FINITE GELFAND PAIRS AND CRACKING POINTS OF THE SYMMETRIC GROUPS

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Abstract. Let Γ be a finite group. Consider the wreath product $G_n := \Gamma^n \rtimes S_n$ and the subgroup $K_n := \Delta_n \times S_n \subseteq G_n$, where S_n is the symmetric group and Δ_n is the diagonal subgroup of Γ^n. For certain values of n (which depend on the group Γ), the pair (G_n, K_n) is a Gelfand pair. It is not known for all finite groups which values of n result in Gelfand pairs. Building off the work of Benson–Ratcliff [4], we obtain a result which simplifies the computation of multiplicities of irreducible representations in certain tensor product representations, then apply this result to show that for $\Gamma = S_k,\ k \geq 5$, (G_n, K_n) is a Gelfand pair exactly when $n = 1, 2$.

1. Introduction

Let G be a finite group and K a subgroup of G. Denote by $L(G)$ the set of complex-valued functions on G. This is an algebra under the convolution product

$$f \star g(x) = \frac{1}{|G|} \sum_{y \in G} f(xy^{-1}) g(y).$$

The pair (G, K) is said to be a *Gelfand pair* if the subalgebra $L(K \backslash G / K)$ of K-biinvariant functions in $L(G)$ is commutative.

Gelfand pairs are well-studied in the context of Lie groups, where there is an analogous definition in terms of the algebra of integrable K-biinvariant functions on the group G. (See, for example, [3].) In the Lie group setting, the Gelfand pair structure can be used to construct irreducible unitary representations of G from representations of the subgroup K. Historically, these techniques played a pivotal role in describing the representation theory of semi-simple Lie groups [8]. In the finite group setting, the theory of Gelfand pairs is less-developed, and has found surprising applications outside of group theory including statistics, experimental design, and combinatorics. For example, in [7], Diaconis uses finite Gelfand pairs to determine the rate at which certain Markov chains converge to stationary distributions, and the authors of [2] apply finite Gelfand pairs to the study of association schemes. We refer to [6] for more information on these applications. In [1], finite Gelfand pairs are used to study parking functions, a useful tool in algebraic combinatorics.

This paper concerns a construction introduced by Aker–Can in [1] which produces families of finite Gelfand pairs associated to a fixed finite group. The construction proceeds as follows. Given a finite group Γ, the symmetric group S_n acts

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on Γ^n by permuting the factors, and we form the wreath product $G_n := \Gamma^n \rtimes S_n$ of Γ with S_n. Let Δ_n be the diagonal subgroup of Γ^n. Then $K_n := \Delta_n \times S_n$ is a subgroup of G_n, and for certain values of n, the pair (G_n, K_n) is a Gelfand pair.

In particular, when Γ is abelian, (G_n, K_n) is a Gelfand pair for all values of n [5]. Such Gelfand pairs are relevant in the study of parking functions when Γ is cyclic [1]. For non-abelian Γ, Benson–Ratcliff establish the following two results.

(1) [4, Theorem 1.2] The pair $(G_{|\Gamma|}, K_{|\Gamma|})$ is not a Gelfand pair.
(2) [4, Theorem 1.1] There is some integer $N(\Gamma)$ with $3 \leq N(\Gamma) \leq |\Gamma|$ such that (G_n, K_n) is a Gelfand pair for $n < N(\Gamma)$, and is not a Gelfand pair for $n \geq N(\Gamma)$.

We refer to $N(\Gamma)$ as the *cracking point* of Γ and say that Γ *cracks* at $N(\Gamma)$.

Aker–Can showed through GAP computations that there are groups for which this upper bound is reached and also groups for which this lower bound is reached [1]. For example, the symmetric group S_3 has a cracking point of 6, whereas the group $GL(2, \mathbb{F}_3)$ has a cracking point of 3. On the other hand, Benson–Ratcliff show that in certain infinite families of groups with no bound on order, the cracking point remains constant. For example, they show that for all odd primes p, the dihedral group D_p has a cracking point of 6 [4]. In general, the relationship between the finite group Γ and its cracking point remains rather mysterious. The main result of this paper is to establish the cracking points of the symmetric groups.

Theorem 1.1. *Let $G_n := (S_k)^n \rtimes S_n$ and $K_n := \Delta_n \times S_n$, where $\Delta_n \subset (S_k)^n$ is the diagonal subgroup. For $k \geq 5$, the pair (G_n, K_n) is a Gelfand pair for $n = 1, 2$ and is not a Gelfand pair for $n \geq 3$; that is, in the notation above, $N(S_k) = 3$ for $k \geq 5$. Moreover, $N(S_4) = 4$ and $N(S_3) = 6$.*

We prove Theorem 1.1 using a general observation (Lemma 2.1) which simplifies the computation of cracking points.

Our paper is structured as follows. In Section 2, we discuss a decomposition of the G_n-representation $L(G_n/K_n)$, following the setup in [4]. This gives us the vocabulary necessary to establish our key observation, Lemma 2.1. In Section 3, we apply Lemma 2.1 to prove our main result.

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2. Background

Let Γ be a finite group and $K_n \subset G_n$ as above. By general results about Gelfand pairs, the pair (G_n, K_n) is a Gelfand pair if and only if the left quasi-regular representation $ind_{K_n}^{G_n}(triv_{K_n})$ of G_n in $L(G_n/K_n)$ is multiplicity free [7, Ch. 3F Thm. 9]. Benson–Ratcliff give a decomposition of the space $L(G_n/K_n)$ into irreducible G_n-representations in [4]. In this section, we review some of the details of this decomposition in order to establish our key lemma.

As $G_n = \Gamma^n \rtimes S_n$, it is perhaps unsurprising that the irreducible representations of G_n can be constructed from those of Γ and certain subgroups of S_n. The construction is as follows. Let $\{\pi_\ell\}_{\ell \in S}$ be the irreducible representations of Γ, where S

is an indexing set in bijection with the conjugacy classes of Γ. The irreducible representations of Γ^n are all of the form $\pi := \pi_{\ell_1}\hat{\otimes}\cdots\hat{\otimes}\pi_{\ell_{n-1}}\hat{\otimes}\pi_{\ell_n}$, where $\hat{\otimes}$ denotes the exterior tensor product, and $\ell_i \in S$ (note that we allow for $\ell_i = \ell_k$ for $i \neq k$). The symmetric group S_n acts on any such π by permuting the factors, and we denote by S_π the stabilizer of π in S_n. Denote by ω the intertwining representation of S_π; that is,

$$\omega : S_\pi \to GL(V_1 \otimes \cdots \otimes V_n), \quad \omega(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

where V_i is the vector space of the representation π_{ℓ_i}. Then for any irreducible representation ρ of S_π, the induced representation $R_{\pi,\rho} := ind_{\Gamma^n \rtimes S_\pi}^{G_n}((\pi\circ\omega)\hat{\otimes}\rho)$ is an irreducible G_n-representation, and all irreducible representations of G_n are of this form [4, Sec. 3.2]. Throughout this paper, for a representation π of a group G, we denote by χ_π its character.

Benson–Ratcliff provide a useful method for determining the multiplicity of $R_{\pi,\rho}$ in $L(G_n/K_n)$. In particular, they show that the dimension of the space of K_n-fixed vectors in $R_{\pi,\rho}$ (which is equal to the multiplicity of $R_{\pi,\rho}$ in $L(G_n/K_n)$) is equal to the dimension of the space of K_π-fixed vectors in $(\pi\circ\omega)\hat{\otimes}\rho$, where $K_\pi := \Delta_n \rtimes S_\pi$ [4, Lem. 3.3]. This can be calculated by taking the inner product of the character of $(\pi\circ\omega)\hat{\otimes}\rho$, with the trivial character on K_π:

(1)
$$\frac{1}{|\Delta_n \times S_\pi|}\sum_{(\delta,\sigma)\in K_\pi}\chi_{\pi\circ\omega}(\delta,\sigma)\chi_\rho(\sigma) = \frac{1}{|S_\pi|}\sum_{\sigma\in S_\pi}\left(\frac{1}{|\Delta_n|}\sum_{\delta\in\Delta_n}\chi_{\pi\circ\omega}(\delta,\sigma)\right)\chi_\rho(\sigma).$$

The inner sum on the right hand side of (1) is a class function on S_π. This class function plays an important role in our story so we give it a name:

$$M_\pi(\sigma) := \frac{1}{|\Delta_n|}\sum_{\delta\in\Delta_n}\chi_{\pi\circ\omega}(\delta,\sigma). \tag{2}$$

Equation (1) determines the coefficient of χ_ρ in the decomposition of M_π into irreducible characters of S_π. Therefore, we see that (G_n, K_n) is a Gelfand pair if and only if for each choice of π, the coefficient of χ_ρ in M_π is less than or equal to 1 for all irreducible representations ρ of S_π.

Now we wish to highlight a key observation concerning $M_\pi(e)$, the value of M_π on the identity $e \in S_\pi$. First, note that for $\delta \in \Delta_n$,

$$\chi_{\pi\circ\omega}(\delta, e) = \prod_{i=1}^{n}\chi_{\pi_{\ell_i}}(\delta).$$

Thus, substituting this into (2), we have

$$M_\pi(e) = \frac{1}{|\Delta_n|}\sum_{\delta\in\Delta_n}\Big(\prod_{i=1}^{n}\chi_{\pi_{\ell_i}}(\delta)\Big) = \frac{1}{|\Gamma|}\sum_{C}\Big(\prod_{i=1}^{n}\chi_{\pi_{\ell_i}}(C)\Big)|C|, \tag{3}$$

where C runs over the conjugacy classes of Γ. The second equality follows from the fact that $\Delta_n \simeq \Gamma$ by the obvious isomorphism. Now, the right hand side of (3) is also equal to the inner product of the Γ-representation $\pi_{\ell_1}\otimes\cdots\otimes\pi_{\ell_{n-1}}$ with π_{ℓ_n} whenever $\chi_{\pi_{\ell_n}}$ is real-valued. Hence we have proven the following result.

Lemma 2.1. *Let Γ be a finite group, and let $\{\pi_\ell\}_{\ell\in S}$ be the irreducible representations of Γ. Then for $\pi = \pi_{\ell_1}\hat{\otimes}\cdots\hat{\otimes}\pi_{\ell_n}$, $M_\pi(e)$ is equal to the multiplicity of π_{ℓ_n} in $\pi_{\ell_1}\otimes\cdots\otimes\pi_{\ell_{n-1}}$, if $\chi_{\pi_{\ell_n}}$ is real-valued.*

Remark 2.2. Note that the product in (3) is not changed by reordering the $\chi_{\pi_{\ell_i}}$. Thus, more generally, we have shown that $M_\pi(e)$ is equal to the multiplicity of π_{ℓ_i} in $\pi_{\ell_1} \otimes \cdots \otimes \pi_{\ell_{i-1}} \otimes \pi_{\ell_{i+1}} \otimes \cdots \otimes \pi_{\ell_n}$ whenever $\chi_{\pi_{\ell_i}}$ is real-valued. For our purposes, we will only consider the case when $i = n$, as in Lemma 2.1.

With this, we can simplify the calculations used in computing cracking points. In particular, we can use Lemma 2.1 to make statements about M_π based solely on the dimensions of π_{ℓ_n} and $\pi_{\ell_1} \otimes \cdots \otimes \pi_{\ell_{n-1}}$, which allows us to circumvent the necessity for complete character tables in some cases. An example of such utility is given in the proof of Theorem 1.1 below.

3. Cracking Points of S_k

In this section we use Lemma 2.1 to compute the cracking points of the symmetric groups. We start with the following observation.

Lemma 3.1. *Let S_π be the stabilizer of $\pi = \pi_{\ell_1} \hat{\otimes} \cdots \hat{\otimes} \pi_{\ell_n}$ in S_n. If $M_\pi(e) > \sum_{\rho \in \widehat{S}_\pi} \dim \rho$, then (G_n, K_n) is not a Gelfand pair.*

Proof. Because M_π is a class function on S_π, it can be expressed uniquely as a linear combination of irreducible characters χ_ρ of S_π:

$$M_\pi = \sum_{\rho \in \widehat{S}_\pi} a_\rho \chi_\rho$$

where $\{a_\rho\}_{\rho \in \widehat{S}_\pi}$ are complex coefficients. Now by (1), $a_\rho = \langle M_\pi, \chi_\rho \rangle$ counts the dimension of the space of K_π-fixed vectors in the $\Gamma^n \rtimes S_\pi$-representation $(\pi \circ \omega) \hat{\otimes} \rho$. Thus, we see that $a_\rho \in \mathbb{Z}^{\geq 0}$ for all $\rho \in \widehat{S}_\pi$. Therefore, if

$$M_\pi(e) = \sum_{\rho \in \widehat{S}_\pi} a_\rho \dim \rho > \sum_{\rho \in \widehat{S}_\pi} \dim \rho$$

there must be some $\rho \in \widehat{S}_\pi$ such that $a_\rho > 1$. Hence, (G_n, K_n) is not a Gelfand pair. □

Proof of Theorem 1. Fix $k \geq 5$, and let π_m be the highest dimensional irreducible representation of S_k. We claim that there is an irreducible representation ψ of S_k such that for $\pi = \pi_m \hat{\otimes} \pi_m \hat{\otimes} \psi$, there is some irreducible character χ_ρ of S_π which has a coefficient greater than 1 in the decomposition of M_π. By [4, Lem. 3.3], this implies that the irreducible G_3-representation $R_{\pi,\rho}$ has multiplicity greater than 1 in $L(G_3/K_3)$, and hence (G_3, K_3) is not a Gelfand pair.

To show that such a representation ψ exists, there are two cases to consider. The first case is when $\psi = \pi_m$ and $S_\pi = S_3$. The other case is when $\psi \neq \pi_m$, in which case $S_\pi = S_2 \times S_1 \simeq S_2$. We will prove that (G_3, K_3) is not a Gelfand pair by showing that $M_\pi(e) > 4$ in the first case, $M_\pi(e) > 2$ in the second case, and then applying Lemma 3.1. By Lemma 2.1, this is equivalent to showing that the coefficient of π_m in $\pi_m \otimes \pi_m$ is greater than 4, or that the coefficient of π_i in $\pi_m \otimes \pi_m$ is greater than 2 for some $\pi_i \in \widehat{S}_k$ different than π_m. To do this, we will show that the following inequality holds for all $k \geq 5$:

$$(\dim \pi_m)^2 > 4 \dim \pi_m + \sum_{\pi_i \in \widehat{S}_k,\ \pi_i \neq \pi_m} 2 \dim \pi_i.$$

As π_m is of maximal dimension in $\widehat{S}_k$, it is enough to show that

$$(\dim \pi_m)^2 \geq 4 \dim \pi_m + 2(p(k) - 1) \dim \pi_m$$

where $p(k)$ is the number of partitions of k, which is equal to the number of irreducible representations of S_k. Simplifying, this amounts to establishing the inequality

$$\dim \pi_m \geq 2p(k) + 2. \tag{4}$$

An asymptotic lower bound is given for $\dim \pi_m$ in [11, Thm.1], namely

$$\dim \pi_m \geq e^{-c\sqrt{k}} \sqrt{k!}$$

where $c = \frac{\pi}{\sqrt{6}}$. Similarly, an asymptotic upper bound was found for $p(k)$ in [9]:

$$p(k) \leq \frac{d}{k} e^{\pi \sqrt{\frac{2k}{3}}}$$

where $d = 5.433$. Combining these results with (4), we see that (G_3, K_3) fails to be a Gelfand pair if

$$e^{-c\sqrt{k}} \sqrt{k!} \geq \frac{2d}{k} e^{\pi \sqrt{\frac{2k}{3}}} + 2.$$

This is equivalent to the condition that the ratio

$$r(k) := \frac{k e^{-c\sqrt{k}} \sqrt{k!} - 2k}{2d e^{\pi \sqrt{\frac{2k}{3}}}}$$

is greater than or equal to 1. A direct calculation shows that this holds for $k = 16$. For $k > 16$, note that, after replacing $k!$ with the Gamma function $\Gamma(k+1)$ restricted to the positive real axis, the derivative $\frac{d}{dk} r(k)$ is positive and hence $r(k)$ is increasing. Thus, $r(k) \geq 1$ for $k \geq 16$ and $N(S_k) = 3$ in that case. The rest we calculate through a case-by-case analysis.

The cracking points of S_k for $k = 4, 5, 6$, and 7 can be computed directly using their character tables. Here we show $N(S_5) = 3$ as an example. To do this, we will calculate M_π directly for a specific choice of an irreducible representation π of $S_5 \times S_5 \times S_5$. Consider the following partial character table of S_5, which contains the characters of the highest dimensional and second highest dimensional irreducible representations:

	(1)	(10)	(15)	(20)	(20)	(24)	(30)
	I	2	2, 2	3	3, 2	5	4
π_1	5	1	1	-1	1	0	1
π_2	6	0	-2	0	0	1	0

Now let $\pi = \pi_2 \hat{\otimes} \pi_2 \hat{\otimes} \pi_1$, which has a stabilizer of $S_\pi \simeq S_2$ in S_3. Then calculating $M_\pi(\sigma)$ directly from (3), we see that $M_\pi(\sigma) = 2$ for both σ in $\widehat{S}_\pi$. Hence, $M_\pi = 2\chi_{triv}$, and (G_3, K_3) is not a Gelfand pair for $\Gamma = S_5$. Similarly, for $k = 6$ and $k = 7$ we take π to be the representation consisting of two copies of the highest dimensional irreducible representation of S_k and one copy of the second highest. Again from (3), we calculate that $M_\pi(e) = 4$ in the case of S_6, and $M_\pi(e) = 5$ for S_7. Thus, by Lemma 3.1, (G_3, K_3) is not a Gelfand pair in either case. Finally, for $\Gamma = S_4$, taking π to be four copies of the standard representation of S_4 suffices to show that (G_4, K_4) is not a Gelfand pair. Calculating the decomposition of M_π into irreducible S_π-representations for each choice of π when $n = 3$, one finds no cases of multiplicity, and hence $N(S_4) = 4$.

For S_8 through S_{15}, we show directly that (4) holds. The computations are contained in the table below. The values for $\dim \pi_m$ are given in [10].

k	$\dim \pi_m$	$2p(k)+2$
8	90	46
9	216	62
10	768	86
11	2310	114
12	7700	156
13	21450	204
14	69498	272
15	292864	354

□

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