n; \tag{3.1.2}$$

$$(h'|h'') = 0 \text{ for } h', h'' \in \mathfrak{h}''. \tag{3.1.3}$$

Since $\alpha_1^\vee, \ldots, \alpha_n^\vee$ are linearly independent, we obtain

$$(\alpha_i^\vee|\alpha_j^\vee) = b_{ij}\varepsilon_i\varepsilon_j \ (i, j = 1, \ldots, n). \tag{3.1.4}$$

As $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, there is no ambiguity in the definition of $(\cdot|\cdot)$. Our first result regarding this form is the following lemma.

Lemma 3.1.1.
(1) The kernel of the restriction of the bilinear form $(\cdot|\cdot)$ to $\mathfrak{h}'$ coincides with $\mathfrak{c}$.
(2) The bilinear form $(\cdot|\cdot)$ is non-degenerate[2] on $\mathfrak{h}$.

Proof. From proposition 1.6.1, we know the center of $\mathfrak{g}(A)$ is

$$\mathfrak{c} = \{h \in \mathfrak{h}' | \langle \alpha_i, h\rangle = 0 \text{ for all } \ i = 1, \ldots n\}.$$

The kernel of $(\cdot|\cdot)$ restricted to $\mathfrak{h}'$ is the set of $h' \in \mathfrak{h}'$ such that $(\alpha|h') = 0$ for all $\alpha \in \mathfrak{h}'$. Write $\alpha = a_1\alpha_1^\vee + \cdots a_n\alpha_n^\vee$. Then the kernel is the set of $h' \in \mathfrak{h}'$ such that

$$(a_1\alpha_1^\vee + \cdots a_n\alpha_n^\vee|h') = (a_1\alpha_1^\vee|h') + \cdots (a_n^\vee\alpha_n|h') = a_1\langle\alpha_i, h'\rangle\varepsilon_1 + \cdots a_n\langle\alpha_n, h'\rangle\varepsilon_n = 0$$

Thus the kernel coincides with the center. Now to show $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{h}$ let $\ell \in \mathfrak{h}$ be such that $(\ell|h) = 0$ for all $h \in \mathfrak{h}$. We can reduce to $\ell \in \mathfrak{h}'$. So $\ell = \sum_{i=1}^n c_i\alpha_i^\vee$. Thus we have

$$0 = \left(\sum_{i=1}^n c_i\alpha_i^\vee|h\right) = \left\langle \sum_{i=1}^n c_i\varepsilon_i\alpha_i, h\right\rangle.$$

Therefore, $\sum_{i=1}^n c_i\varepsilon_i\alpha_i = 0$ and hence $c_i = 0$.

□

[2]A bilinear form $(\cdot|\cdot)$ on a vector space V is *non-degenerate* if $(x|y) = 0$ for all $y \in V$ implies $x = 0$.

Since the bilinear form is non-degenerate on $\mathfrak{h}$ we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by

$$\langle \nu(h), h_1 \rangle = (h|h_1),\ h, h_1 \in \mathfrak{h},$$

and the induced bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. Now from (3.1.2) we have that

$$\langle \nu(\alpha_i^\vee), h_1 \rangle = (\alpha_i^\vee, h_1) = \langle \alpha_i, h_1 \rangle \varepsilon_i = \langle \varepsilon_i \alpha_i, h_1 \rangle.$$

So we conclude

$$\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i \ i = 1, \dots n. \tag{3.1.5}$$

Therefore using (3.1.4) we obtain:

$$(\alpha_i|\alpha_j) = b_{ij},\ i, j = 1, \dots, n, \tag{3.1.6}$$

$$(\alpha_i|\alpha_i) = a_{ii}/\varepsilon_i,\ i = 1, \dots n. \tag{3.1.7}$$

3.2. **A symmetric bilinear form on $\mathfrak{g}(A)$.** Following section 2.2 in [2] we will extend the form we defined in the previous section to all of $\mathfrak{g}(A)$.

Theorem 3.2.1. *Let $\mathfrak{g}(A)$ be a Lie algebra associated to a symmetrizable matrix A. Fix a decomposition* (3.1.1) *of A. Then there exists a non-degenerate symmetric bilinear $\mathbb{C}$-valued form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ such that the following hold.*

1. *The form $(\cdot|\cdot)$ is invariant, i.e. $([x, y]|z) = (x|[y, z])$ for all $x, y, z \in \mathfrak{g}(A)$.*
2. *The form restricted to $\mathfrak{h}$ is defined by* (3.1.2) *and* (3.1.3)*, and is non-degenerate.*
3. *The pairing $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.*
4. *The form restricted to $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ is non-degenerate for $\alpha \neq 0$, and hence $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non-degenerately paired by $(\cdot|\cdot)$.*
5. *The bracket $[x, y] = (x|y)\nu^{-1}(\alpha)$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Delta$.*

Proof. Consider the principal $\mathbb{Z}$-gradation

$$\mathfrak{g}(A) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

and set $\mathfrak{g}(N) = \bigoplus_{j=-N}^{N} \mathfrak{g}_j$ for $N \in \mathbb{Z}_{\geq 0}$. We will give a process for extending our form $(\cdot|\cdot)$ to $\mathfrak{g}(A)$. Let $(\cdot|\cdot)$ be the symmetric bilinear form on $\mathfrak{g}(0) = \mathfrak{h}$ to be the form from section 2.1. Now extend it to $\mathfrak{g}(1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by setting

$$(e_i|f_j) = \delta_{ij}\varepsilon_i \ (i, j = 1, \dots, n);\ \ (\mathfrak{g}_0|\mathfrak{g}_{\pm 1}) = 0;\ \ (\mathfrak{g}_{\pm 1}|\mathfrak{g}_{\pm 1}) = 0. \tag{2.2.1}$$

This extended form on $\mathfrak{g}(1)$ satisfies condition (1) as long as both $[x, y]$ and $[y, z]$ are in $\mathfrak{g}(1)$. It is sufficient to check condition (1) when $x = e_i$, $y = f_j$, and $z = \alpha_k^\vee$. We have

$$([e_i, f_j]|\alpha_k^\vee) = (\delta_{ij}\alpha_i^\vee|\alpha_k^\vee) = \delta_{ij} b_{ik} \varepsilon_i \varepsilon_j$$

and

$$(e_i|[f_j, \alpha_k^\vee]) = (e_i|\alpha_j(\alpha_k^\vee) f_j) = \delta_{ij} a_{kj} \varepsilon_i.$$

Since $A = DB$, $a_{kj} = \varepsilon_k b_{kj}$. The matrix B is symmetric, thus $b_{kj} = b_{jk}$. Therefore if $i \neq j$, then $([e_i, f_j]|\alpha_k^\vee) = 0 = (e_i|[f_j, \alpha_k^\vee])$, and if $i = j$, then $([e_i, f_j]|\alpha_k^\vee) = b_{ik}\varepsilon_i\varepsilon_k = (e_i|[f_j, \alpha_k^\vee])$.

Now we can extend $(\cdot|\cdot)$ to a bilinear form on the space $\mathfrak{g}(N)$ via induction on N so that it will satisfy the property that $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ if $|i|, |j| \leq N$ and $i + j \neq 0$. Additionally we will extend the form so that condition (1) will be satisfied so long as $[x, y]$ and $[y, z]$ are in $\mathfrak{g}(N)$. Suppose $(\cdot|\cdot)$ is already defined on $\mathfrak{g}(N-1)$ with the desired properties. Then to extend this form to $\mathfrak{g}(N)$ we only need to define $(x|y)$ for $x \in \mathfrak{g}_{\pm N}$, $y \in \mathfrak{g}_{\mp N}$. Write $y = \sum_i [u_i, v_i]$ where u_i and v_i are homogeneous elements of nonzero degree which lie in $\mathfrak{g}(N-1)$. Then we have $[x, u_i] \in \mathfrak{g}(N-1)$. Set

$$(x|y) = \sum_i ([x, u_i]|v_i).$$

This is well-defined regardless of choice of expressions for x and y. Thus by induction we have constructed a bilinear form on $\mathfrak{g}$ such that (1) and (2) hold. For condition (3), take some $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, and $y \in \mathfrak{g}_\beta$. Then we have

$$\begin{aligned} 0 &= ([h,x]|y) + (x|[h,y]) \\ &= (\alpha(h)x|y) + (x|\beta(h)y) \\ &= \alpha(h)(x|y) + \beta(h)(x|y) \\ &= (x|y)(\alpha(h) + \beta(h)). \end{aligned}$$

Therefore $(x|y) = 0$ if $\alpha + \beta \neq 0$. For condition (5), let $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ where $\alpha \in \Delta$. Then,

$$\begin{aligned} ([x,y] - (x|y)\nu^{-1}(\alpha)|h) &= ([x,y]|h) - ((x|y)\nu^{-1}(\alpha)|h) \\ &= ([x,y|h]) - (x|y)(\nu^{-1}(\alpha)|h) \\ &= (x|[y,h]) - (x|y)\langle \nu(\nu^{-1}(\alpha)), h\rangle \\ &= (x|[y,h]) - (x|y)\langle \alpha, h\rangle \\ &= (x|\alpha(h)y) - (x|y)\alpha(h) \\ &= \alpha(h)(x|y) - \alpha(h)(x|y) \\ &= 0 \end{aligned}$$

for every $h \in \mathfrak{h}$. Notice since $[x,y] - (x|y)\nu^{-1}(\alpha) \in \mathfrak{h}$, by (2) we must have $[x,y] - (x|y)\nu^{-1}(\alpha) = 0$. Finally if condition (4) fails, then by condition (3), the form $(\cdot|\cdot)$ is degenerate. Let $\mathfrak{i} = \ker(\cdot|\cdot)$. Then $\mathfrak{i}$ is a nonzero ideal such that $\mathfrak{i} \cap \mathfrak{h} = 0$ but this contradicts the definition of $\mathfrak{g}(A)$.

□

3.3. **Rewriting a generalized Cartan matrix in terms of** $(\cdot|\cdot)$**.** Following section 2.3 in [2], we rewrite a generalized Cartan matrix in terms of the form built in section 3.2.

Let A be a symmetrizable generalized Cartan matrix and fix a decomposition

$$A = \operatorname{diag}(\varepsilon_1 \ldots, \varepsilon_n)(b_{ij})_{i,j=1}^n \tag{3.3.1}$$

where ε_i are positive rational numbers and (b_{ij}) is a symmetric rational matrix. Such a decomposition always exists[3].

Remark 3.3.1. If A is indecomposable, then the matrix $\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ is uniquely determined by (3.3.1) up to a constant factor.

Now fix a non-degenerate bilinear symmetric form $(\cdot|\cdot)$ associated to the decomposition (3.3.1) which we defined in section 2.1. From (3.1.6) and (3.1.7) we have

$$(\alpha_i|\alpha_i) = 2/\varepsilon_i > 0 \text{ for } \ i = 1, \ldots, n; \tag{3.3.2}$$

$$(\alpha_i|\alpha_j) = a_{ij}/\varepsilon_i \leq 0 \text{ for } i \neq j; \tag{3.3.3}$$

$$\alpha_i^\vee = \frac{2}{(\alpha_i|\alpha_i)\nu^{-1}(\alpha_i)}. \tag{3.3.4}$$

Thus we obtain the usual expression for the generalized Cartan matrix:

$$A = \left(\frac{2(\alpha_i|\alpha_i)}{(\alpha_i|\alpha_i)}\right)_{i,j=1}^n.$$

By Theorem 3.2.1, we can extend this bilinear form $(\cdot|\cdot)$ from $\mathfrak{h}$ to a unique form on $\mathfrak{g}(A)$ satisfying all of the properties of Theorem 3.2.1.

[3] A proof of this is found in [2]

Definition 3.3.1. The form $(\cdot|\cdot)$ on the Kac-Moody algebra $\mathfrak{g}(A)$ provided by Theorem 3.2.1 and satisfying (3.3.2) is called the *standard invariant form*.

3.4. **Two results in $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$.** This section, which follows section 2.4 in [2], gives two results which are useful in computations concerning the generalized Casimir operator.

Let A be a symmetrizable matrix, and let $\mathfrak{g}(A)$ be the associated Lie algebra and $(\cdot|\cdot)$ be a bilinear form on $\mathfrak{g}(A)$ provided by Theorem 3.2.1. If we have some root α, by Theorem 3.2.1.4, we can choose dual bases $\{e_\alpha^{(i)}\}$ and $\{e_{-\alpha^{(i)}}\}$ of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$; i.e. bases such that $(e_\alpha^{(i)}|e_{-\alpha}^{(j)}) = \delta_{ij}$ for $i, j = 1, \dots, \mathrm{mult}\alpha$. For $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ we can write

$$(x|y) = \sum_i (x|e_{-\alpha}^{(i)})(y|e_\alpha^{(i)}). \tag{3.4.1}$$

We now arrive at the first important result of this section.

Lemma 3.4.1. *If $\alpha, \beta \in \Delta$ and $z \in \mathfrak{g}_{\beta-\alpha}$, then we have*

$$\sum_s e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] = \sum_s [e_{-\beta}^{(s)}, z] \otimes e_\beta^{(s)} \tag{3.4.2}$$

in $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$.

Proof. First define the bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$ by $(x \otimes y|x_1 \otimes y_1) = (x|x_1)(y|y_1)$. Choose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\beta}$. Then by Theorem 3.2.1.1 and (3.4.1),

$$\begin{aligned}
\sum_s (e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}]|e \otimes f) &= \sum_s (e_{-\alpha}^{(s)}|e)([z, e_\alpha^{(s)}]|f) \\
&= \sum_s (e_{-\alpha}^{(s)}|e)(e_\alpha^{(s)}|[f, z]) \\
&= (e|[f, z]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_s ([e_{-\beta}^{(s)}, z] \otimes e_\beta^{(s)}|e \otimes f) &= \sum_s ([e_{-\beta}^{(s)}, z]|e)(e_\beta^{(s)}|f) \\
&= \sum_s (e_{-\beta}^{(s)}|[z, e])(e_\beta^{(s)}|f) \\
&= ([z, e]|f) \\
&= (e|[f, z]).
\end{aligned}$$

Therefore, $(\sum_s e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] - \sum_s [e_{-\beta}^{(s)}, z] \otimes e_\beta^{(s)}|e \otimes f) = 0$ for all $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\beta}$. Since this form is non-degenerate, we conclude $\sum_s e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] - \sum_s [e_{-\beta}^{(s)}, z] \otimes e_\beta^{(s)} = 0$. $\square$

We obtain one last result in this section.

Corollary 3.4.1. *In the notation of Lemma 3.4.1 we have*

$$\sum_s [e_{-\alpha}^{(s)}, [z, e_\alpha^{(s)}]] = -\sum_s [[z, e_{-\beta}^{(s)}], e_\beta^{(s)}] \text{ in } \mathfrak{g}(A), \tag{3.4.3}$$

$$\sum_s e_{-\alpha}^{(s)} [z, e_\alpha^{(s)}] = -\sum_s [z, e_{-\beta}^{(s)}] e_\beta^{(s)} \text{ in } U(\mathfrak{g}(A)). \tag{3.4.4}$$

Proof. Apply the following linear maps to (3.4.2).

$$\mathfrak{g}(A) \otimes \mathfrak{g}(A) \to \mathfrak{g}(A)$$
$$x \otimes y \mapsto [x, y]$$

$$\mathfrak{g}(A) \otimes \mathfrak{g}(A) \to U(\mathfrak{g}(A))$$
$$x \otimes y \mapsto xy$$

□

3.5. **The generalized Casimir operator.** In this section we will define the generalized Casimir operator. We follow section 2.5 in [2].

Definition 3.5.1. Let $\mathfrak{g}(A)$ be a Lie algebra associated to a matrix A, $\mathfrak{h}$ the Cartan subalgebra, and $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ the root space decomposition with respect to $\mathfrak{h}$. A $\mathfrak{g}(A)$-module V is called *restricted* if for every $v \in V$, we have $\mathfrak{g}_\alpha(v) = 0$ for all but a finite number of positive roots α.

Let A be a symmetrizable matrix and $(\cdot|\cdot)$ be a bilinear form provided by Theorem 3.2.1. We will now begin the process of defining the generalized Casimir operator. Let V be a restricted $\mathfrak{g}(A)$-module, then we first introduce a linear function $\rho \in \mathfrak{h}^*$ by

$$\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii} \ (i = 1, \dots, n).$$

If $\det A = 0$, this will not define ρ uniquely, so pick an arbitrary one if $\det A = 0$. Now from (3.1.5) and (3.1.7) we have that

$$(\rho|\alpha_i) = \frac{1}{2}(\alpha_i|\alpha_i) \ (i = 1, \dots, n). \tag{3.5.1}$$

For each positive root α choose a basis $\{e_\alpha^{(i)}\}$ of the space $\mathfrak{g}_\alpha$ and let $\{e_{-\alpha}^{(i)}\}$ be a dual basis of $\mathfrak{g}_{-\alpha}$. Define an operator Ω_0 on V by

$$\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}.$$

This is independent of the choice of basis and a well defined sum as V is a restricted $\mathfrak{g}(A)$-module. This leads us to the definition of the generalized Casimir operator.

Definition 3.5.2. Let $u_1, u_2, \dots$ and $u^1, u^2, \dots$ be dual bases of $\mathfrak{h}$. The *generalized Casimir operator* is defined by

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0.$$

To end this section we will record a few formulas which are useful in future computations. First,

$$\sum_i \langle \lambda, u^i \rangle \langle \mu, u_i \rangle = (\lambda|\mu), \tag{3.5.2}$$

which is a result from

$$\lambda = \sum_i \langle \lambda, u^i \rangle \nu(u_i) = \sum_i \langle \lambda, u_i \rangle \nu(u^i). \tag{3.5.3}$$

Take $x \in \mathfrak{g}_\alpha$ then

$$\begin{aligned}[\sum_i u^i u_i, x] &= \sum_i \langle \alpha, u^i \rangle x u_i + \sum_i u^i \langle \alpha, u_i \rangle x \\ &= \sum_i \langle \alpha, u^i \rangle \langle \alpha, u_i \rangle x + x(\sum_i u^i \langle \alpha, u_i \rangle + u_i \langle \alpha, u^i \rangle).\end{aligned}$$

Remark 3.5.1. In the computation above we used the fact that $[ab, c] = [a, c]b + a[b, c]$ for proof of this see Proposition 2.2.1.

Using this calculation we get

$$[\sum_i u^i u_i, x] = x((\alpha|\alpha) + 2\nu^{-1}(\alpha)) \text{ for } x \in \mathfrak{g}_\alpha. \tag{3.5.4}$$

3.6. **The action of the Casimir operator on restricted $\mathfrak{g}$-modules.** Section 2.6 in [2] presents an important result about the generalized Casimir operator, namely when V is a restricted module, Ω will commute with the action of $\mathfrak{g}(A)$ on V.

We begin by considering the root space decomposition of $U(\mathfrak{g}(A))$ with respect to $\mathfrak{h}$:

$$U(\mathfrak{g}(A)) = \bigoplus_{\beta \in Q} U_\beta, \text{ where } U_\beta = \{x \in U(\mathfrak{g}(A)) | [h, x] = \langle \beta, h \rangle x \text{ for all } \ h \in \mathfrak{h}\}.$$

Set $U'_\beta = U(\mathfrak{g}'(A)) \cap U_\beta$, so that $U(\mathfrak{g}'(A)) = \bigoplus_\beta U'_\beta$. We now arrive at the following theorem:

Theorem 3.6.1. *Let $\mathfrak{g}(A)$ be a Lie algebra with a symmetrizable Cartan matrix.*

(1) if V is a restricted $\mathfrak{g}'(A)$-module and $u \in U'_\alpha$ then:

$$[\Omega_0, u] = -u(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)). \tag{3.6.1}$$

(2) If V is a restricted $\mathfrak{g}(A)$-module, then Ω commutes with the action of $\mathfrak{g}(A)$ on V.

Proof. We will first show Part (2) follows directly from part (1). To show part (2), we need to show for every $X \in \mathfrak{g}(A)$ and $v \in V$ that $\Omega.X.v = X.\Omega.v$. Now we can make two simplifications. Since $U(\mathfrak{g}(A)) = \bigoplus_{\beta \in Q} U_\beta$, we have $X = X_{\alpha_1} + \cdots X_{\alpha_\ell}$ with $X_{\alpha_i} \in U_{\alpha_i}$. Since the action is linear, it will suffice to show Ω commutes with X_{α_i}. We also notice for all $h \in \mathfrak{h}$ we have $[\Omega_0, h] = 0$, so we just need to show Ω commutes with $u \in U'_\alpha$. Using (3.5.4), we have

$$\begin{aligned}
[\Omega, u] &= [2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0, u] \\
&= 2[\nu^{-1}(\rho), u] + [\sum_i u^i u_i, u] + [\Omega_0, u] \\
&= 2\langle \alpha, \nu^{-1}(\rho) \rangle u + u((\alpha|\alpha) + 2\nu^{-1}(\alpha)) + [\Omega_0, u] \\
&= u(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)) + [\Omega_0, u].
\end{aligned}$$

If $\Omega.u.v = u.\Omega.v$, we need $[\Omega, u] = 0$ i.e.

$$[\Omega_0, u] = -u(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha))$$

which is part (1). We will now show part (1) is true. We begin by making a simplifying reduction that if (3.6.1) holds for $u_\alpha \in U'_\alpha$ and $u_\beta \in U'_\beta$, then it holds for $u_\alpha u_\beta$. This is true as

$$\begin{aligned}
[\Omega_0, u_\alpha u_\beta] &= [\Omega_0, u_\alpha] u_\beta + u_\alpha [\Omega_0, u_\beta] \\
&= -u_\alpha(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)) u_\beta - u_\alpha u_\beta (2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta)) \\
&= -u_\alpha u_\beta (2)(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)) + 2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta) \\
&= -u_\alpha u_\beta (2(\rho|\alpha + \beta) + (\alpha + \beta|\alpha + \beta) + 2\nu^{-1}(\alpha + \beta)).
\end{aligned}$$

Therefore, since the e_{α_i} and $e_{-\alpha_i}$ generate $\mathfrak{g}'(A)$, any element in U'_{α_i} is a product of the e_{α_i}'s. Thus is suffices to check (3.6.1) for $u = e_{\alpha_i}$. So

$$\begin{aligned}[\Omega_0, e_{\alpha_i}] &= [2\sum_{\alpha\in\Delta_+}\sum_j e_{-\alpha}^{(j)}e_{\alpha}^{(j)}, e_{\alpha_i}] \\ &= 2\sum_{\alpha\in\Delta_+}\sum_j [e_{-\alpha}^{(j)}e_{\alpha}^{(j)}, e_{\alpha_i}] \\ &= 2\sum_{\alpha\in\Delta_+}\sum_j (e_{-\alpha}^{(j)}[e_{\alpha}^{(j)}, e_{\alpha_i}] + [e_{-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha}^{(j)}) \\ &= 2[e_{-\alpha_i}e_{\alpha_i}, e_{\alpha_i}] + 2\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j (e_{-\alpha}^{(j)}[e_{\alpha}^{(j)}, e_{\alpha_i}] + [e_{-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha}^{(j)}).\end{aligned}$$

We will now expand the red section from above.

$$\begin{aligned}2[e_{-\alpha_i}e_{\alpha_i}, e_{\alpha_i}] &= 2(e_{-\alpha_i}[e_{\alpha_i}, e_{\alpha_i}] + [e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i}) \\ &= 2(e_{-\alpha_i}|e_{\alpha_i})\nu^{-1}(\alpha_i)e_{\alpha_i} \\ &= -2\nu^{-1}(\alpha_i)e_{\alpha_i}\end{aligned}$$

where the last two equality's come from Theorem 3.2.1.5 and that $\{e_{-\alpha_i}\}$ and e_{α_i} are dual bases. Now we will focus on the blue section.

$$2\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j (e_{-\alpha}^{(j)}[e_{\alpha}^{(j)}, e_{\alpha_i}] + [e_{-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha}^{(j)}) = 2\left(\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j e_{-\alpha}^{(j)}[e_{\alpha}^{(j)}, e_{\alpha_i}] + \sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j [e_{-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha}^{(j)}\right).$$

Further restricting to the green section we can apply Corollary 3.4.1 with $z = e_{\alpha_i} \in \mathfrak{g}_{\alpha-\beta}$ where $\beta = (\alpha - \alpha_i)$ then,

$$\begin{aligned}\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j e_{-\alpha}^{(j)}[e_{\alpha}^{(j)}, e_{\alpha_i}] &= \sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j [e_{\alpha_i}, e_{\alpha_i-\alpha}^{(j)}]e_{\alpha-\alpha_i}^{(j)} \\ &= -\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j [e_{\alpha_i-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha-\alpha_i}^{(j)}.\end{aligned}$$

Now by Lemma 1.3.1 the collection of $\alpha \in \Delta_+\backslash\{\alpha_i\}$ is the same collection of $\alpha - \alpha_i$ with $\alpha \in \Delta_+\backslash\{\alpha_i\}$. Thus we can re-index our summation as

$$-\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j [e_{\alpha_i-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha-\alpha_i}^{(j)} = -\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_j [e_{-\alpha}^{(j)}, e_{\alpha_i}]e_{\alpha}^{(j)}.$$

Therefore the entire blue section goes to zero. Therefore by (3.5.1),

$$[\Omega_0, e_{\alpha_i}] = -2\nu^{-1}(\alpha_i)e_{\alpha_i} = -2e_{\alpha_i}\nu^{-1}(\alpha_i) - 2(\alpha_i|\alpha_i)e_{\alpha_i} = -e_{\alpha_i}(2(\rho|\alpha_i) + (\alpha_i|\alpha_i) + 2\nu^{-1}(\alpha_i)).$$

□

We end this section with the following Corollary.

Corollary 3.6.1. *If, under the hypotheses of Theorem 3.6.1.2, there exists $v \in V$ such that $e_i(v) = 0$ for all $i = 1, \dots, n$, and $h(v) = \langle\Lambda, h\rangle v$ for some $\Lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$, then*

$$\Omega(v) = (\Lambda + 2\rho|\Lambda)v. \tag{3.6.2}$$

If, furthermore, $U(\mathfrak{g}(A))v = V$, then

$$\Omega|_V = (\Lambda + 2\rho|\Lambda)I_V. \tag{3.6.3}$$

Proof. We get (3.6.2) from the definition of Ω and formula (3.5.2). Formula (3.6.3) follows from (3.6.2) and Theorem 3.6.1. □

3.7. **A Hermitian form for $\mathfrak{g}(A)$.** In this section we define a Hermitian form for $\mathfrak{g}(A)$. This section is based on section 2.7 in [2].

Let A be an $n \times n$ matrix over $\mathbb{R}$. Let $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^{\vee})$ be a realization of the matrix A over $\mathbb{R}$, i.e. $\mathfrak{h}_{\mathbb{R}}$ is a vector space of dimension $2n - \ell$ over $\mathbb{R}$ so that $(\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^{\vee})$ is a realization of A over $\mathbb{C}$.

Consider the complex Lie algebra $\mathfrak{g}(A)$ as a real Lie algebra. Denote by ω_0 by the antilinear automorphism of $\mathfrak{g}(A)$ determined by:

$$\omega_0(e_i) = -f_i,\ \omega_0(f_i) = -e_i\ (i = 1, \dots, n),\ \omega_0(h) = -h \text{ for } h \in \mathfrak{h}_{\mathbb{R}}.$$

Call ω_0 the *antilinear Cartan involution.*

Definition 3.7.1. We define the *compact form* $\kappa(A)$ of $\mathfrak{g}(A)$ as the fixed point set of ω_0.

The compact form is a real Lie algebra whose complexification is $\mathfrak{g}(A)$.

Definition 3.7.2. A *Hermitian form* on a vector space V over the complex numbers is a function $f : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and $a, b \in \mathbb{R}$

(1) $f(au + bv, w) = af(u, w) + bf(v, w)$,
(2) $f(u, v) = \overline{f(v, u)}$.

Let A be a symmetrizable matrix over $\mathbb{R}$ and $(\cdot|\cdot)$ a standard form on $\mathfrak{g}(A)$. We define a Hermitian form on $\mathfrak{g}(A)$ by:

$$(x|y)_0 := -(x|\omega_0(y)).$$

From Theorem 3.2.1 we have the following properties of this Hermitian form:

(1) The restriction of $(\cdot|\cdot)_0$ to $\mathfrak{g}_\alpha$ is non-degenerate for all $\alpha \in \Delta \cup \{0\}$;
(2) The pairing $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta)_0 = 0$ if $\alpha \neq \beta$;
(3) The operators adu and $-ad\omega_0(u)$ for $u \in \mathfrak{g}(A)$ are adjoint to each other, i.e. for all $x, y \in \mathfrak{g}(A)$
$$([u, x]|y)_0 = -(x|[\omega_0(u), y])_0;$$
(4) The restriction of $(\cdot|\cdot)_0$ to $\kappa(A)$ is a non-degenerate invariant $\mathbb{R}$-bilinear form.

3.8. **The Lie algebra $\mathfrak{g}(0)$.** In this section we will consider the most degenerate example, the Lie algebra $\mathfrak{g}(0)$ associated to the $n \times n$ zero matrix. We will follow results found in section 2.8 in [2] In this example $[e_i, e_j] = 0$, $[f_i, f_j] = 0$, and $[e_i, f_j] = \delta_{ij}\alpha_i^{\vee}$, $(i, j = 1 \dots, n)$. Thus,

$$\mathfrak{g}(0) = \mathfrak{h} \oplus \sum_i \mathbb{C}e_i \oplus \sum_i \mathbb{C}f_i.$$

The center of $\mathfrak{g}(0)$ is $\mathfrak{c} = \sum_{i=1}^n \mathbb{C}\alpha_i^{\vee}$ and $\dim \mathfrak{h} = 2n$. Therefore we can choose elements $d_1, \dots, d_n \in \mathfrak{h}$ such that

$$\mathfrak{h} = \mathfrak{c} + \sum_{i=1}^n \mathbb{C}d_i,$$

and

$$[d_i, e_j] = \delta_{ij}e_j,\ [d_i, f_j] = -\delta_{ij}f_j\ (i, j = 1 \dots, n).$$

We define a non-degenerate symmetric invariant bilinear form on the basis of $\mathfrak{g}(0)$ by:

$$(e_i|f_i) = 1,\ (\alpha_i|d_i) = 1, \text{ all the others } =0.$$

In this case we have $\rho = 0$ with Casimir operator

$$\Omega = 2\sum_i \alpha_i^{\vee} d_i + 2\sum_i f_i e_i$$

Set $\mathfrak{c}_1 = \sum \mathbb{C}(\alpha_i^{\vee}) - \alpha_j^{\vee}) \subset \mathfrak{c}$.

Definition 3.8.1. The Lie algebra $\mathfrak{g}'(0)/\mathfrak{c}_1$ is a *Heisenberg Lie algebra of order* n which is a Lie algebra with basis e_i, f_i for $i = 1 \ldots, n$ and z such that $[e_i, f_j] = \delta_{ij} z$ $(i, j = 1, \ldots, n)$, and all other brackets are zero.

References

[1] Nima Amini. Infinite-dimensional lie algebras.
[2] Victor G Kac. *Infinite-dimensional Lie algebras.* Cambridge university press, 1990.
[3] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204. Springer Science & Business Media, 2012.