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UNITARY DUAL OF $SL_2(\mathbb{R})$

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Abstract

In this thesis we study the unitary representations of $SL_2(\mathbb{R})$ through the passage to $(\mathfrak{g}, K)$-modules. We classify the irreducible admissible $(\mathfrak{g}, K)$-modules of $SL_2(\mathbb{R})$ and demonstrate Casselman's embedding theorem by showing that all irreducible admissible $(\mathfrak{g}, K)$-modules embed into the Harish-Chandra module of the principal series representations.

Contents

Introduction

Lie groups are objects which have both analytic and algebraic structure. They are often described as "continuous symmetries" and are ubiquitous in nature. For this reason we have seen the theory of unitary representations of Lie groups play an important role in the study of quantum physics since its introduction in the early twentieth century. These objects are also of interest to mathematicians.

The representation theory of finite groups was classified near the end of the 19-th century. Mathematicians like Frobenius and Schur then began to study the representation theory of non-abelian finite groups. In the 1920s, the mathematician/physicist Weyl extended these ideas to the representation theory of compact Lie groups. This was an achievement with strong relevance to both mathematics and physics. Weyl was one of the first to use algebraic methods to study representations of Lie groups by considering Lie algebras. In the 1930s the study of the representations theory of non-compact Lie groups began to flourish and was of interest to both physicists and mathematicians. In particular a group of interest was $SL_2(\mathbb{R})$. This group was studied because it is isomorphic to $SO^+(1,2)$, a simpler version of the restricted Lorentz group $SO^+(1,3)$. These groups are symmetries of Minkowski spacetime. In 1947 Bargmann identified all of the unitary representations of $SL_2(\mathbb{R})$ in [Bar47] under this motivation.

The Soviet mathematician Israel Gelfand also studied representation theory under a more mathematical motivation. In the 1930s, Gelfand outlined a program to apply the use of symmetry to study a wide range of mathematical problems. The setting for the program is a space of interest X, with an action of a group G. A description of Gelfand's program can be found in [Vog00], but the basic steps are

1. Attach a vector space V to X to translate questions about X to questions about V. Often V is the vector space of functions on X. The action of G on X will lift to an action on V.
2. Decompose the vector space V into a set $\{V_i\}$ of G-invariant subspaces and translate questions about V into questions about V_i.
3. Understand each V_i and answer the questions we have about them.
4. Translate the results for each V_i back to V and to X.

Gelfand classified the unitary representations of $SL_2(\mathbb{R})$ in 1946 in [GN46] and [GN47].

A large encompassing class of Lie groups called real reductive groups are in general non-compact and non-abelian, yet have a representation theory which can be studied in general. This class of groups includes $SL_2(\mathbb{R})$. It was Harish-Chandra who pioneered the ideas for studying this class of groups in [Har51a], [Har51b], [Har54] and [Har57] during the 50s with great inspiration from Weyl's techniques for studying compact groups.

In this thesis we follow the program developed by Harish-Chandra to classify the unitary representations of real reductive groups through the example of $SL_2(\mathbb{R})$. This group is the simplest example of a non-compact, non-abelian Lie group. However, in applying the general techniques to $SL_2(\mathbb{R})$ we concretely see much of the story for the general picture for real reductive groups.

The term "unitary dual" refers to the equivalence classes of irreducible unitary representations of a Lie group. For real reductive groups, unitary representations can always be constructed from irreducible representations via a "continuous" direct sum called the "direct integral". This is why we focus on understanding the irreducible unitary representations.

Representation theory also has a deep connection with automorphic forms and number theory through the Langland's correspondence. This is a more modern motivation for studying the representations of real reductive groups.

One of the major insights of Harish-Chandra is that the study of unitary representations of a real reductive G can be reduced to the study of the so called $(\mathfrak{g}, K)$-modules of G. In this thesis we will discuss this equivalence and apply the theory $(\mathfrak{g}, K)$-modules to the classification of irreducible unitary representations of $SL_2(\mathbb{R})$.

In the quest to find all unitary representations, we will enlarge our target to finding all "admissible representations". The class of all admissible representations is a strictly larger class of Hilbert space representations that contain all unitary representations. It would initially seem that this would be a more challenging task, however, it turns out that this larger class of admissible representations has a more natural parameterisation. In 1973 Robert Langlands gave a parameterisation of the admissible representations of real reductive groups in terms of quotients of principal series representations. The key point is that in general there is no known simple relationship between the parameters given by Langland's classification and whether or not they are "unitarizable". After one attains a parameterisation for the admissible representations, one needs to discern which of these representations admit an inner product to become unitary.

The steps that we will take to classify the irreducible admissible representations of $SL_2(\mathbb{R})$ can be broken down as follows.

Take V, an irreducible admissible representation of $SL_2(\mathbb{R})$,

- Decompose as a representation of the compact subgroup SO_2.
- Induce an action of the complexification of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ on a subspace of V.
- Use this action of $\mathfrak{g}^{\mathbb{C}}$ and K to construct a $(\mathfrak{g}, K)$-module.
- Classify the $(\mathfrak{g}, K)$-modules by using a convenient basis of $\mathfrak{g}^{\mathbb{C}}$.

The goal of finding the unitary dual is at its heart both analytic and algebraic. Since this problem is important in physics, it was first attacked using analytic methods. This is how Bargmann first classified the unitary dual of $SL_2(\mathbb{R})$. However, in studying real reductive groups in general, Harish-Chandra's introduction of $(\mathfrak{g}, K)$-modules showed that this problem can be reduced to becoming far more algebraic. This extends Weyl's approach to studying compact Lie groups.

There have been textbooks written on the representation theory of reductive groups such as [Kna86] and [Wal88] from the 1980s. Since the original motivation

for studying these objects was mathematical physics, most resources approach this topic analytically. The original contribution of this thesis is that we specialize to the example of $SL_2(\mathbb{R})$ and take a more algebraic approach and assume little knowledge of harmonic analysis. Since the nature of the question is analytic, we offload a large portion of the analytic details to Theorem 1.4.5, a corollary of the Peter-Weyl theorem. The statement of this theorem is almost purely algebraic and is easily understandable. Consequently, many results which are usually proven analytically, in this thesis are instead proven using either algebraic and/or basic analytic techniques. This is possible because $SL_2(\mathbb{R})$ is a basic example.

In the Chapter 1 we establish some basic facts about $SL_2(\mathbb{R})$ as a Lie group and define Banach space and unitary representations. We the define admissible representations and shift our focus to classifying these. In Chapter 2 we establish the principal series representations as type of induced representation and prove its admissibility. We then describe the structure of the K-finite vectors of the principal series in detail. In Chapter 3 we introduce the Lie algebra and show how to take a Banach space representation and induce a representation of the Lie algebra on the K-finite vectors of the Banach space. We then define $(\mathfrak{g}, K)$-modules and state two critical theorems of Harish-Chandra which reduce the problem of classifying unitary representations to classifying admissible $(\mathfrak{g}, K)$-modules. In Chapter 4 we discuss an important characteristic of irreducible admissible $(\mathfrak{g}, K)$-modules, their infinitesimal character. We then construct an element of the center, the Casimir element. In the final chapter we classify the $(\mathfrak{g}, K)$-modules of $SL_2(\mathbb{R})$ and combine with our results on the structure of the Harish-Chandra module of the principal series to demonstrate the embedding theorem of Casselman. We conclude with a discussion on unitarity.

The unpublished set of notes [Mil] from Dragan Milicic takes a modern and more algebraic approach to the representation theory of $SL_2(\mathbb{R})$ than classical textbooks such as [Kna86] and [Wal88]. This aligns with one of the objectives of this thesis. For this reason, the majority of the proofs in this thesis are based on proofs found in these notes. The original contribution of this thesis is to provide exposition to these proofs in context with the overall story as well as completing missing steps. We also deviate from the notes in some areas where we forego the more abstract route in favour of a more explicit approach. In particular we deviate from [Mil] in our treatment of Casselman's embedding theorem. Instead of taking the abstract approach in [Mil], we classify the irreducible admissible $(\mathfrak{g}, K)$-modules and use our description of the structure of the principal series to demonstrate Casselman's embedding theorem by studying the irreducible subspaces of the principal series representation.

The notes [Mil] also stand out from standard textbooks like [Kna86] and [Wal88] by taking an alternate definition of the principal series. This definition is not equivalent but serves the same purpose and we provide an explanation of this choice. We also point out where this choice simplifies analytic arguments.

The story which we wish to illustrate in this thesis is: Whilst the question of determining the unitary representations of $SL_2(\mathbb{R})$ is both analytic and algebraic, the problem reduces to being an algebraic one.

Chapter 1

The Lie group $SL_2(\mathbb{R})$

1.1 Basic structure theory

In this section we define $SL_2(\mathbb{R})$ as a Lie group and prove its basic properties. All results in this section were proved independently. The group $SL_2(\mathbb{R})$ is the group of 2×2 matrices with determinant 1 and real entries:

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{R} \right\},$$

with the group operation given by matrix multiplication.

Definition 1.1.1. *A* **Lie group** *G is a group which is also a smooth manifold such that the group action and inversion are smooth maps. More precisely, The multiplication map*

$$G \times G \to G; \quad (g_1, g_2) \mapsto g_1 g_2$$

of manifolds is a smooth map from the product manifold $G \times G$ to G and the inversion map

$$(\cdot)^{-1} : G \to G; \quad g \mapsto g^{-1}$$

is a smooth map from the manifold G to itself.

Definition 1.1.2. *A* **subgroup** *of a Lie group G is a subgroup H which is also a sub-manifold.*

We now proceed by showing that $SL_2(\mathbb{R})$ is a matrix Lie group. First we prove that $SL_2(\mathbb{R})$ is a manifold.

Theorem 1.1.3. *([Lee03], Corollary 5.13) Let $f : X \to Y$ be a smooth map of manifolds X and Y. The* **Regular Level Set Theorem** *states that if $f(x) = y$ for some $x \in X, y \in Y$ and if the differential of f at x $Df : T_xM \to T_yM$ is surjective, then $f^{-1}(y)$ is a sub-manifold of X with dimension* $\dim X - \dim Y$.

Proposition 1.1.4. *The group $SL_2(\mathbb{R})$ is a 3-manifold over $\mathbb{R}$.*

Proof. To show $SL_2(\mathbb{R})$ is a manifold we use Theorem 1.1.3. To apply the theorem we notice that by definition, $SL_2(\mathbb{R})$ is the pre-image of 1 under the determinant map

$$\det : \mathbb{R}^4 \to \mathbb{R}; (x_1, x_2, x_3, x_4) \mapsto (x_1 x_4 - x_2 x_3).$$

The determinant is clearly a smooth map between the manifolds $\mathbb{R}^4$ and $\mathbb{R}$. The differential (Jacobian) of the determinant map at the point (x_1, x_2, x_3, x_4) is

$$D \det(x_1, x_2, x_3, x_4) = (x_4, -x_3, -x_2, x_1).$$

If the determinant is 1, then $(x_4, -x_3, -x_2, x_1)$ has at least one non-zero entry so the linear map $D\det$ is surjective onto $\mathbb{R}$ and thus $SL_2(\mathbb{R})$ is a sub-manifold of dimension three.

QED

Lemma 1.1.5. *The group $SL_2(\mathbb{R})$ is a Lie group*

Proof. Firstly, matrix multiplication is a function from $\mathbb{R}^8$ to $\mathbb{R}^4$ and all entries are polynomials:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \mapsto \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Hence, matrix multiplication is smooth. Secondly, the inversion map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

also has all entries as polynomials, so it is also smooth. *QED*

Now we will establish some basic properties of $SL_2(\mathbb{R})$ as a Lie group.

Proposition 1.1.6. *The group $SL_2(\mathbb{R})$ is not compact.*

Proof. First note that the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}^+$$

is in $SL_2(\mathbb{R})$. In particular a is unbounded, so we can find a sequence of elements of $SL_2(\mathbb{R})$ which are unbounded in the euclidean topology. Therefore $SL_2(\mathbb{R})$ is unbounded and not compact. *QED*

Proposition 1.1.7. *The group $SL_2(\mathbb{R})$ is path-connected.*

Proof. The first part of this proof will be to show that there exists a path from any element of $SL_2(\mathbb{R})$ to an element of the subgroup

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \leq \theta < 2\pi \right\}. \tag{1.1.1}$$

Consider the arbitrary element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Since the determinant of this matrix must be one, we can split into two cases, either a and d are both non-zero or b and c are both non-zero. For each of these two cases we will define a different but similar path to $SO_2(\mathbb{R})$.

1. If a and d are both non-zero, define the path $\gamma : [0,1] \to SL_2(\mathbb{R})$ with component functions $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and rule

$$\gamma(t) = \begin{pmatrix} t\frac{a}{|a|} + (1-t)a & (1-t)b \\ (1-t)c & f(t) \end{pmatrix}$$

where $f(t) = \frac{1-\gamma_2(t)\gamma_3(t)}{\gamma_1(t)}$. This path is constructed to linearly interpolate the first three entries to an element of $SO_2(\mathbb{R})$ and the last entry ensures that the determinant remains 1 and the path is well defined on the manifold $SL_2(\mathbb{R})$. By assumption a is non zero so γ_1 is never zero and f is well defined. Also, since f is a composition of continuous functions, f is continuous. Thus by the construction of f, $\gamma(t)$ has determinant 1 and γ is well defined and continuous. Also note that

$$\gamma(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \gamma(1) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SO_2(\mathbb{R}).$$

2. Now, if we assume b and c are non-zero we define similarly $\hat{\gamma} : [0,1] \to SL_2(\mathbb{R})$ with component functions $\hat{\gamma} = (\hat{\gamma_1}, \hat{\gamma_2}, \hat{\gamma_3}, \hat{\gamma_4})$ and rule

$$t \mapsto \begin{pmatrix} (1-t)a & t\frac{b}{|b|} + (1-t)b \\ g(t) & (1-t)d \end{pmatrix}$$

where $g(t) = \frac{\hat{\gamma_1}(t)\hat{\gamma_4}(t)-1}{\hat{\gamma_2}(t)}$. Since $\hat{\gamma_2}$ is never zero, g is well defined. Also, since g is a composition of continuous functions, it is continuous. Thus by the construction of g, $\gamma(t)$ has determinant one and γ is well defined. Note that we have

$$\gamma(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \gamma(1) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO_2(\mathbb{R})$$

We have just proven that any point in $SL_2(\mathbb{R})$ is path connected to a point in $SO_2(\mathbb{R})$. Thus in order to prove that $SL_2(\mathbb{R})$ is path connected, it now suffices to prove that that the subgroup $SO_2(\mathbb{R})$ is path connected.

By definition

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \leq \theta < 2\pi \right\},$$

so it is clear $SO_2(\mathbb{R})$ is path connected because we can let the parameter θ vary continuously to construct a continuous path between any two points. *QED*

1.2 Iwasawa decomposition

Our study of the representation theory of $SL_2(\mathbb{R})$ will heavily leverage the representation theory of simpler subgroups of $SL_2(\mathbb{R})$. In this section we state and prove the Iwasawa decomposition which lets us split $SL_2(\mathbb{R})$ into a product of a "compact" matrix, a diagonal matrix and a unipotent matrix. This proof is based on [Mil] and Lemma 1.2.2 was proven independently. Let

$$K = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \leq \theta < 2\pi \right\},$$

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

For this next section $G = SL_2(\mathbb{R})$.

Lemma 1.2.1. (Iwasawa Decomposition). *The map*

$$K \times A \times N \to G; \quad (k, a, n) \mapsto kan \tag{1.2.1}$$

is a diffeomorphism.

Proof. Consider the following matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} t & y \\ 0 & t^{-1} \end{pmatrix}. \tag{1.2.2}$$

where $t > 0, 0 \leq \theta < 2\pi$ and $y \in \mathbb{R}$. The determinant of the right hand side is one, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Also by equating entries,

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} t\cos(\theta) \\ -t\sin(\theta) \end{pmatrix}. \tag{1.2.3}$$

We would like to show that any matrix in $SL_2(\mathbb{R})$ can be written uniquely in the form of (1.2.2). Now we will split into two cases, either $a = 0$ or $a \neq 0$. Firstly assume $a = 0$. By (1.2.3), we have $t\cos(\theta) = 0$. Since $t > 0$, we get $\cos(\theta) = 0$. This implies that $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Thus, $\sin(\theta) = \pm 1$. By multiplying out (1.2.2) on the right we get

$$\begin{pmatrix} 0 & \pm t^{-1} \\ \pm t & \pm y \end{pmatrix}.$$

This is clearly an arbitrary element of $SL_2(\mathbb{R})$ when $a = 0$. Now assume that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an arbitrary element of $SL_2(\mathbb{R})$ when $a \neq 0$. Pick $t > 0$ as

$$t = \sqrt{a^2 + c^2}. \tag{1.2.4}$$

Pick $\theta_0 = \arctan(\left|\frac{c}{a}\right|)$. We now get

$$\begin{pmatrix} |a| \\ |c| \end{pmatrix} = \begin{pmatrix} t\cos(\theta_0) \\ t\sin(\theta_0) \end{pmatrix}. \tag{1.2.5}$$

Thus for all $a, c \in \mathbb{R}$, we can pick θ in the unique quadrant such that equation (1.2.3) holds. These choices of $t > 0$ and $0 \leq \theta < 2\pi$ are the unique and the only way (1.2.3) holds. Equating the entry of b in equation (1.2.2) gives us

$$b = y\cos(\theta) + t^{-1}\sin(\theta). \tag{1.2.6}$$

Now, $\cos(\theta) \neq 0$, because otherwise $a = 0$. If θ and t have already been picked it is clear that we can pick a unique $y \in \mathbb{R}$ such that for all $b \in \mathbb{R}$, (1.2.6) holds. Since a, b, c, d are entries of a matrix in $SL_2(\mathbb{R})$ and we have freely picked a, b, c then d is determined. We have just shown that we can choose $t > 0$ and θ to make equation (1.2.2) hold. Therefore every matrix in $SL_2(\mathbb{R})$ can be written uniquely in the form of (1.2.2).

We can also rewrite (1.2.2) as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & y/t \\ 0 & 1 \end{pmatrix}$$

and clearly

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K, \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in A, \quad \begin{pmatrix} 1 & y/t \\ 0 & 1 \end{pmatrix} \in N.$$

Thus we have proved that the map in (1.2.1) is a bijection.

Now we need to prove is that the map and its inverse are smooth. Equation (1.2.4) implies that t depends on a and c smoothly as a and c are never both zero on $SL_2(\mathbb{R})$. Equation (1.2.3) implies that $\cos(\theta)$ and $\sin(\theta)$ (the entries of k) depend smoothly on a and c. Consider

$$\begin{pmatrix} y \\ t^{-1} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \tag{1.2.7}$$

Finally equation 1.2.7 gives us that y depends smoothly on a, b, c, d via matrix multiplication. This means that the map $g \mapsto (k, a, n)$ is smooth.

Obviously the inverse of this map is smooth because matrix multiplication is smooth, therefore the map is in fact a diffeomorphism.

QED

This decomposition allows us to write any element $g \in SL_2(\mathbb{R})$ uniquely as a product kan where $k \in K, a \in A, n \in N$.

Lemma 1.2.2. (Reverse Iwasawa decomposition). *Every $g \in SL_2(\mathbb{R})$ also has a unique decomposition where g is of the form nak for some $n \in N, a \in A, k \in K$.*

Proof. For every $g \in G$, by the Iwasawa decomposition we know g^{-1} is uniquely of the form $g^{-1} = kan$ for some $k \in K, a \in A, n \in N$. Therefore $g = (g^{-1})^{-1} = n^{-1}a^{-1}k^{-1}$. Thus we also get that every g is of the form $g = n'a'k'$ for some $n', a', k' \in N, A, K$ respectively. *QED*

1.3 Unitary Representations

In this section we define every type of representations that will be studied in this thesis and define the unitary dual. Since Lie groups have additional analytic manifold structure, we will focus on those representations which are compatible with this analytic structure.

Definition 1.3.1. *A* **Banach space representation** *of a Lie group G is a pair (π, V) where V is a complex Banach space and π is a group homomorphism into the invertible endomorphisms of V, $\pi : G \to GL(V)$ such that the function $g \mapsto \pi(g)\xi$ is continuous for all $\xi \in V$.*

Notice that in the above definition we give the additional constraint that our representations acknowledge the analytic structure of the manifold. In this paper, all the group representations we will be dealing with will be Banach space representations. We also remark that if the vector space is finite dimensional, then $GL(V)$ has a manifold structure and the above definition is equivalent to saying $\pi : G \to GL(V)$ is a smooth Lie group homomorphism. This is the standard definition of a Lie group representation.

Definition 1.3.2. *A* **subrepresentation** *of a G-representation (π, V) is a G-representation (π', W) where W is a Banach subspace of V.*

Definition 1.3.3. *A* **unitary representation** *of G is a pair (π, V) where V is a complex Hilbert space and $\pi : G \to U(V)$ is a group homomorphism into the invertible unitary endomorphisms of V, such that the function $g \mapsto \pi(g)\xi$ is continuous for all $\xi \in V$. An endomorphism $U \in GL(V)$ is unitary if $UU^* = U^*U = I$ where U^* denotes the adjoint.*

These unitary representations are the ones which we aim to classify.

Definition 1.3.4. *A representation (π, V) of a Lie group G is* **irreducible** *if the only G-stable subspaces of V are $\{0\}$ and V. More precisely, an irreducible representation is a representation such that there is no proper, nontrivial subspace $W \subset V$ such that $\pi(g)w \in W$ for all $w \in W$.*

Definition 1.3.5. *Two representations (π, V) and (π', V') of G are* **equivalent** *if there is a bounded linear $E : V \to V'$ with a bounded inverse such that*

$$\pi'(g)E = E\pi(g)$$

for all $g \in G$.

If π and π' are unitary representations, then they are **unitarily equivalent** *if they are equivalent via an operator E that is unitary.*

The unitary equivalence is what defines the equivalence classes in the unitary dual. We can now confidently say that we understand what the "Unitary dual of $SL_2(\mathbb{R})$" actually means.

Definition 1.3.6. *Let $H \subset G$ be Lie subgroups and (π, V) a Banach space representation of H. The* **induced representation** $\mathrm{Ind}_H^G(\pi)$ *is the vector space $C(G)$ of all*

continuous functions from G to X such that $f(hg) = \pi(h)f(g)$ for all $h \in H, g \in G$. Explicitly,

$$\mathrm{Ind}_H^G(\pi) := \{f : G \to V : f \in C(G), f(hg) = \pi(h)f(g) \text{ for all } h \in H, g \in G\}. \tag{1.3.1}$$

Induced representations allow us to construct representations of G from simpler subgroups. We will later see that inside the structure of a particular induced representation, lies all other irreducible unitary representations

Lemma 1.3.7. *The induced representation $\mathrm{Ind}_H^G(\pi)$ has a well-defined right action of G as follows*

$$R(g) : \mathrm{Ind}_H^G(\pi) \to \mathrm{Ind}_H^G(\pi); \quad (R(g)f)(x) = f(xg)$$

for all $g, x \in G$.

Proof. Firstly we need to check $R(g)$ actually maps into $\mathrm{Ind}_H^G(\pi)$ for all $g \in G$. Because G is a Lie group, the map $x \mapsto xg$ is continuous. Thus $R(g)f$ is a composition of continuous functions so $R(g)f$ is continuous. Also,

$$(R(g')f)(hg) = f(hgg') = \sigma(h)f(gg') = \sigma(h)(R(g')f)(g)$$

so $R(g')f \in \mathrm{Ind}_H^G(\pi)$. Now to check associativity of the action,

$$(R(h)(R(g)f))(x) = (R(g)f)(xh) = f(xhg) = (R(hg)f)(x).$$

QED

The existence of this action shows that the induced representation is a representation of G in the classical way, however, above we claimed that all of the representations that we will deal with will be Banach space representations. So far we haven't given the induced representation a Banach space structure, We will not yet prove that the induced representation is a Banach space representation. Later on when we invoke this definition we will define the appropriate norm and prove that it is a Banach space representation with respect to this norm.

1.4 Admissible representations

As mentioned in the introduction, to find all unitary representations, we enlarge our target to finding all "admissible representations". The class of all admissible representations is a strictly larger class of Hilbert space representations that contain all unitary representations, but is more manageable and has a more natural parameterisation.

For the time being the reader should reframe the goal in their mind to classifying the admissible representations of $SL_2(\mathbb{R})$. In this section we will define what an admissible representation is. But first we state and discuss some results about the representation theory of compact groups.

The proofs of Lemmas 1.4.10, 1.4.12 and 1.4.13 are based on the proof of Theorem 15.4 from [Mar11].

Firstly we recall the definition of the algebraic direct sum.

Definition 1.4.1. *The* **algebraic direct sum** *of a set of vector spaces $\mathcal{A}$ is*

$$\bigoplus_{\alpha\in\mathcal{A}} V_\alpha := \{(v_\alpha) : v_\alpha \in V_\alpha \text{ and all but finitely many } v_\alpha \text{ are zero}\}.$$

Now we state the Hilbert space direct sum.

Definition 1.4.2. *([Sep07], p57) The* **Hilbert space direct sum** *of a set of Hilbert spaces $\mathcal{A}$ is*

$$\widehat{\bigoplus_{\alpha\in\mathcal{A}}} V_\alpha := \{(v_\alpha) : v_\alpha \in V_\alpha \text{ and } \sum_{\alpha\in\mathcal{A}} \|v_\alpha\|_\alpha^2 < \infty\}.$$

Traditionally for vector spaces we can only add finitely many elements together but when we also have a Hilbert space structure we can make sense of the infinite convergent sum of elements. The Hilbert space direct sum can be thought of extending the usual direct sum by allowing infinite convergent sums of elements. This definition can be thought of a good example of where we see the worlds of algebra and analysis meet.

Lemma 1.4.3. *([Wal88], Lemma 1.4.8) Let (π, V) be a Hilbert space representation of a compact Lie group K. There exists an inner product $(\cdot,\cdot)$ on V that gives the original topology on V and is such that relative to $(\cdot,\cdot)$, π is unitary.*

Theorem 1.4.4. *([Sep07], Lemma 3.13.) For a compact group K, every irreducible Hilbert space representation is finite dimensional.*

The statement of this theorem in [Sep07] has an extra unitary assumption. However, we can drop this unitarity assumption in the by Lemma 1.4.3.

Theorem 1.4.5. *([Sep07], Corollary 3.15) Let (π, V) be a representation of a compact Lie group K on a Hilbert space, There exists finite dimensional irreducible sub-representations of K, $V_\alpha \subseteq V$ so that*

$$V = \widehat{\bigoplus_{\alpha}} V_\alpha.$$

Here each irreducible representation can appear multiple times (or infinitely often) in the direct sum decomposition.

Again, we drop the unitary assumption in the original statement of this theorem by Lemma 1.4.3.

Definition 1.4.6. *We define the* **multiplicity** *of an irreducible sub-representation V_α of V to be the number summands of V which are isomorphic to V_α as representations of K in the direct sum decomposition in 1.4.5. We denote this as*

$$\text{mult}(V_\alpha : V).$$

Definition 1.4.7. *([Cas13], p.6) Let (π, V) be a Banach space representation of a Lie group G. Let $V^{(K)}$ denote the subspace of vectors that are contained in a K-stable subspace of finite dimension. These are called the* **K finite vectors** *of the representation.*

Our aim is to take the analytic problem of studying unitary representations and apply algebraic techniques. This definition is important because this subspace is the subspace which can be studied algebraically.

Now we are ready to define admissibility.

Definition 1.4.8. *([Cas13], p.7) A Banach space representation (π, V) of a Lie group G is called* **admissible** *if the space of K-finite vectors, $V^{(K)}$ is a direct sum of irreducible representations of K, each occurring with finite multiplicity; i.e,*

$$V^{(K)} = \bigoplus_i V_i$$

and $\text{mult}(V_i : V^{(K)}) < \infty$.

When we have a Hilbert space representation, the condition that the space of K-finite vectors is a direct sum of irreducible representations of K is automatically satisified. The key part of this definition is the finite multiplicity condition. In this thesis the Banach space representations that we consider (namely the principal series representations) are actually not too far away from being a Hilbert space. This is also our first time seeing the compact subgroup K play a role, and we will see it prominently from here on. The reason for this is that the representation theory of compact groups is well-behaved, as demonstrated from the theorems stated in this section.

We now state without proof an important theorem of Harish-Chandra.

Theorem 1.4.9. *([Wal88], Theorem 3.4.10) Let (π, V) be an irreducible unitary representation of a Lie group G. Then (π, V) is admissible*

Now since we claim that the representation theory of the compact subgroup K is important, now is a good time to study the representation theory of the compact subgroup SO_2 of $SL_2(\mathbb{R})$. It is clear that SO_2 is isomorphic to

$$S^1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$$

as Lie groups since the map

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\theta}$$

is a Lie group isomorphism. So we are reduced to study the representation theory of S^1.

Firstly, we are going to characterise the irreducible representations of S^1 and to do so we first prove a couple lemmas.

Lemma 1.4.10. *Any continuous homomorphism $t : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ is of the form $t(x) = nx$ for some $n \in \mathbb{Z}$.*

Proof. Let $n = t(1)$. For all $q \in \mathbb{N}$,

$$qt\left(\frac{1}{q}\right) = t(1) = n$$

and thus

$$t\left(\frac{1}{q}\right) = \frac{n}{q}.$$

Clearly for all $p \in \mathbb{N}$,

$$t\left(\frac{p}{q}\right) = \frac{np}{q},$$

therefore for all $x \in \mathbb{Q}/2\pi\mathbb{Z}$,

$$t(x) = nx$$

and since $\mathbb{Q}/2\pi\mathbb{Z}$ is dense in $\mathbb{R}/2\pi\mathbb{Z}$ and t is continuous we have that $t(x) = nx$ for some $n \in \mathbb{R}/2\pi\mathbb{Z}$. Now $t(2\pi) = 2n\pi = 1$ as elements in the quotient, therefore $n \in \mathbb{Z}$. *QED*

Lemma 1.4.11. *Irreducible representations of S^1 are one dimensional*

Proof. Irreducible representations of compact groups are finite dimensional by 1.4.4. This means that if we have W, an irreducible representation of S^1 then S^1 acts on W by complex matrices. Since S^1 is commutative, we can pick a basis $\{e_i\}_{i=1}^n$ to simultaneously diagonalize these matrices and now S^1 acts by diagonal matrices. However, diagonal matrices will always fix the subspace $\text{span}\{e_1\}$. Therefore since W is irreducible, the whole space must be $\text{span}\{e_1\}$ so it must actually be one-dimensional. *QED*

Lemma 1.4.12. *Any continuous homomorphism $\rho : S^1 \to \mathbb{C}^\times$ is of the form*

$$\rho(e^{i\theta}) = e^{in\theta}$$

for some $n \in \mathbb{Z}$.

Proof. Firstly for all $\theta \neq 0$ and for all $m \in \mathbb{N}$,

$$1 = \rho(e^{i2m\pi}) = \rho((e^{i\theta})^{\frac{2\pi m}{\theta}}) = \rho(e^{i\theta})^{\frac{2\pi m}{\theta}}.$$

Thus,

$$\left|\rho(e^{i\theta})\right|^{2\pi m/\theta} = 1. \tag{1.4.1}$$

In particular m can be arbitrarily large so the only way 1.4.1 can hold is if

$$\left|\rho(e^{i\theta})\right| = 1.$$

This implies that ρ maps into S^1. Hence for all $e^{i\theta} \in S^1$,

$$\rho(e^{i\theta}) = e^{it(\theta)}$$

for some map $t : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$.

Now ρ is a homomorphism so

$$\rho(e^{i(\theta+\phi)}) = \rho(e^{i\theta})\rho(e^{i\phi})$$

so $t(\theta+\phi) = t(\theta)+t(\phi)$. Because the complex Arg function is continuous and from the above equation, t is a continuous homomorphism. Therefore Lemma 1.4.10 implies ρ is of the form

$$\rho(e^{i\theta}) = e^{in\theta}$$

for some $n \in \mathbb{Z}$. *QED*

These last two lemmas give us a classification for the irreducible representations of S^1.

Theorem 1.4.13. *The irreducible representations of S^1 are all one dimensional and of the form (π_n, V_n) for some $n \in \mathbb{Z}$, with*

$$\pi_n(k(\theta))v = e^{in\theta}v \tag{1.4.2}$$

for all $k(\theta) \in S^1$ and $v \in V_n$.

Chapter 2

Principal series representations

For this chapter $G = SL_2(\mathbb{R})$ and K, A, N are as defined in 1.2.

To understand the admissible representations we will need to study representations of easier subgroups of $SL_2(\mathbb{R})$. We are already seeing the compact subgroup K play an important role, but another important subgroup of $SL_2(\mathbb{R})$ will be the "parabolic subgroup" P. It will turn out that all admissible representations can in fact be realised as quotients of induced representations of P. In this next section we will define P and construct these induced representations of P.

Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^*, b \in \mathbb{R} \right\}. \tag{2.0.1}$$

Clearly P is a subgroup of G. Since $K \cap P = \{\pm I\}$, we also have the corresponding Iwasawa decomposition of P: MAN, where $M = \{\pm I\}$.

We are now going to build a representation of G from a one dimensional representation of P, and show that such "parabolically induced" representations are always admissible.

2.1 The Banach space structure of the induced representation

All proofs in this section were done independently. Let (σ, U) be a one-dimensional representation of P.

Lemma 2.1.1. *Each function $f \in \mathrm{Ind}_P^G(\sigma)$ is completely determined by its restriction to K.*

Proof. By the reverse Iwasawa decomposition (Lemma 1.2.2) we have for all $g \in G$, $f(g) = f(nak) = \sigma(na)f(k)$ so f is uniquely determined by its restriction to K. *QED*

Lemma 2.1.2. *If $h : K \to U$ is a continuous function such that $h(mk) = \sigma(m)h(k)$ for all $m \in M$, then $h = \hat{h}|_K$ for some $\hat{h} \in \mathrm{Ind}_P^G(\sigma)$.*

Proof. Define

$$\hat{h} : G \to U; \quad \hat{h}(g) = \sigma(na)h(k),$$

where $g = nak$ for some $n \in N, a \in A, k \in K$. Firstly, by construction, $\hat{h}$ restricts to h. Secondly, by construction $\hat{h}$ is defined in terms of $\sigma(na)$ acting on $h(k)$, and

because h is continuous and the action of σ is continuous, $\hat{h}$ is also continuous. Thirdly, we need to show that

$$\hat{h}(pg) = \sigma(p)\hat{h}(g)$$

for all $p \in P, g \in G$. Note that M is a scalar multiple of the identity so elements of M commute with all other elements of G. Now let $p \in P$ and $g \in G$, then from the reverse Iwasawa decomposition of p and g we have

$$\hat{h}(pg) = \hat{h}(n_1 a_1 m_1 n_2 a_2 k_2).$$

Because P is a subgroup, $n_1 a_1 m_1 n_2 a_2 \in P$, so $n_1 a_1 m_1 n_2 a_2 = nam$ for some $n \in N, a \in A, m \in M$. Therefore,

$$\begin{aligned}
\hat{h}(pg) &= \hat{h}(namk_2) \\
&= \sigma(na)h(mk_2) \\
&= \sigma(na)\sigma(m)h(k_2) \\
&= \sigma(nam)h(k_2) \\
&= \sigma(n_1 a_1 m_1 n_2 a_2)h(k_2) \\
&= \sigma(n_1 a_1 m_1)\sigma(n_2 a_2)h(k_2) \\
&= \sigma(p)\sigma(n_2 a_2)h(k_2) \\
&= \sigma(p)\hat{h}(g).
\end{aligned}$$

QED

When we first defined induced representations, we promised to define a Banach space structure when needed. We will now follow through on this promise for the induced representation $\mathrm{Ind}_P^G(\sigma)$.

Lemma 2.1.3. *The induced representation* $\mathrm{Ind}_P^G(\sigma)$ *is a Banach space with norm* $\|\cdot\| : \mathrm{Ind}_P^G(\sigma) \to \mathbb{R}_{\geq 0}$ *given by*

$$\|f\| = \sup_{k \in K} \|f(k)\|_U,$$

where $\|\cdot\|_U : U \to \mathbb{R}_{\geq 0}$ *is the norm on* U*.*

Proof. Because K is compact, the norm is well-defined. What is left is to show that the space is complete. Let f_n be a Cauchy sequence in $\mathrm{Ind}_P^G(\sigma)$ and $C(K)$ be the vector space of continuous functions from K to U. Because $C(K)$ is complete with the norm $\|\cdot\|$ restricted to K, we know $f_n|_K$ is convergent to some function $q \in C(K)$. Now observe that for all $m \in M$ and $k \in K$

$$f_n|_K(mk) = \sigma(m) f_n|_K(k)$$

so

$$\begin{aligned}\lim_{n\to\infty} f_n|_K(mk) &= \lim_{n\to\infty} \sigma(m) f_n|_K(k) \\ q(mk) &= \sigma(m) \lim_{n\to\infty} f_n|_K(k) \\ &= \sigma(m) q(k),\end{aligned}$$

because the action of $\sigma(m)$ is continuous. Thus, Lemma 2.1.2 applies and there exists some $\hat{q} \in \mathrm{Ind}_P^G(\sigma)$ for which $\hat{q}|_K = q$. Clearly $f_n \to \hat{q}$ in the norm we have defined and thus $\mathrm{Ind}_P^G(\sigma)$ is complete and a Banach space. *QED*

Lemma 2.1.4. *The induced representation* $\mathrm{Ind}_P^G(\sigma)$ *is a Banach space representation of* G*.*

Proof. We want to show that if we have a sequence in G such that $x_n \to x$, then $R(x_n)f \to R(x)f$ for all $f \in \mathrm{Ind}_P^G(\sigma)$.

So, let (x_n) be a sequence in G with $x_n \to x$. We want to show the following goes to zero

$$\|R(x_n)f - R(x)f\| = \sup_{k\in K} \|f(kx_n) - f(kx)\|_U.$$

Observe that the above is equivalent to showing that s_n converges uniformly to s where $s_n(k) := f(kx_n)$ and $s(k) := f(kx)$. Firstly we will show that $kx_n \to kx$ uniformly as functions of k. Consider

$$\|kx_n - kx\|_G = \|k(x_n - x)\|_G$$

where $\|\cdot\|_G$ is the norm on G coming from the euclidean topology of $\mathbb{R}^4$. We would like to bound the above in terms of the norm of k and the norm of $(x_n - x)$. This can be done due to the following: Because k and $(x_n - x)$ are two matrices, they have an operator norm $\|\cdot\|_{OP}$. All norms on a finite dimensional vector space are equivalent, so for all $g \in G$ we get the following

$$c\|g\|_{OP} \leq \|g\|_G \leq C\|g\|_{OP}$$

for some positive constants $c, C \in \mathbb{R}_{>0}$.

Now we know that for operator norms $\|AB\|_{OP} \leq \|A\|_{OP}\|B\|_{OP}$, so we deduce

$$\begin{aligned}\|k(x_n - x)\|_G &\leq C\|k(x_n - x)\|_{OP} \\ &\leq C\|k\|_{OP}\|x_n - x\|_{OP} \\ &\leq \frac{C}{c^2}\|k\|_G\|x_n - x\|_G.\end{aligned}$$

but

$$\|k\|_G = \left\| \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\|_G = \sqrt{2}.$$

Therefore we have shown

$$\|kx_n - kx\|_G \leq 2\frac{C}{c^2}\|x_n - x\|_G$$

for all $k \in K$. Because the constant $2C/c^2$ does not depend on k, we get that $kx_n \to kx$ uniformly as functions of k. Because f is continuous, $s_n \to s$ uniformly as functions of k. Therefore

$$\sup_{k\in K} \|f(kx_n) - f(kx)\|_U \to 0.$$

QED

2.2 Admissibility of principal series

Our aim is to show that this new induced Banach space representation $\mathrm{Ind}_P^G(\sigma)$ is admissible. Even though this is a Banach space representation, it isn't too far away from being a Hilbert space representation and admits an inner product. Propositions 2.2.1, 2.2.2 were proved independently, while the rest of the results are based on [Mil] with the help of Anna Romanov.

Proposition 2.2.1. *The formula* (2.2.1) *defines an inner product on* $\mathrm{Ind}_P^G(\sigma)$.

$$\langle f, h\rangle = \int_K f(k)\overline{h(k)}dk. \tag{2.2.1}$$

Proof. Since K is compact, it has finite measure M and by the extreme value theorem, for every continuous function f on K, $\sup_{k\in K}|f(k)| < \infty$. Therefore for all $f \in \mathrm{Ind}_P^G(\sigma)$ we have

$$\begin{aligned}\|f\|_2^2 = \langle f, f\rangle &= \int_K |f(k)|^2 dk \\ &\leq M \sup_{k\in K} |f(k)|^2 < \infty.\end{aligned}$$

So by Holders inequality, for all $f, h \in \mathrm{Ind}_P^G(\sigma)$

$$\begin{aligned}\langle f, h\rangle &= \int_K f(k)\overline{h(k)}dk \\ &\leq \int_K |f(k)h(k)|dk \\ &= \|fh\|_1 \\ &\leq \|f\|_2\|h\|_2 \\ &< \infty\end{aligned}$$

so the inner product is finite.

We also want to show that for all $f \in \mathrm{Ind}_P^G(\sigma)$ that $\langle f, f\rangle = 0$ if and only if $f = 0$. Since f is continuous, we have that $\langle f, f\rangle = 0$ implies that $f(k) = 0$ for all $k \in K$. However, from 2.1.1 we get that

$$f(g) = \sigma(na)f(k) = 0$$

for all $g \in G$ and thus $\langle f, f \rangle = 0$ if and only if $f = 0$. The linearity, conjugate symmetry and positive definiteness properties are all straight forward to check so the inner product is well defined. *QED*

This inner product gives a different topology on $\mathrm{Ind}_P^G(\sigma)$ compared to the norm we defined in 2.1.3. However, since uniform convergence implies L^2 convergence on compact sets, if we take the Hilbert space completion of $\mathrm{Ind}_P^G(\sigma)$ with respect to this inner product, then we get a Hilbert space representation of G. In other words the continuity requirement in 1.3.1 still holds. Denote the Hilbert space completion of $\mathrm{Ind}_P^G(\sigma)$ as $\overline{\mathrm{Ind}_P^G(\sigma)}$. For context, textbooks [Kna86] and [Wal88] work with this definition instead. In this thesis we choose the Banach space definition because in a number of places it simplifies the analysis because we don't need to deal with the technical details of L^2 spaces. It is enough to use this definition because both $\mathrm{Ind}_P^G(\sigma)$ and $\overline{\mathrm{Ind}_P^G(\sigma)}$ are both "infinitesimally" equivalent. In fact they will both admit the same so called Harish-Chandra modules. However, the Hilbert space structure is useful so in this next part we will utilise it to prove the first condition of admissibility from Definition 1.4.8.

Since $\overline{\mathrm{Ind}_P^G(\sigma)}$ is a Hilbert space representation of G, using Lemma 1.4.5 we can decompose this space as a representation of K and there exists irreducible finite dimensional sub-representations of K, $\{V_\alpha\}$ such that

$$\overline{\mathrm{Ind}_P^G(\sigma)} = \widehat{\bigoplus_\alpha} V_\alpha.$$

Let $V^{(K)}$ denote the space of K-finite vectors of $\mathrm{Ind}_P^G(\sigma)$ and $\overline{V}^{(K)}$ denote the space of K-finite vectors of $\overline{\mathrm{Ind}_P^G(\sigma)}$.

Proposition 2.2.2. *The space of K-finite vectors $\overline{V}^{(K)}$ is the algebraic direct sum,*

$$\overline{V}^{(K)} = \bigoplus_\alpha V_\alpha.$$

Proof. We prove by showing the inclusion in both directions. First assume that $v \in \overline{V}^{(K)}$. By Definition 1.4.7, $v \in L$ for some K-stable finite dimensional subspace $L \subseteq \overline{\mathrm{Ind}_P^G(\sigma)}$.

Since L is K-stable and a Hilbert space, it is a representation of K, thus by 1.4.5,

$$L = \widehat{\bigoplus_\beta} L_\beta$$

for some finite dimensional sub-representations of K, L_β. However, since L is finite dimensional, this Hilbert space direct sum is equal to the algebraic direct sum so

$$L = \bigoplus_\beta L_\beta.$$

Notice that L_β are also irreducible K-subrepresentations of $\overline{\mathrm{Ind}_P^G(\sigma)}$ so each $L_\beta \in \{V_\alpha\}$. Therefore, because $v \in L$,

$$v \in \bigoplus_\alpha V_\alpha.$$

This proves the inclusion in one direction.

Now assume that $v \in \bigoplus_\alpha V_\alpha$. This implies that v is a finite sum of elements,

$$v = \sum_{i=1}^{N} v_i$$

where each v_i is in some $V_{\alpha(i)} \in \{V_\alpha\}$. Therefore

$$v \in \bigoplus_{i=1}^{N} V_{\alpha(i)}.$$

Since each $V_{\alpha(i)}$ is finite dimensional and K-stable, $\bigoplus_{i=1}^N V_{\alpha(i)}$ is finite dimensional and K-stable so, $v \in \overline{V}^{(K)}$ by Definition 1.4.7 and we are done.

QED

Now observe by Definition 1.4.7, because $\mathrm{Ind}_P^G(\sigma)$ is a subspace of $\overline{\mathrm{Ind}_P^G(\sigma)}$ we have $V^{(K)}$ is a subspace of $\overline{V}^{(K)}$. Thus by Proposition 2.2.2,

$$V^{(K)} \subseteq \bigoplus_\alpha V_\alpha.$$

This implies that $V^{(K)}$ is an algebraic direct sum of irreducible representations. Thus, we have satisfied the first requirement of admissibility in Definition 1.4.8.

What remains is to show that that each irreducible representation occurs with finite multiplicity. In particular, we want to show that for all irreducible subrepresentations V,

$$\mathrm{mult}_K(V : V^{(K)}) < \infty.$$

Lemma 2.2.3. *If V is an irreducible K-subrepresentation of W, then*

$$\mathrm{mult}_K(V : W) = \dim \mathrm{Hom}_K(V, W).$$

Proof. Because W is a representation of a compact group K it splits into a direct sum of irreducibles $\bigoplus_i V_i$. Using the Hom and direct sum identity we have that

$$\mathrm{Hom}_K(V, W) = \mathrm{Hom}_K(V, \bigoplus_i V_i) \cong \bigoplus_i \mathrm{Hom}_K(V, V_i).$$

Since V and all V_i are irreducible and finite dimensional, due to Schur's Lemma we have,

$$\dim \mathrm{Hom}_K(V, V_i) = \begin{cases} 1 \text{ if } V \cong V_i \\ 0 \text{ if } V \ncong V_i \end{cases}.$$

Therefore in $\bigoplus_i \mathrm{Hom}_K(V, V_i)$, one dimension gets picked up for every copy of V in W and we get that $\dim \bigoplus_i \mathrm{Hom}(V, V_i) = \mathrm{mult}(V : W)$. *QED*

So, from lemma 2.2.3, to show that $\mathrm{Ind}_P^G(\sigma)$ is admissible we would like to show that for all irreducible subrepresentations V,

$$\dim \mathrm{Hom}_K(V, V^{(K)}) < \infty.$$

Since $V^{(K)}$ is a subspace of $\mathrm{Ind}_P^G(\sigma)$ it is enough to prove

$$\dim \mathrm{Hom}_K(V, \mathrm{Ind}_P^G(\sigma)) < \infty$$

and this is what we will do.

Every representation (π_1, X_1) of a Lie group G_1 can be restricted to a representation of a Lie subgroup $H_1 \subset G_1$. We denote the restriction as $\mathrm{Res}_{H_1}^{G_1}(\pi_1)$.

A fundamental theorem in representation theory is the following.

Theorem 2.2.4. *([Sep07], Theorem 7.47)* **(Frobenius reciprocity).** *Let G be a Lie group and H a closed subgroup of G. If V is a representation of H and W is a representation of G, then as vector spaces*

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(W), V). \tag{2.2.2}$$

Notice that above in Lemma 2.2.3 our Hom-space is over K. Therefore, to employ Frobenius reciprocity, we need to realise $\mathrm{Ind}_P^G(\sigma)$ as a representation induced to K. First, note that $K \cap P = M$. Therefore, for all $f \in \mathrm{Ind}_P^G(\sigma)$ we have

$$f(mk) = \sigma(m) f(k),$$

for all $m \in M$, $k \in K$. Thus $f|_K$ is an induced representation of $\sigma|_M$ and $f|_K \in \mathrm{Ind}_M^K(\sigma|_M)$. We also define the norm on $\mathrm{Ind}_M^K(\sigma)$ to be $\|f\| = \sup_{k \in K} \|f(k)\|_U$ which gives $\mathrm{Ind}_M^K(\sigma)$ a Banach space structure.

Therefore, we have the restriction map $\phi : \mathrm{Ind}_P^G(\sigma) \to \mathrm{Ind}_M^K(\sigma|_M)$ defined by $f \mapsto f|_K$. Clearly restriction will be a homomorphism as K-representations so ϕ is a homomorphism of K-representations. Since each $f \in \mathrm{Ind}_P^G(\sigma)$ is determined by its restriction to K (Lemma 2.1.1), we have that ϕ is an injective homomorphism. Moreover, due to Lemma 2.1.2 we have that ϕ is also surjective. Therefore ϕ is a bijection and thus an isomorphism.

Now we have reduced the problem to showing that

$$\dim \mathrm{Hom}_K(V, \mathrm{Ind}_M^K(\sigma|_M)) < \infty.$$

Now we can apply Frobenius reciprocity (Theorem 2.2.4), giving us

$$\mathrm{Hom}_K(V, \mathrm{Ind}_M^K(\sigma|_M)) \cong \mathrm{Hom}_M(\mathrm{Res}_M^K(V), \sigma|_M).$$

By 1.4.11, V is one dimensional, but $\sigma|_M$ is also one dimensional. Thus, the homomorphism space between them must be at most one dimensional. Id est,

$$\dim \mathrm{Hom}_M(\mathrm{Res}^K_M V, \sigma|_M) \leq 1$$

and thus

$$\dim \mathrm{Hom}_M(V, \mathrm{Ind}^K_M(\sigma|_M)) \leq 1 < \infty.$$

This proves that $\mathrm{Ind}^G_P(\sigma)$ is an admissible representation of G, and all irreducible representations have multiplicity at most 1 in $V^{(K)}$.

We have proved the following theorem

Theorem 2.2.5. *The K-finite vectors $V^{(K)}$ of the induced representation $\mathrm{Ind}^G_P(\sigma)$ are a direct sum of irreducible subrepresentations of K. Additionally, any K-subrepresentation of $V^{(K)}$ appears with multiplicity at most 1. In particular, $\mathrm{Ind}^G_P(\sigma)$ is admissible.*

Definition 2.2.6. *Let $\epsilon = \pm 1$ and $\lambda \in \mathbb{C}$. Define for all non-zero $a \in \mathbb{R}$*

$$\mathrm{sign}(a) = \begin{cases} 2 \text{ if } a > 0 \\ 1 \text{ if } a < 0 \end{cases}.$$

Now, define the representation $(\sigma_{\epsilon,\lambda}, \mathbb{C})$ of P with

$$\sigma_{\epsilon,\lambda} : P \to \mathbb{C}^\times$$

as

$$\sigma_{\epsilon,\lambda} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = (\epsilon)^{\mathrm{sign}(a)} |a|^{\lambda+1}. \tag{2.2.3}$$

The admissible representation of G, $\mathrm{Ind}^G_P(\sigma_{\epsilon,\lambda})$ is called the **principal series representation** *associated to parameters (ϵ, λ).*

These principal series representations are the induced representations that all of the other admissible representations will embed into.

2.3 The K-finite vectors of the principal series

In this section we describe the space $V^{(K)}$ of K-finite vectors for the principal series explicitly. The proofs in this section take inspiration from pages 65-67 of [Vog81].

Let $V^{(K)}$ be the K-finite vectors of $\mathrm{Ind}^G_P(\sigma_{\epsilon,\lambda})$. We have showed $\mathrm{Ind}^G_P(\sigma_{\epsilon,\lambda})$ is admissible so by definition it is a direct sum of irreducible subrepresentations of K. We will spend this next section proving which irreducible representations appear in this direct sum. Theorem 1.4.13 classifies the irreducible representations of K. Therefore there exists irreducible K-subrepresentations $\{V_n\}$

$$V^{(K)} = \bigoplus_n V_n \tag{2.3.1}$$

such that $R(k(\theta))v = e^{in\theta}v$ for all $k(\theta) \in K$ and $v \in V_n$. The goal of this section is to refine the decomposition (2.3.1) and determine the values of n.

Theorem 2.3.1. *Let $V^{(K)}$ be the space of K-finite vectors of the principal series representation $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$. If $\epsilon = 1$*

$$V^{(K)} = \bigoplus_{n \in 2\mathbb{Z}} V_n$$

and if $\epsilon = -1$

$$V^{(K)} = \bigoplus_{n \in 2\mathbb{Z}} V_{n+1}$$

Proof. Since $-I \in P$, for all $f \in \mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ by 1.3.1 for all $g \in G$,

$$f(g) = f((-I)(-Ig)) = \sigma_{\epsilon,\lambda}(-I)f(-Ig). \tag{2.3.2}$$

Now,

$$\begin{aligned}\sigma_{\epsilon,\lambda}(-I) &= \sigma_{\epsilon,\lambda}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\\ &= \epsilon\end{aligned}$$

by definition of $\sigma_{\epsilon,\lambda}$. Therefore by (2.3.2),

$$f(g) = \epsilon f(-g). \tag{2.3.3}$$

Let

$$k(\theta) := \begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}$$

be an arbitrary element of K. Since

$$k(\pi)k(\theta) = -k(\theta)$$

it follows from (2.3.3) that we have for all $k(\theta) \in K$,

$$R(k(\pi))f(k(\theta)) = \epsilon f(k(\theta)).$$

Now assume additionally that $f \in V^{(K)}$. This implies

$$f = \sum_{j=1}^{N} f_n$$

with $f_n \in V_n$.

Since the f_n are a basis, we get

$$R(k(\pi))f_n(k(\theta)) = \epsilon f_n(k(\theta)).$$

Because $f_n \in V_n$, by definition of V_n 1.4.13 we have

$$e^{in\pi} f_n(k(\theta)) = \epsilon f_n(k(\theta))$$

which implies

$$f_n(k(\theta))(e^{in\pi} - \epsilon) = 0$$

If n is odd and $\epsilon = 1$ then $f_n(k(\theta)) = 0$.

Now, if $f_n(k(\theta)) = 0$ then f_n is zero on K. But for all $g \in G$,

$$f_n(g) = f_n(nak) = \sigma_{\epsilon,\lambda}(na) f_n(k) = \sigma_{\epsilon,\lambda}(na)0 = 0$$

so f_n is zero on G. Because f was taken as an arbitrary element of $V^{(K)}$, this implies that if n is odd then V_n is empty. Correspondingly if $\epsilon = 1$ and n is odd then V_n is empty.

Now, by definition $f \in \mathrm{Ind}_M^K(\sigma_{\epsilon,\lambda}|_M)$ if and only if f is continuous and

$$f(mk) = \sigma_{\epsilon,\lambda}|_M(m) f(k)$$

for all $m \in M = \{\pm I\}$ and $k \in K$.

This condition is obviously true if $m = I$, therefore $f \in \mathrm{Ind}_M^K(\sigma_{\epsilon,\lambda}|_M)$ if and only if f is continuous and

$$\begin{aligned} f(-Ik) &= \sigma_{\epsilon,\lambda}|_M(-I) f(k) \\ &= \epsilon f(k) \end{aligned}$$

which we restate as

$$f(-k) = \epsilon f(k). \tag{2.3.4}$$

Consider when $\epsilon = 1$. For all even integers m we take take $f_m : K \to \mathbb{C}$ to be defined by

$$f_m(k(\theta)) = e^{im\theta}$$

for all $k(\theta) \in K$. The function f_m satisfies 2.3.4. Since f_m is also continuous $f_m \in \mathrm{Ind}_M^K(\sigma_{\epsilon,\lambda}|_M)$. Take $\hat{f_m} \in \mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ to be the lifting of f_m to $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ as

defined in lemma 2.1.2. Take some arbitrary $k(\theta_1) \in K$. For all $g \in G$,

$$\begin{aligned} R(k(\theta_1))\hat{f_m}(g) &= R(k(\theta_1))\hat{f_m}(nak) \\ &= \hat{f_m}(nak(\theta)k(\theta_1)) \\ &= \hat{f_m}(nak(\theta+\theta_1) \\ &= \sigma_{\epsilon,\lambda}(na)\hat{f_m}(k(\theta+\theta_1)) \\ &= \sigma_{\epsilon,\lambda}(na)e^{im(\theta+\theta_1)} \\ &= \sigma_{\epsilon,\lambda}(na)e^{im\theta}e^{im\theta_1} \\ &= \sigma_{\epsilon,\lambda}(na)\hat{f_m}(k)e^{im\theta_1} \\ &= \hat{f_m}(g)e^{im\theta_1} \\ &= e^{im\theta_1}\hat{f_m}(g). \end{aligned}$$

Therefore $\hat{f_m} \in V_m$ and V_m is non-empty.

Correspondingly, the same argument shows that if $\epsilon = -1$ then for all odd integers m, $\hat{f_m} \in V_m$ and V_m is non-empty.

We have now proved the theorem. *QED*

We now have a very explicit description of the K-finite vectors of the principal series. This will be of great use particularly in the last chapter of this thesis when we observe the irreducible subspaces of the principal series and find all of the irreducible representations.

Chapter 3

The Lie algebra

3.1 Basic structure theory

In this section we introduce the Lie algebra, a purely algebraic object with its own representation theory which we will use to study the unitary representations of $SL_2(\mathbb{R})$. This strategy was used by Weyl to study unitary representations of compact groups. We will use it to study unitary representations of $SL_2(\mathbb{R})$.

Definition 3.1.1. *The* **Lie algebra** $\mathfrak{g}$ *of a matrix Lie group G is the tangent space at the identity T_IG.*

The Lie algebra is the tangent space of a real manifold so it is naturally a vector space over $\mathbb{R}$.

Proposition 3.1.2. *The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ is*

$$\{X \in M_{2,2}(\mathbb{R}) : \operatorname{tr}(X) = 0\}.$$

Proof. The tangent space at the identity of G can be defined as

$$\{\gamma'(0) : \text{ where } \gamma : [-1,1] \to G \text{ is a smooth path with } \gamma(0) = I \ \}.$$

Let $\gamma : [-1,1] \to G$ be an arbitrary smooth path. Note that $\det(\gamma(t)) = 1$ so $\frac{d}{dt}\det(\gamma(t)) = 0$. By the chain rule,

$$\begin{aligned}\frac{d}{dt}\det(\gamma(t))|_{t=0} &= D\det \circ D\gamma|_{t=0} \\ &= \operatorname{Jac}(\det) \circ \gamma'(t)^T|_{t=0} \\ &= \gamma'(0) \cdot (\nabla \det)(0)|_{t=0}\end{aligned}$$

where in the last step we take the transpose of a one dimensional matrix. This is just the directional derivative of the determinant map in the direction of $\gamma'(0)$ at the identity. So,

$$0 = \lim_{h\to 0} \frac{\det(I + h\gamma'(0)) - \det(I)}{h}.$$

Using the trace formula for the determinant $\det(I + \epsilon X) = 1 + \epsilon \operatorname{tr}(X) + O(\epsilon^2)$ we get,

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{1 + h \operatorname{tr}(\gamma'(0)) + O(h^2) - \det(I)}{h} \\ &= \lim_{h \to 0} \frac{h \operatorname{tr}(\gamma'(0)) + O(h^2)}{h} \\ &= \operatorname{tr}(\gamma'(0)) + \lim_{h \to 0} \frac{O(h^2)}{h} \end{aligned}$$

and thus, $\operatorname{tr}(\gamma'(0)) = 0$. Therefore

$$T_I G = \{X \in M_2(\mathbb{R}) : \operatorname{tr}(X) = 0\}.$$

QED

In differential geometry, we view the tangent space as a "linearization" of a manifold. We often study the tangent space because affine spaces are much simpler than non-affine spaces. In Lie theory the manifolds have the extra algebraic structure of a group. We will soon see that the tangent space of the identity (the Lie algebra) also inherits some algebraic structure from the group. Lie algebras also have a representation theory which is related to the representation theory of the corresponding Lie group.

We will now discuss the kind of algebraic structure that Lie algebras have.

Definition 3.1.3. *A* **Lie bracket** *on a vector space V is a bilinear map*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the following conditions

- *The Jacobi identity:*

  $$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

  for all $X, Y, Z \in V$
- *Alternating:*

  $$[X, X] = 0$$

  for all $X \in V$.

Lemma 3.1.4. *A Lie algebra $\mathfrak{g}$ of a matrix Lie group G is equipped with a* **Lie bracket** $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *given by the matrix commutator*

$$[X, Y] = XY - YX.$$

Proof. We prove the lemma for $SL_2(\mathbb{R})$ only. First we prove that the matrix commutator is well defined on $\mathfrak{sl}_2(\mathbb{R})$. Let

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \quad a, b, c, x, y, z \in \mathbb{R}$$

by arbitrary elements of $\mathfrak{sl}_2(\mathbb{R})$. We have that the commutator of these elements is

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix} - \begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \\
&= \begin{pmatrix} ax+bz & \cdots \\ \cdots & cy+ax \end{pmatrix} - \begin{pmatrix} ax+cy & \cdots \\ \cdots & bz+ax \end{pmatrix} \\
&= \begin{pmatrix} bz-cy & \cdots \\ \cdots & cy-bz \end{pmatrix}
\end{aligned}$$

which has zero trace. Therefore $XY - YX \in \mathfrak{sl}_2(\mathbb{R})$ for all $X, Y \in \mathfrak{sl}_2(\mathbb{R})$.

Clearly the commutator is bilinear by the distributive property of matrices, and it is alternating by $[X, X] = X^2 - X^2 = 0$.

The remaining property to show is the Jacobi identity:

$$\begin{aligned}
&[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
&= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) - (ZX - XZ)Y \\
&+ Z(XY - YX) - (XY - YX)Z \\
&= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY \\
&+ XZY + ZXY - ZYX - XYZ + YXZ \\
&= 0.
\end{aligned}$$

QED

Definition 3.1.5. *The* **matrix exponential** *of a matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is defined as the power series*

$$\exp(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

where $X^0 := I$. This series is always convergent so the matrix exponential is well defined.

Theorem 3.1.6. *Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. The matrix exponential defines a map from the Lie algebra to the Lie group,*

$$\exp : \mathfrak{g} \to G$$

by just restricting the matrix exponential map to $\mathfrak{g}$.

Proof. We prove for $SL_2(\mathbb{R})$ only. The matrix exponential has the following trace formula

$$\det(\exp A) = e^{\operatorname{tr} A}.$$

The proof for this can be found in [Hal15]. Since $\mathfrak{sl}_2(\mathbb{R})$ is a space of traceless matrices, we have for all $X \in \mathfrak{sl}_2(\mathbb{R})$

$$\det(\exp X) = e^0 = 1.$$

So, the matrix exponential does indeed map into G. *QED*

The following proposition and proof are from [Hal15].

Proposition 3.1.7. *([Hal15], Proposition 2.4) The path* $\exp(tX)$ *has tangent vector* X *at* $t = 0$.

Proof. We prove by differentiating the power series for $\exp(tX)$ term by term. This is justified because each entry of the matrix exponential $\exp(tX)$ is given by a convergent power series and you can differentiate a convergent power series term by term. For more details see [Hal15]. Therefore we have,

$$\begin{aligned}\frac{d}{dt}\exp(tX)|_{t=0} &= \frac{d}{dt}I + tX + \frac{t^2X^2}{2} + \cdots|_{t=0} \\ &= X.\end{aligned}$$

QED

Definition 3.1.8. *A* **representation** (ρ, V) *of a Lie algebra* $\mathfrak{g}$ *is a vector space* V *over* $\mathbb{C}$ *and a linear map* $\rho : \mathfrak{g} \to \mathrm{End}(V)$ *such that*

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

As mentioned before, the representations of a Lie group G are related to the representations of its Lie algebra $\mathfrak{g}$. Notice that all definitions about Lie algebras are purely algebraic and have no analytic conditions like our definitions for Lie groups and their representations. This is an example of how the representation theory of Lie algebras is simpler than the representation theory of Lie groups. This is exactly why we are interested in studying Lie algebra representations.

3.2 Lie algebra representations from Lie group representations

We know from differential geometry that a smooth map between manifolds can be differentiated to give a map between the tangent spaces of the manifolds. As mentioned prior in 1.3, if we have a finite dimensional representation (ϕ, V) of a Lie group G then $\phi : G \to GL(V)$ is a smooth Lie group homomorphism. Thus the differential of this map at the identity is a map between tangent spaces and thus a map from the Lie algebra $\mathfrak{g}$ to $\mathrm{End}(V)$, which is the tangent space of $GL(V)$. This map, in fact, defines a Lie algebra representation of $\mathfrak{g}$ on V. This is the standard theory which we aim to generalize to Banach space representations of G, and we will see that the extra analytic condition of a Banach space representation is what is needed to generalize this strategy to the infinite dimensional setting. The key challenge is that it is not clear what the differential means for a map $\phi : G \to GL(V)$ when V is infinite dimensional since $GL(V)$ does not have a manifold structure.

In this section we will show how a Banach space representation (ϕ, V) of $SL_2(\mathbb{R})$ induces a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ on the subspace of "smooth" vectors in V. The proofs in this section are based on [Mil] and [Kna86] with missing steps filled in with the assistance of Alessandro Ottazzi.

Definition 3.2.1. *([Kna86], p.52) A function* f *from an open subset* S *of* $\mathbb{R}^n$ *to a Banach space* V *is called* **differentiable** *at* $x_0 \in S$ *if there exists a linear map* $f'(x_0) : \mathbb{R}^n \to V$ *such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{|x - x_0|} = 0.$$

If f is differentiable at every point in S then $x \mapsto f'(x)$ is a map from S to $\mathrm{End}(\mathbb{R}^n, V)$ *and we say $f \in C^1$ if $x \mapsto f'(x)$ is continuous. We may then say that $f \in C^2$ if $x \mapsto f'(x)$ is in C^1 and so on. We say that $f \in C^\infty$ if $f \in C^k$ for all $k \geq 1$.*

The language of the above definition is suggestive of a generalization to manifolds and this is the definition that we need

Definition 3.2.2. *A function f from a smooth manifold G to a Banach space V is called* **differentiable** *at a point $x_0 \in G$ if for some chart of x_0, say $R : G \to U \subset \mathbb{R}^n$, we have that the map $f \circ R^{-1} : U \to V$ is a differentiable function. We can define the corresponding classes of C^1 and C^∞ in the same ways as above in 3.2.1.*

The definition 3.2.2 is well defined and doesn't depend on the chart since transition maps of smooth manifolds are smooth and thus they will preserve differentiability.

Lemma 3.2.3. *([Kna86], p.52) The following properties hold for Definition 3.2.2:*

(i) The function f is of class C^k if and only if all its partial derivatives up to and including order k exist and are continuous.

(ii) The chain rule holds.

We now have to appropriate definitions to talk about differentiation in the infinite dimensional context.

Definition 3.2.4. *A* **smooth vector** *of a G-representation (ϕ, V) is an element $\xi \in V$ such that the map*

$$G \to V; \quad g \mapsto \phi(g)\xi$$

is in C^∞.

We denote the set of smooth vectors by $C^\infty(\phi)$

Theorem 3.2.5. *The set of smooth vectors $C^\infty(\phi)$ is a dense subspace of V.*

The above theorem hints as to why having an action of $\mathfrak{g}$ on the subspace $C^\infty(\phi)$ still captures useful information about the original representation.

For the rest of this section

$$G = SL_2(\mathbb{R}), \quad \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}), \quad K = SO_2.$$

We now want to show that the subspace $C^\infty(\phi)$ has the structure of a representation of $\mathfrak{g}$ as in Definition 3.1.8. First define $f : \mathfrak{g} \to V$ by

$$f(X) = \phi(\exp X)\xi.$$

We define an action of $\mathfrak{g}$ on a vector $\xi \in C^\infty(\phi)$ by

$$\tilde{\phi}(X)\xi := f'(0)(X).$$

Now since $\mathfrak{g}$ can be identified with $\mathbb{R}^n$ for some n, and the exponential map is smooth, we have that f is in C^∞ and the derivative makes sense.

By definition of $f'(0)$ we have the following,

$$\lim_{X\to 0} \frac{f(X) - f(0) - f'(0)(X)}{|X|} = 0.$$

For $X \in \mathfrak{g}$ with $|X| = 1$ we can reformulate the above limit as

$$\begin{aligned}
0 &= \lim_{t\to 0^+} \frac{f(tX) - f(0) - f'(0)(tX)}{|tX|} \\
&= \lim_{t\to 0^+} \frac{f(tX) - f(0) - f'(0)(tX)}{t} \\
&= \lim_{t\to 0^+} \frac{f(tX) - f(0)}{t} - \frac{f'(0)(tX)}{t} \\
&= \lim_{t\to 0^+} \frac{f(tX) - f(0)}{t} - \frac{tf'(0)(X)}{t} \\
&= \lim_{t\to 0^+} \frac{f(tX) - f(0)}{t} - f'(0)(X) \\
&= \lim_{t\to 0^+} \frac{\phi(\exp(tX))\xi - \xi}{t} - f'(0)(X).
\end{aligned}$$

Therefore,

$$f'(0)(X) = \lim_{t\to 0} \frac{\phi(\exp(tX))\xi - \xi}{t} = \frac{d}{dt}\phi(\exp(tX))\xi|_{t=0}.$$

We now have three different forms for the action of the Lie algebra,

$$\begin{aligned}
\tilde{\phi}(X)\xi &= f'(0)(X) \\
&= \frac{d}{dt}\phi(\exp tX)\xi|_{t=0} \\
&= \lim_{t=0} \frac{\phi(\exp(tX))\xi - \xi}{t}.
\end{aligned}$$

We will now show that $\tilde{\phi}$ defines a representation of the Lie algebra.

To show that the action is well defined we need to show that for all $\xi \in C^\infty(\phi)$, $\tilde{\phi}(X)\xi \in C^\infty(\phi)$. Let $g \in G$ and consider

$$\phi(g)\tilde{\phi}(X)\xi = \phi(g)\frac{d}{dt}\phi(\exp tX)\xi|_{t=0}.$$

Since $\phi(g)$ acts continuously we have that

$$\begin{aligned}
\phi(g)\tilde{\phi}(X)\xi &= \frac{d}{dt}\phi(g)\phi(\exp tX)\xi|_{t=0} \\
&= \frac{d}{dt}\phi(g\exp tX)\xi|_{t=0}.
\end{aligned}$$

Since group multiplication is smooth, the right hand side is a composition of smooth functions. Thus it is also smooth and $\phi(g)\tilde{\phi}(X)\xi$ is a smooth map. Therefore, $\tilde{\phi}(X)\xi \in C^\infty(\phi)$.

Clearly $\tilde{\phi}(X)$ is a linear map on $C^\infty(\phi)$ so $\tilde{\phi}$ is a map into $\mathrm{End}(C^\infty(\phi))$.

Since $f'(0)$ is a linear map by definition, we have that $\tilde{\phi}$ is linear.

It remains to prove that $\tilde{\phi}$ preserves the Lie bracket. This proof is more involved than the previous ones so we start with some preliminary Lemmas. First, we are going to establish the product rule for matrices.

Lemma 3.2.6. *Let $A : \mathbb{R} \to M_{2,2}$ and $B : \mathbb{R} \to M_{2,2}$ be differentiable functions. If $C : \mathbb{R} \to M_{2,2}$ is defined as the matrix product*

$$C(t) = A(t)B(t)$$

then for the derivative of C we have,

$$C'(t) = A'(t)B(t) + A(t)B'(t).$$

Proof. The (k, l)-th entry of the matrix $C(t)$ is

$$c_{kl}(t) = \sum_{m=1}^{2} a_{km}(t)b_{ml}(t).$$

By the standard product rule, we have

$$c'_{kl}(t) = \sum_{m=1}^{2} a'_{km}(t)b_{ml}(t) + \sum_{m=1}^{2} a_{km}(t)b'_{ml}(t).$$

Therefore, we conclude that

$$C'(t) = A'(t)B(t) + A(t)B'(t).$$

QED

Lemma 3.2.7. *Let $y \in G$ and $X \in \mathfrak{g}$. We have that $yXy^{-1} \in \mathfrak{g}$ and for all $\xi \in C^\infty(\phi)$,*

$$\tilde{\phi}(yXy^{-1})\xi = \phi(y)\tilde{\phi}(X)\phi(y^{-1})\xi.$$

Proof. For the first claim, trace is basis invariant so $\mathrm{tr}(yXy^{-1}) = \mathrm{tr}(X) = 0$ and $yXy^{-1} \in \mathfrak{g}$.

Next, on the right hand side for all $\xi \in C^\infty(\phi)$ we have

$$\begin{aligned}\phi(y)\tilde{\phi}(X)\phi(y^{-1})\xi &= \frac{d}{dt}\phi(y)\phi(\exp(tX))\phi(y^{-1})\xi|_{t=0} \\ &= \frac{d}{dt}\phi(y\exp(tX)y^{-1})\xi|_{t=0}.\end{aligned}$$

Clearly both $y\exp(tX)y^{-1}$ and $\exp(t(yXy^{-1}))$ have tangent vector yXy^{-1} at the identity. Therefore,

$$\frac{d}{dt} y \exp(tX) y^{-1}|_{t=0} = \frac{d}{dt} \exp(t(yXy^{-1}))|_{t=0}$$

and by the chain rule we have

$$\begin{aligned}\phi(y)\tilde{\phi}(X)\phi(y^{-1})\xi &= \frac{d}{dt} \exp(t(yXy^{-1}))\xi|_{t=0} \\ &= \tilde{\phi}(yXy^{-1})\xi.\end{aligned}$$

QED

Lemma 3.2.8. *The path* $\exp(-tY)X\exp(tY)$ *has tangent vector* $[X,Y]$ *at* $t=0$.

Proof. Using the product rule for matrices,

$$\begin{aligned}\frac{d}{dt}\exp(-tY)X\exp(tY)|_{t=0} &= -YX\exp(t\cdot 0) + \exp(t\cdot 0)XY \\ &= -YX + XY \\ &= [X,Y].\end{aligned}$$

QED

Lemma 3.2.8 establishes that for all $\xi \in C^\infty(\phi)$,

$$\begin{aligned}\tilde{\phi}([X,Y])\xi &= \frac{\partial}{\partial s}\phi(\exp(s[X,Y]))\xi|_{s=0} \\ &= \frac{\partial}{\partial t}\phi(\exp(s\frac{\partial}{\partial t}\exp(-tY)X\exp(tY)))\xi|_{s,t=0} \\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi(\exp(s\exp(-tY)X\exp(tY)))\xi|_{s,t=0} \\ &= \frac{\partial}{\partial t}\tilde{\phi}(\exp(-tY)X\exp(tY))\xi|_{t=0} \\ &= \frac{\partial}{\partial t}\phi(\exp(-tY))\tilde{\phi}(X)\phi(\exp(tY))\xi|_{t=0}.\end{aligned}$$

Where the last equality follows from lemma 3.2.7. Now, in the same spirit as the proof of the product rule, we have the following.

$$\begin{aligned}&\frac{\partial}{\partial t}\phi(\exp(-tY))\tilde{\phi}(X)\exp(tY)\xi|_{t=0} \\ =&\lim_{t\to 0}\frac{1}{t}\left(\phi(\exp(-tY))\tilde{\phi}(X)\phi(\exp(tY))\xi - \tilde{\pi}(X)\xi\right) \\ =&\lim_{t\to 0}\frac{1}{t}\Big(\phi(\exp(-tY))\tilde{\phi}(X)\phi(\exp(tY))\xi - \tilde{\phi}(X)\phi(\exp(tY))\xi \\ &+\tilde{\phi}(X)\phi(\exp(tY))\xi - \tilde{\phi}(X)\xi\Big) \\ =&\lim_{t\to 0}\frac{\phi(\exp(-tY)\xi - \xi)}{t}\tilde{\phi}(X)\phi(\exp(tY)) + \frac{\tilde{\phi}(X)\phi(\exp(tY))\xi - \tilde{\phi}(X)\xi}{t} \\ =&-\tilde{\phi}(Y)\tilde{\phi}(X)\xi + \tilde{\phi}(X)\tilde{\phi}(Y)\xi.\end{aligned}$$

We have shown that $\tilde{\phi}$ is a Lie algebra representation on the subspace of smooth vectors.

We state the following theorem

Proposition 3.2.9. *([Kna86], Proposition 8.5) For an admissible representation (π, V), every K-finite vector is a smooth vector and the space of K-finite vectors is stable under $\tilde{\pi}(\mathfrak{g})$.*

We now have a way of taking an admissible Banach space representation (ϕ, V) of G and inducing a representation of the Lie algebra $\mathfrak{g}$ of G on $V^{(K)}$, the subspace of K-finite vectors of V.

Definition 3.2.10. *We call the representation of $\mathfrak{g}$ and K on $V^{(K)}$ the* **Harish-Chandra module** *of (ϕ, V) and denote the Harish-Chandra module of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ as $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$.*

3.3 $(\mathfrak{g}, K)$-modules

Finite dimensional representations of a Lie group have a close relationship with representations of the Lie algebra. We state the following facts which demonstrate this. If (ϕ, V) is a finite dimensional representation of G, then ϕ is a smooth homomorphism from G to $GL(V)$. Its differential at the identity gives a representation $D\phi$ of the Lie algebra $\mathfrak{g}$ on V. Moreover, $D\phi$ uniquely determines ϕ. We also have that any representation of the Lie algebra $\mathfrak{g}$ on V is the differential of some representation of G on V. Discussion of this can be found in [Kna86].

For infinite dimensional unitary representations we do not have such a close correspondence. However, we have shown that any Banach space representation V of G can be differentiated to get a representation of the Lie algebra $\mathfrak{g}$ on the subspace of smooth vectors and K-finite vectors of V. Harish-Chandra showed that representations of $\mathfrak{g}$ which are also representations of K and have some additional conditions do have similar correspondences as those mentioned above in the finite dimensional case. This leads to the definition of $(\mathfrak{g}, K)$-modules. In this section we define $(\mathfrak{g}, K)$-modules and discuss the results of Harish-Chandra which give the infinite dimensional analogues of the nice correspondences from the finite dimensional case.

Now we are going to specialize to $SL_2(\mathbb{R})$ and mention which results and definitions are similar in general and which results simplify for $SL_2(\mathbb{R})$.

Before we state the definition, we introduce some notation. Let V be a vector space and m be a positive integer or infinity.

$$V^{\oplus m} := \underbrace{V \oplus V \oplus \cdots}_{m \text{ times}}.$$

Definition 3.3.1. ($(\mathfrak{g}, K)$-modules). *Let (π, V) be a representation of K and let $(\tilde{\pi}, V)$ be a representation of $\mathfrak{g}$. We say V is a representation of $(\mathfrak{g}, K)$ or a $(\mathfrak{g}, K)$-module if the following conditions hold*

1. *V is the algebraic direct sum of finite dimensional irreducible representations of K, i.e*

$$V = \bigoplus_{n\in\mathbb{Z}} V_n^{\oplus \operatorname{mult}(V_n:V)} \tag{3.3.1}$$

where V_i are the one-dimensional irreducible representations of K as defined in 1.4.13.

2. *If $Y \in \mathfrak{k}$ where $\mathfrak{k}$ is the Lie algebra of K, then $\frac{d}{dt}\pi(\exp(tY))v|_{t=0} = \tilde{\pi}(Y)v$.*

Often we will just say that (π, V) is a $(\mathfrak{g}, K)$-module and it is understood that $\tilde{\pi}$ will denote the representation of $\mathfrak{g}$.

Condition 2 is a compatibility requirement between how $\mathfrak{g}$ and K act. In particular it ensures that the differential of the K action agrees with action of $\mathfrak{g}$ restricted to the corresponding compact elements.

The general definition of $(\mathfrak{g}, K)$-modules has a third compatibility condition but this condition is not required for $SL_2(\mathbb{R})$ since $SL_2(\mathbb{R})$ is connected (proved in Proposition 1.1.7).

Notice that we take the algebraic direct sum instead of the Hilbert space direct sum. This shows that our definition is really an algebraic one.

An admissible $(\mathfrak{g}, K)$-module is defined in the same way as an admissible representation of G as defined in 1.4.8.

Definition 3.3.2. *In the setting of 3.3.1, if each $V_n^{\oplus \operatorname{mult}(V_n:V)}$ is finite dimensional (i.e each multiplicity of V_n is finite), then V is an* **admissible** *$(\mathfrak{g}, K)$-module.*

It is clear by definitions that the Harish-Chandra module of an admissible representation is an admissible $(\mathfrak{g}, K)$-module.

Definition 3.3.3. *Let (π, V) be a $(\mathfrak{g}, K)$-module. The set of* **K-types** *of (π, V) is*

$$\{n \in \mathbb{Z} : V_n \neq 0\}.$$

Definition 3.3.4. *An isomorphism of two $(\mathfrak{g}, K)$-modules is a map which is both an isomorphism of $\mathfrak{g}$-representations and an isomorphism as representations of K. If the $(\mathfrak{g}, K)$-modules of two representations (π, V) and (π', V') of G are isomorphic, we say that (π, V) and (π', V') are* **infinitesimally equivalent**.

We state without proof the following important theorem of Harish-Chandra. Recall the definition of unitary equivalence from Definition 1.3.5.

Theorem 3.3.5. *Two irreducible unitary representations of G are unitarily equivalent if and only if they are infinitesimally equivalent.*

Definition 3.3.6. *A $(\mathfrak{g}, K)$-module is said to be* **irreducible** *if it is irreducible as a representation of $\mathfrak{g}$.*

Recall the definition of unitary equivalence 1.3.5. This is another important result of Harish-Chandra

Theorem 3.3.7. *([Vog87], Theorem 2.15) Two unitary representations of G are equivalent if and only if they are infinitesimally equivalent. I.e their Harish-Chandra modules are isomorphic.*

We also have a notion of unitarity for $(\mathfrak{g}, K)$-modules, and it is a result from Harish-Chandra that every irreducible "infinitesimally unitary" $(\mathfrak{g}, K)$-module is

the Harish-Chandra module of a unique irreducible unitary representation of G where $\mathfrak{g}$ is the Lie algebra of G and K is the maximal compact subgroup of G. We state and discuss this result further in the last section on unitarity. If we combine this result with Theorem 3.3.7 then we see how the problem of studying the equivalence classes of irreducible unitary representations reduces to studying the equivalences classes of admissible $(\mathfrak{g}, K)$-modules, and then determining which are unitary.

The reason for switching our analysis to $(\mathfrak{g}, K)$-modules is that these objects can be studied algebraically.

Before we continue, we are going to introduce the complexification of the Lie algebra. Since Lie algebras are vector spaces, we can complexify a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ as $\mathfrak{g}^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ to get a complex vector space. Elementary tensors of $\mathfrak{g}^{\mathbb{C}}$ are of the form

$$(a + ib) \otimes_{\mathbb{R}} X$$

where $a, b \in \mathbb{R}$ and $X \in \mathfrak{g}$. We can rewrite this as

$$\begin{aligned}(a + ib) \otimes_{\mathbb{R}} X &= a \otimes_{\mathbb{R}} X + ib \otimes_{\mathbb{R}} X \\ &= 1 \otimes_{\mathbb{R}} aX + i \otimes_{\mathbb{R}} bX\end{aligned}$$

so we can say that elements of $\mathfrak{g}^{\mathbb{C}}$ are

$$X + iY$$

where $X, Y \in \mathfrak{g}$. For this reason the complexification is often written as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Proposition 3.3.8. *The complexification $\mathfrak{g}^{\mathbb{C}}$ is also a Lie algebra in the obvious way with the bracket $[\cdot, \cdot]_{\mathfrak{g}^{\mathbb{C}}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ defined by*

$$[X_1 + iY_1, X_2 + iY_2]_{\mathfrak{g}^{\mathbb{C}}} := [X_1, X_2] - [Y_1, Y_2] + i[X_1, Y_2] + i[Y_1, X_2].$$

This can be proven by simply checking that $[\cdot, \cdot]_{\mathfrak{g}^{\mathbb{C}}}$ inherits the conditions 3.1.3 of a Lie bracket from $[\cdot, \cdot]$.

In this thesis all our representations are complex vector spaces. This means we can lift our a representation (ρ, V) of the real Lie algebra $\mathfrak{g}$ to a representation $(\tilde{\rho}, V)$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ by defining the action as follows,

$$\tilde{\rho}(X + iY)v = \rho(X)v + i\rho(Y)v. \tag{3.3.2}$$

We can also go the other way. Since $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$, any representation of the complexification $\mathfrak{g}^{\mathbb{C}}$ can be restricted to a representation of $\mathfrak{g}$. So far we have reduced the objective of this thesis to studying the isomorphism classes of irreducible admissible $(\mathfrak{g}, K)$-modules. Now every $(\mathfrak{g}, K)$-module can be complexified to a $(\mathfrak{g}^{\mathbb{C}}, K)$-module. These are easier to classify, however we still need the following Lemma.

Lemma 3.3.9. *A $(\mathfrak{g}, K)$-module is irreducible if and only if the corresponding $(\mathfrak{g}^{\mathbb{C}}, K)$-module is irreducible.*

Proof. Clearly if a $(\mathfrak{g}, K)$-sub-module is stable under $\mathfrak{g}^{\mathbb{C}}$, it is stable under the Lie sub-algebra $\mathfrak{g}$.

If a $(\mathfrak{g}, K)$-sub-module is stable under $\mathfrak{g}$ then equation 3.3.2 shows that it must be stable under $\mathfrak{g}^{\mathbb{C}}$.

Therefore, a $(\mathfrak{g}, K)$-module has a non-trivial $\mathfrak{g}$-invariant subspace if and only if it has a non-trivial $\mathfrak{g}^{\mathbb{C}}$ invariant subspace. QED

This shows that to classify the isomorphism classes of irreducible admissible $(\mathfrak{g}, K)$-modules it is enough to classify the isomorphism classes $(\mathfrak{g}^{\mathbb{C}}, K)$-modules, their corresponding complexifications.

3.4 Universal Enveloping Algebra

In this section we introduce the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ where $\mathfrak{g}$ is any Lie algebra. This is an associative algebra which contains the Lie algebra, yet has equivalent representation theory to the Lie algebra. The reason why we want to consider the universal enveloping algebra is because we want to study the center of $\mathcal{U}(\mathfrak{g})$. The Lie algebra also has a notion of a center, however for $\mathfrak{sl}_2(\mathbb{R})$ the center is trivial. The center of $\mathcal{U}(\mathfrak{g})$ has more structure.

In this chapter our goal is to construct an algebra that will naturally contain $\mathfrak{g}$ with equivalent representation theory. To do this, we would like to be able to multiply elements of $\mathfrak{g}$; however, Lie algebras are vector spaces, so don't come equipped with a multiplication operation. We are concerned with the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ which are traceless matrices and these do have a multiplication operation. However $\mathfrak{sl}_2(\mathbb{R})$ is not closed under matrix multiplication since the product of two traceless matrices is not in general traceless. For this reason, we will construct an algebra which is larger than $\mathfrak{g}$ but naturally contains it. The first natural attempt to add a multiplication is to try a "dumb" multiplication where we "formally" multiply elements of $\mathfrak{g}$ together. This leads us to the definition of the tensor algebra of $\mathfrak{g}$.

Definition 3.4.1. *For $k \in \mathbb{N}$, define $T^k(\mathfrak{g}) := \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{k \text{ times}}$. Let $\mathfrak{g}$ be a Lie algebra which is a vector space over $\mathbb{F} = (\mathbb{C}$ or $\mathbb{R})$. The* **Tensor algebra** *of $\mathfrak{g}$ is*

$$\begin{aligned} T(\mathfrak{g}) &:= \mathbb{F} \oplus \bigoplus_{n=1}^{\infty} T^n(\mathfrak{g}) \\ &= \mathbb{F} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \dots . \end{aligned}$$

This is an algebra with multiplication given by the tensor product.

The above definition satisfies one of our requirements because $T(\mathfrak{g})$ is an algebra which contains $\mathfrak{g}$. However the definition does not address the bracket structure of $\mathfrak{g}$ in any way so we shouldn't expect it to have the same representation theory.

We improve the previous definition by introducing the universal enveloping algebra.

Definition 3.4.2. *Take the set*

$$S = \{x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g}\}$$

and define $J = \langle S \rangle$ to be the two-sided ideal generated by S in $T(\mathfrak{g})$. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ *of a Lie algebra $\mathfrak{g}$ is*

$$\mathcal{U}(\mathfrak{g}) := \frac{T(\mathfrak{g})}{J}.$$

Theorem 3.4.3. (Poincare-Birkoff-Witt) *Let $X_1, \ldots, X_m$ be a basis for the Lie algebra $\mathfrak{g}$. Then*

$$\{X_1^{i_1} X_2^{i_2} \cdots X_m^{i_m} : i_1, \ldots, i_n \geq 0\}$$

is a basis for $\mathcal{U}(\mathfrak{g})$.

Notice that the universal enveloping algebra is a much larger object than the corresponding Lie algebra. For instance, the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is 3 dimensional as a vector space but the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{R}))$ is infinite dimensional.

We will now show how the representation theory of Lie algebras are equivalent to the representation theory of the corresponding universal enveloping algebras.

Let $\mathfrak{g}$ be a Lie algebra and (π, V) be a representation of the corresponding universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (as a module over the underlying ring).

For all $x, y \in \mathfrak{g}$,

$$\pi([x, y]) = \pi(x \otimes y - y \otimes x).$$

Since π is an algebra representation,

$$\begin{aligned} \pi([x, y]) &= \pi(x \otimes y) - \pi(y \otimes x) \\ &= \pi(x)\pi(y) - \pi(y)\pi(x) \end{aligned}$$

Therefore, when restricted to $\mathfrak{g}$, (π, V) also defines a representation of $\mathfrak{g}$.

Now on the other hand if we start with a representation (π, V) of $\mathfrak{g}$, we can extend π to the elements of $\mathcal{U}(\mathfrak{g})$.

We now want to define $\tilde{\pi} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$. To do this, we will define $\tilde{\pi}$ on elements of $T(\mathfrak{g})$ and see that this map is zero on the ideal J as defined above. This will show that $\tilde{\pi}$ is indeed well defined on $\mathcal{U}(\mathfrak{g})$.

To define $\tilde{\pi}$ on elements of $T(\mathfrak{g})$ it is enough to define it on a scalar $\lambda \in \mathbb{R}$ and an element $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ with $x_i \in \mathfrak{g}$ and extend linearly:

$$\begin{aligned} &\tilde{\pi}(\lambda) := \lambda I \\ &\tilde{\pi}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) := \pi(x_1)\pi(x_2) \cdots \pi(x_n). \end{aligned} \tag{3.4.1}$$

Now to show well definedness we need to check that $\tilde{\pi}$ is zero on the ideal J. It is enough to check that it is zero on the generators.

Now,

$$\begin{aligned}\tilde{\pi}\left(x \otimes y - y \otimes x - [x,y]\right) &= \pi(x)\pi(y) - \pi(y)\pi(x) - \pi([x,y]) \\ &= 0\end{aligned}$$

since π is a Lie algebra homomorphism. Thus, $\tilde{\pi}$ is zero on J and thus well defined. By construction $\tilde{\pi}$ is a ring homomorphism and thus defines a representation of $\mathcal{U}(\mathfrak{g})$.

We extend the bracket $[\cdot,\cdot] : \mathfrak{g} \to \mathfrak{g}$ to a bracket of $[\cdot,\cdot] : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ by defining

$$[a,b] := a \otimes b - b \otimes a$$

for $a, b \in \mathcal{U}(\mathfrak{g})$.

Lemma 3.4.4. *For the bracket $[\cdot,\cdot] : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, we have the following relation*

$$[a, b \otimes c] = b \otimes [a,c] + [a,b] \otimes c$$

for all $a, b, c \in \mathcal{U}(\mathfrak{g})$.

Proof. The right hand side of the equation can be expanded as

$$\begin{aligned}b \otimes [a,c] + [a,b] \otimes c &= b \otimes (a \otimes c - c \otimes a) + (a \otimes b - b \otimes a) \otimes c \\ &= b \otimes a \otimes c - b \otimes c \otimes a + a \otimes b \otimes c - b \otimes a \otimes c \\ &= a \otimes b \otimes c - b \otimes c \otimes a \\ &= [a, b \otimes c].\end{aligned}$$

QED

For convenience we will denote the center $\mathfrak{Z}(\mathcal{U}(\mathfrak{g}))$ as $\mathfrak{Z}(\mathfrak{g})$ and when multiplying elements of $\mathcal{U}(\mathfrak{g})$ the tensor product symbol $\otimes$ will be omitted.

Chapter 4

Infinitesimal character

4.1 Diximier's Lemma

Now that we have seen that $\mathcal{U}(\mathfrak{g})$-modules are the same as $\mathfrak{g}$-modules we will establish some facts about $\mathcal{U}(\mathfrak{g})$-modules which demonstrate how their center carries important data about a representation. Lemma 4.1.2 was proven independently, the proof of Lemma 4.1.1 is adapted from Corollary 8.13 of [Kna86] and the rest of the proofs are based on [Mil]. **For the remainder of this thesis,**

$$G = SL_2(\mathbb{R}), \quad \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}), \quad K = SO_2.$$

Recall the definition of the subgroup $P = MAN$ from 2.0.1. Let $\mathfrak{p}, \mathfrak{a}, \mathfrak{n}$ be the Lie algebras of P, A and N respectively

Lemma 4.1.1. (Diximier's Lemma). *Let (π, V) be an irreducible admissible representation of $(\mathfrak{g}^{\mathbb{C}}, K)$. There exists $\chi \in \mathrm{Hom}(Z(\mathfrak{g}^{\mathbb{C}}), \mathbb{C})$ such that*

$$\tilde{\pi}(Z) = \chi(Z)I.$$

The mapping χ is called the **infinitesimal character** *of (π, V).*

This theorem can be seen as a generalization of Schur's Lemma. Before we prove the above theorem we need some auxiliary results.

Lemma 4.1.2. *Let V be a $(\mathfrak{g}^{\mathbb{C}}, K)$-module with K-types V_n. Then for all $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ and $v \in V_n^{\oplus \mathrm{mult}(V_n:V)}$, we have*

$$\tilde{\pi}(z)v \in V_n^{\oplus \mathrm{mult}(V_n:V)}$$

and

$$\pi(k)\tilde{\pi}(z) = \tilde{\pi}(z)\pi(k).$$

Proof. For the first claim we have that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$$

where $\mathfrak{k}$ is the Lie algebra of K. By condition 2 in 3.3.1,

$$\begin{aligned}
\tilde{\pi}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}v &= \frac{d}{dt}\exp(t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix})v\Big|_{t=0}\\
&= \frac{d}{dt}\exp\left(\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}\begin{pmatrix}it & 0\\ 0 & -it\end{pmatrix}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}^{-1}\right)v\Big|_{t=0}\\
&= \frac{d}{dt}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{it} & 0\\ 0 & e^{-it}\end{pmatrix}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}^{-1}v\Big|_{t=0}\\
&= \frac{d}{dt}\begin{pmatrix}-ie^{it} & ie^{-it}\\ e^{it} & e^{-it}\end{pmatrix}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}\frac{-1}{2i}v\Big|_{t=0}\\
&= \frac{d}{dt}\begin{pmatrix}-ie^{it}-ie^{-it} & -e^{it}+e^{-it}\\ e^{it}-e^{-it} & -ie^{it}-ie^{-it}\end{pmatrix}\frac{-1}{2i}v\Big|_{t=0}\\
&= \frac{d}{dt}\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}v\Big|_{t=0}\\
&= \frac{d}{dt}e^{int}v|_{t=0}\\
&= ine^{int}v|_{t=0}\\
&= inv. \qquad (4.1.1)
\end{aligned}$$

Since for all $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$,

$$\begin{aligned}
\tilde{\pi}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\tilde{\pi}(z)v &= \tilde{\pi}(z)\tilde{\pi}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}v\\
&= \tilde{\pi}(z)inv\\
&= in\tilde{\pi}(z)v
\end{aligned}$$

then by equation (4.1.1) we have that $\tilde{\pi}(z)v \in V_n^{\oplus \operatorname{mult}(V_n:V)}$. So the first claim is proven.

For the second claim, since $\tilde{\pi}(z)v \in V_n^{\oplus \operatorname{mult}(V_n:V)}$, for some $0 \leq \theta < 2\pi$,

$$\begin{aligned}
\pi(k)\tilde{\pi(z)}v &= e^{in\theta}\tilde{\pi}(z)v\\
&= \tilde{\pi}(z)e^{in\theta}v\\
&= \tilde{\pi}(z)\pi(k)v.
\end{aligned}$$

Therefore the claim is true for all $v \in V_n^{\oplus \operatorname{mult}(V_n:V)}$ for all $n \in \mathbb{Z}$ and since these elements are a basis for V, the results holds. *QED*

Now we can prove Diximier's Lemma 4.1.1.

Proof. Let V be an irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module with K-types V_n and $L \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$. From 4.1.2, $V_n^{\oplus \operatorname{mult}(V_n:V)}$ is stable under L. Since V is admissible, $V_n^{\oplus \operatorname{mult}(V_n:V)}$ is finite dimensional. Therefore L has an action on a finite dimensional subspace of V. Thus, L has an eigenvector v because all complex matrices have an eigenvector. Let c be the eigenvalue of v.

The map $L - cI$ has non-trivial kernel and is also in the center. This kernel is also $\mathfrak{g}$ stable since if w is in the kernel, then for all $X \in \mathfrak{g}$,

$$X((L - cI)w) = 0$$

so

$$(L - cI)Xw = 0$$

and Xw is in the kernel. But the original space is assumed to be irreducible. Thus the $\mathfrak{g}$-stable kernel is actually the whole space and $L - cI$ must be the zero map. Therefore

$$L = cI$$

and all elements of $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ act as scalars.

QED

It should be noted that in this proof we used the fact that each V_n is finite dimensional, which is one of the key points in the definition for admissibility.

We will see that only very few irreducible $(\mathfrak{g}, K)$-modules have the same infinitesimal character.

Definition 4.1.3. *Let*

$$\Xi = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ is the vector space of traceless matrices. The above matrices Γ, Ξ, Π are a basis for $\mathfrak{g}$. This can be seen since they are all linearly independent, any linear combination has zero trace and $\mathfrak{g}$ is three dimensional. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, the above is also a basis for $\mathfrak{g}^{\mathbb{C}}$ as a vector space over the complex numbers

By the Poincare-Birkoff-Witt theorem (3.4.3), a basis for $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ is

$$\{\Gamma^i \Xi^j \Pi^k : i, j, k \geq 0\}.$$

Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebra of A and N respectively.

Lemma 4.1.4. *The Lie algebra $\mathfrak{a}^{\mathbb{C}}$ of A is generated by Ξ and the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ of N is generated by Γ.*

Proof. This is easily seen by computing the Lie algebras of A and N. First, we have that for $t \neq 0$

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

is an arbitrary element of A. Now we compute an element of the tangent space at identity,

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\Big|_{t=0} &= \begin{pmatrix} 1 & 0 \\ 0 & -t^{-2} \end{pmatrix}\Big|_{t=0} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Since A is one dimensional, then $\mathfrak{a}$ is one dimensional so the tangent space must be generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Or equivalently Ξ. An element of N is of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where $t \in \mathbb{R}$. Now compute an element of the tangent space at identity,

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big|_{t=0} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Big|_{t=0} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Similarly, N is one dimensional so $\mathfrak{n}$ is one dimensional and generated by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

or equivalently Γ. Finally, clearly the complexifications are also generated by the same elements as complex vector spaces. *QED*

By the Poincare-Birkoff-Witt theorem (3.4.3), $\mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ has a basis

$$\{\Xi^k : k \geq 0\}.$$

and $\mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ has a basis

$$\{\Gamma^i \Xi^j \Pi^k : i > 0, j, k \geq 0\}.$$

Corollary 4.1.5. $\mathcal{U}(\mathfrak{a}^{\mathbb{C}}) \cap \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) = \{0\}$

Proof. Clearly the Corollary is true by observing the basis for both spaces. *QED*

4.2 The Harish-Chandra homomorphism

In the next section we will prove some results about the center of $\mathcal{U}(\mathfrak{g})$. First we need the following bracket identities for later computations. All proofs are based on [Mil] with missing steps filled in. In particular Lemma 4.2.5.

Lemma 4.2.1.

$$[\Xi, \Gamma^n] = n\Gamma^n, \quad [\Xi, \Pi^n] = -n\Pi^n$$

Proof. We prove by induction in n: For $n = 1$

$$\begin{aligned}[\Xi, \Gamma] &= \Gamma \\ [\Xi, \Pi] &= -\Pi.\end{aligned}$$

Assume the Lemma is true for $n = m - 1$ and prove true for $n = m$:

$$\begin{aligned}[\Xi, \Gamma^m] &= \Xi\Gamma^m - \Gamma^m\Xi \\ &= \Xi\Gamma^{m-1}\Gamma - \Gamma^{m-1}\Xi\Gamma + \Gamma^{m-1}\Xi\Gamma - \Gamma^{m-1}\Gamma\Xi \\ &= [\Xi, \Gamma^{m-1}]\Gamma + \Gamma^{m-1}[\Xi, \Gamma] \\ &= (m-1)\Gamma^m + \Gamma^m = m\Gamma^m\end{aligned}$$

Correspondingly,

$$\begin{aligned}[\Xi, \Pi^m] &= \Xi\Pi^m - \Pi^m\Xi \\ &= \Xi\Pi^{m-1}\Pi - \Pi^{m-1}\Xi\Pi + \Pi^{m-1}\Xi\Pi - \Pi^{m-1}\Pi\Xi \\ &= [\Xi, \Pi^{m-1}]\Pi + \Pi^{m-1}[\Xi, \Pi] \\ &= (-m+1)\Pi^m - \Pi^m = -m\Pi^m.\end{aligned}$$

Thus we are done. *QED*

Let $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$. By Theorem 3.4.3,

$$z = \sum_{j,k,l} a_{j,k,l}\Gamma^j\Xi^k\Pi^l.$$

But because z is in the center,

$$\begin{aligned}0 = [\Xi, Z] &= \sum_{j,k,l} a_{j,k,l}[\Xi, \Gamma^j\Xi^k\Pi^l] \\ &= \sum_{j,k,l} a_{j,k,l}\left([\Xi, \Gamma^j]\Xi^k\Pi^l + \Gamma^j[\Xi, \Xi^k\Pi^l]\right)\end{aligned}$$

by using the bracket identities from Lemma 3.4.4. Therefore, again using the bracket identities,

$$\begin{aligned}0 &= \sum_{j,k,l} a_{j,k,l}\left([\Xi, \Gamma^j]\Xi^k\Pi^l + \Gamma^j\left(\Xi^k[\Xi, \Pi^l] + [\Xi, \Xi^k]\Pi^l\right)\right) \\ &= \sum_{j,k,l} a_{j,k,l}\left([\Xi, \Gamma^j]\Xi^k\Pi^l + \Gamma^j\Xi^k[\Xi, \Pi^l]\right).\end{aligned}$$

Now using the identities from Lemma 4.2.1,

$$0 = \sum_{j,k,l} a_{j,k,l}(j-l)\Gamma^j\Xi^k\Pi^l.$$

Note that when $j \neq l$ we have that $a_{j,k,l} = 0$ by equating with the coefficients with the zero vector on the left hand side of the equation.

Thus we have that for an arbitrary element $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$,

$$z = \sum_{j,k} a_{j,k,j} \Gamma^j \Xi^k \Pi^j.$$

Because $\mathfrak{n}^{\mathbb{C}}$ is generated by Γ, the basis elements of $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ are either in $\mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ or $\mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$. Therefore we have the following Lemma.

Lemma 4.2.2. $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}}) \subset \mathcal{U}(\mathfrak{a}^{\mathbb{C}}) \oplus \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$

Let $\gamma : \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}}) \to \mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ be the projection map from the above direct sum decomposition.

Lemma 4.2.3. *The projection map $\gamma : \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}}) \to \mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ is an algebra homomorphism.*

Proof. Let $z_1, z_2 \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$. Notice that $z - \gamma(z) \in \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ for all $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$. Consider

$$z_1 z_2 - \gamma(z_1)\gamma(z_2) = (z_1 - \gamma(z_1))z_2 + \gamma(z_1)(z_2 - \gamma(z_2)).$$

Since the basis of $\mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ is generated by powers of Ξ, $\mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ is a commutative algebra. Because z_2 is in the center of $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ and $\gamma(z_2) \in \mathcal{U}(\mathfrak{a}^{\mathbb{C}})$ we get

$$\gamma(z_1)(z_2 - \gamma(z_2)) = (z_2 - \gamma(z_2))\gamma(z_1)$$

and so

$$z_1 z_2 - \gamma(z_1)\gamma(z_2) = (z_1 - \gamma(z_1))z_2 + (z_2 - \gamma(z_2))\gamma(z_1) \in \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}}).$$

Thus $z_1 z_2 - \gamma(z_1)\gamma(z_2) \in \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ and

$$\gamma\left(z_1 z_2 - \gamma(z_1)\gamma(z_2)\right) = 0.$$

Because γ is linear and idempotent, we have that

$$\gamma(z_1 z_2) - \gamma(z_1)\gamma(z_2) = 0.$$

and γ is an algebra homomorphism. *QED*

Definition 4.2.4. *The projection map γ is the* **Harish-Chandra** *homomorphism.*

Now we are going to apply some of our work on $(\mathfrak{g}, K)$-modules to principal series representations. Let $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ be the Harish-Chandra module of the principal series representation as defined in 3.2.10. Recall the action of G from 1.3.7. We denote the action of an element $X \in \mathfrak{g}$ (or $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$) on a K-finite vector $f \in V^{(K)}$ as $\tilde{R}(X)f$. Recall by definition for $X \in \mathfrak{g}$,

$$\tilde{R}(X)f = \lim_{t \to 0} \frac{R(\exp(tX))f - f}{t}.$$

Now if we consider the evaluation at the identity of this function, $(\tilde{R}(X)f)(1)$, this is a limit of a sequence of functions in $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$. Because the space $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ is equipped with a "uniform norm", the limit of the sequence of functions also converges pointwise on K. Therefore,

$$\begin{aligned}(\tilde{R}(X)f)(1) &= \lim_{t\to 0}\left(\frac{R(\exp(tX))f - f}{t}\right)(1)\\ &= \lim_{t\to 0}\frac{R(\exp(tX))f(1) - f(1)}{t}.\end{aligned}$$

Here we explicitly used the Banach space structure of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$. This is why we choose this norm in 2.1.3, and why this norm makes the analysis easier.

Lemma 4.2.5. *For all $Y \in \mathcal{U}(\mathfrak{p}^{\mathbb{C}})$,*

$$(\tilde{R}(Y)f)(1) = \tilde{\sigma}_{\epsilon,\lambda}(Y)f(1).$$

Proof. First prove for $Y \in \mathfrak{p}$. Let $f \in \mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$. By definition,

$$\begin{aligned}(\tilde{R}(Y)f)(1) &= \lim_{t\to 0}\frac{R(\exp(tY))f(1) - f(1)}{t}\\ &= \lim_{t\to 0}\frac{f(\exp(tY)) - f(1)}{t}.\end{aligned}$$

Because $\exp(tY) \in P$, by 1.3.1 we get that $f(\exp(tY)) = \sigma_{\epsilon,\lambda}(\exp(tY))f(1)$. Thus we get

$$(\tilde{R}(Y)f)(1) = \lim_{t\to 0}\frac{\sigma_{\epsilon,\lambda}(\exp(tY))f(1) - f(1)}{t}$$

Now, the right hand side of this equation is just the definition of the induced action $\tilde{\sigma}_{\epsilon,\lambda}$ at Y on the vector $f(1)$.

$$(\tilde{R}(Y)f)(1) = \tilde{\sigma}_{\epsilon,\lambda}(Y)f(1).$$

From the way that we define an action of $\mathfrak{p}^{\mathbb{C}}$ in 3.3.2 the result also holds for all $Y \in \mathfrak{p}^{\mathbb{C}}$. Similarly by the way that we define the action of $\mathcal{U}(\mathfrak{p}^{\mathbb{C}})$ in (3.4.1) the result holds for all $Y \in \mathcal{U}(\mathfrak{p}^{\mathbb{C}})$. *QED*

Theorem 4.2.6. *Let $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ be the Harish-Chandra module of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ for all $f \in \mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ and $z \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$*

$$\tilde{R}(z)f = \tilde{\sigma}_{\epsilon,\lambda}(\gamma(z))f.$$

In other words the Harish-Chandra module of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ has infinitesimal character $\tilde{\sigma}_{\epsilon,\lambda} \circ \gamma$.

Before we prove, notice that Diximier's Lemma (4.1.1) assumed that we had an irreducible admissible $(\mathfrak{g}, K)$-module. We have proven that the Harish-Chandra

module of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ is admissible, but it is not in general irreducible. So, the fact that it has an infinitesimal character is not following from Diximier's.

Proof. Let $Y \in \mathfrak{n}$. Since $\mathfrak{n} \subseteq \mathfrak{p}$ by lemma 4.2.5 we have

$$(\tilde{R}(Y)f)(1) = \tilde{\sigma}_{\epsilon,\lambda}(Y)f(1).$$

It is clear from equation (2.2.3) that $\sigma_{\epsilon,\lambda}|_N = \mathrm{id}$ so

$$\begin{aligned}(\tilde{R}(Y)f)(1) &= \lim_{t\to 0} \frac{\sigma_{\epsilon,\lambda}(\exp(tY))f(1) - f(1)}{t} \\ &= \lim_{t\to 0} \frac{\sigma_{\epsilon,\lambda}(f(1) - f(1)}{t} \\ &= 0. \end{aligned} \tag{4.2.1}$$

Now clearly from how we define the action of $\mathfrak{n}^{\mathbb{C}}$ in 3.3.2, the equation (4.2.1) holds for all $Y \in \mathfrak{n}^{\mathbb{C}}$.

Let $z \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}})$. This implies $z - \gamma(z) \in \mathfrak{n}^{\mathbb{C}}\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$, so if $z - \gamma(z) \neq 0$ then

$$z - \gamma(z) = YX$$

for some $Y \in \mathfrak{n}^{\mathbb{C}}$ and $X \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}})$.

$$\begin{aligned}(\tilde{R}(z - \gamma(z))f)(1) &= (\tilde{R}(XY)f)(1) \\ &= (\tilde{R}(Y)(\tilde{R}(X)f))(1) \\ &= 0 \end{aligned} \tag{4.2.2}$$

from equation (4.2.1) since $Y \in \mathfrak{n}^{\mathbb{C}}$. Therefore we have that

$$\begin{aligned}(\tilde{R}(z)f)(1) &= (\tilde{R}(\gamma(z) + z - \gamma(z))f)(1) \\ &= (\tilde{R}(\gamma(z))f)(1) + \tilde{R}(z - \gamma(z))f))(1) \\ &= (\tilde{R}(\gamma(z))f)(1)\end{aligned}$$

and since $\gamma(z) \in \mathcal{U}(\mathfrak{a}^{\mathbb{C}}) \subset \mathcal{U}(\mathfrak{p}^{\mathbb{C}})$, then $\gamma(z) \in \mathcal{U}(\mathfrak{p}^{\mathbb{C}})$ and we have from Lemma 4.2.5

$$\tilde{R}(\gamma(z))f)(1) = \tilde{\sigma}_{\epsilon,\lambda}(\gamma(z))f(1).$$

So, equation (4.2.2) gives

$$(\tilde{R}(z)f)(1) = \tilde{\sigma}_{\epsilon,\lambda}(\gamma(z))f(1).$$

Now for all $k \in K$

$$\begin{aligned}(\tilde{R}(z)f)(k) &= (R(k)\tilde{R}(z)f)(1) \\ &= (\tilde{R}(z)R(k)f)(1)\end{aligned}$$

Since $\pi(k)$ and $\tilde{\pi}(z)$ commute by Lemma 4.1.2,

$$\begin{aligned}(\tilde{R}(z)f)(k) &= \tilde{\sigma}_{\epsilon,\lambda}(\gamma(z))R(k)f(1)\\ &= \tilde{\sigma}_{\epsilon,\lambda}(\gamma(z))f(k)\end{aligned}$$

and by Lemma 2.1.1 we have proven the theorem.

QED

4.3 Casimir element

In this section we will construct an element in the center. This element is known as the Casimir element. We will see that the infinitesimal character almost entirely determines an irreducible admissible representation so having an explicit element of the center will prove very useful in Chapter 5. All proofs in the section are based on [Mil].

Definition 4.3.1. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ admits an action on $\mathfrak{g}^{\mathbb{C}}$ called the* **adjoint representation**. *An element of $A \in \mathfrak{g}^{\mathbb{C}}$ acts on an element $B \in \mathfrak{g}^{\mathbb{C}}$ via*

$$(\operatorname{ad} A)B := [A, B].$$

It should be noted that under this action the center can be equivalently defined as

$$\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}}) = \{z \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) : \operatorname{ad} X z = 0 \text{ for all } X \in \mathfrak{g}^{\mathbb{C}}\ \}.$$

Now, the tensor product $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$ has a $\mathfrak{g}^{\mathbb{C}}$-module structure via the action

$$X \cdot (Y \otimes Z) := [X, Y] \otimes Z + Y \otimes [X, Z] \tag{4.3.1}$$

for $X, Y, Z \in \mathfrak{g}^{\mathbb{C}}$. Note that this action is the same as the adjoint action due to how we defined the bracket on $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ in 3.4.4. In particular elements of $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$ are naturally elements of $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ and the $\mathfrak{g}^{\mathbb{C}}$-module structure 4.3.1 is compatible with the multiplication on $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$.

Additionally, $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ has a $\mathfrak{g}^{\mathbb{C}}$-module structure via the action

$$X \cdot T := -T \circ \operatorname{ad} X + \operatorname{ad} X \circ T$$

for all $X \in \mathfrak{g}^{\mathbb{C}}$ and $T \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$. We will now show that $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, g^{\mathbb{C}})$ and $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$ are isomorphic as $\mathfrak{g}^{\mathbb{C}}$-modules.

Let $\hat{\mathfrak{g}^{\mathbb{C}}}$ denote the vector space dual of $\mathfrak{g}^{\mathbb{C}}$. Corollary 5.5 from chapter 3 of [Lan02] describes the natural isomorphism of vector spaces between $\hat{\mathfrak{g}^{\mathbb{C}}} \otimes \mathfrak{g}^{\mathbb{C}}$ and $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ given by the isomorphism

$$\phi : \hat{\mathfrak{g}^{\mathbb{C}}} \otimes \mathfrak{g}^{\mathbb{C}} \to \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$$

defined by

$$\phi(Y \otimes Z) = A_{Y,Z}$$

for all $Y \in \hat{\mathfrak{g}^{\mathbb{C}}}, Z \in \mathfrak{g}^{\mathbb{C}}$ where $A_{Y,Z} \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ is defined by

$$A_{Y,Z}(X) = Y(X)Z.$$

Since $\mathfrak{g}^{\mathbb{C}}$ is finite dimensional, we can identify $\hat{\mathfrak{g}^{\mathbb{C}}}$ with $\mathfrak{g}^{\mathbb{C}}$ by treating an element in $X \in \mathfrak{g}^{\mathbb{C}}$ as an element in $\hat{\mathfrak{g}^{\mathbb{C}}}$ by

$$X(Y) = \operatorname{tr}(XY) \tag{4.3.2}$$

for all $Y \in \mathfrak{g}^{\mathbb{C}}$. (For context this is slightly different to the natural isomorphism of $\hat{\mathfrak{g}^{\mathbb{C}}}$ and $\mathfrak{g}^{\mathbb{C}}$ in which we take the transpose of X). We now have a linear isomorphism between $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$ and $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$. We are interested in this isomorphism because it turns out that these spaces are also isomorphic as $\mathfrak{g}^{\mathbb{C}}$-modules and it is easier to find an "invariant" element in $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$. This will be the Casamir element that we are looking for.

The mapping $\langle X, Y \rangle \mapsto \operatorname{tr}(XY)$ is called the **Killing form** and is important in the theory of Lie algebras.

Now we will prove that the isomorphism is actually a $\mathfrak{g}^{\mathbb{C}}$-module isomorphism. To prove this, we want to show that

$$X \cdot \phi(Y \otimes Z)(U) = \phi(X \cdot Y \otimes Z)(U)$$

for all $X, U \in \mathfrak{g}^{\mathbb{C}}$.

$$\begin{aligned}
X \cdot \phi(Y \otimes Z)(U) &= X \cdot A_{Y,Z}(U) \\
&= -A_{Y,Z}(\operatorname{ad} X)U + \operatorname{ad} X A_{Y,Z}(U) \\
&= -A_{Y,Z}[X, U] + [X, A_{Y,Z}(U)] \\
&= -\operatorname{tr}(Y[X, U])Z + \operatorname{tr}(YU)[X, Z] \\
&= \operatorname{tr}([Y, X]U)Z + \operatorname{tr}(YU)[X, Z] \\
&= A_{[Y,X],Z}(U) + A_{Y,[X,Z]}(U) \\
&= \phi([Y, X]Z)(U) + \phi(Y[X, Z])(U) \\
&= \phi([Y, X]Z + Y[X, Z])(U) \\
&= \phi(X \cdot [Y, Z])(U).
\end{aligned}$$

Thus, ϕ is a $\mathfrak{g}^{\mathbb{C}}$-module homomorphism.

From Theorem 6.1 of chapter 3 of [Lan02], any basis $\{v_i\}_{i \in I}$for a finite dimensional vector space has a corresponding dual basis $\{\tilde{v_i}\}_{i \in I} \subseteq \hat{\mathfrak{g}^{\mathbb{C}}}$ such that,

$$\tilde{v_i}(v_j) = \delta_{i,j}. \tag{4.3.3}$$

where $\delta_{i,j}$ is the kronecker delta. Recall that we identify $\hat{g}$ with $\mathfrak{g}^{\mathbb{C}}$ by letting elements act via (4.3.2). Thus (4.3.3) can be stated as

$$\operatorname{tr}(\tilde{v_i} v_j) = \delta_{i,j}. \tag{4.3.4}$$

Notice that from (4.3.3), $\tilde{v_i}$ acts on a vector as the projection onto v_i.

Let $\{X_i\}_{i=1}^3$ be a basis for $\mathfrak{g}^{\mathbb{C}}$ and $\{\tilde{X}_i\}_{i=1}^3$ its dual basis. Now for all $Y \in \mathfrak{g}^{\mathbb{C}}$,

$$\phi(\sum_{i=1}^{3} \tilde{X}_i \otimes X_i)(Y) = \sum_{i=1}^{3} \tilde{X}_i(Y) X_i = Y.$$

This implies that

$$\phi(\sum_{i=1}^{3} \tilde{X}_i \otimes X_i) = \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} \tag{4.3.5}$$

and for all $X \in \mathfrak{g}^{\mathbb{C}}$,

$$\begin{aligned} X \cdot \phi(\sum_{i=1}^{3} \tilde{X}_i \otimes X_i) &= X \cdot \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} \\ &= -\mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} \operatorname{ad} X + \operatorname{ad} X \, \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} \\ &= 0. \end{aligned}$$

Thus for all $X \in \mathfrak{g}^{\mathbb{C}}$,

$$X \cdot \sum_{i=1}^{3} \tilde{X}_i \otimes X_i = 0. \tag{4.3.6}$$

However, X is acting via the adjoint action 4.3.1, and the element

$$\sum_{i=1}^{3} \tilde{X}_i \otimes X_i$$

is also naturally an element of $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ and 4.3.6 implies that

$$\operatorname{ad} X(\sum_{i=1}^{3} \tilde{X}_i \otimes X_i) = 0$$

for all $X \in \mathfrak{g}^{\mathbb{C}}$

Therefore $\sum_{i=1}^{3} \tilde{X}_i \otimes X_i \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$.

Recall the basis for $\mathfrak{g}^{\mathbb{C}}$, $\{\Gamma, \Xi, \Pi\}$ as defined in 4.1.3. The corresponding dual basis is

$$\begin{aligned} \tilde{\Gamma} &:= -\Pi \\ \tilde{\Xi} &:= 2\Xi \\ \tilde{\Pi} &:= -\Gamma. \end{aligned}$$

We get that

$$c := \sum_{i=1}^{3} \tilde{X}_i \otimes X_i = 2\Xi^2 - \Gamma\Pi - \Pi\Gamma \in \mathfrak{Z}(\mathfrak{g}^{\mathbb{C}}). \tag{4.3.7}$$

Using the identity $\Gamma\Pi - \Pi\Gamma = -2\Xi$,

$$c = 2\Xi^2 - 2\Xi - 2\Gamma\Pi$$

Since ϕ is an isomorphism, due to (4.3.5) it is clear that the Casimir element c is independent of the choice of basis $\{X_i\}$.

Definition 4.3.2. *We call the element of the center* $c := \sum_{i=1}^{3} \tilde{X}_i \otimes X_i$ *the* **Casimir element** *of* $\mathfrak{g}^{\mathbb{C}}$.

It is clear that

$$\gamma(c) = 2\Xi^2 - 2\Xi. \tag{4.3.8}$$

Chapter 5

The irreducible $(\mathfrak{g}, K)$-modules of $SL_2(\mathbb{R})$

5.1 Classification of $(\mathfrak{g}, K)$-modules

In this section we are going to classify the irreducible $(\mathfrak{g}^{\mathbb{C}}, K)$-modules of $SL_2(\mathbb{R})$. The proofs are based on chapter one of [Vog81].

In this chapter it will become apparent why we complexify $\mathfrak{g}$ and why infinitesimal character is so important.

We introduce a new basis for $\mathfrak{g}^{\mathbb{C}}$ as

$$\begin{aligned} H &= -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ X &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \\ Y &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right). \end{aligned} \tag{5.1.1}$$

The commutation relations are

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{5.1.2}$$

Let (π, V) be an admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module with K-types V_n. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$$

In the proof of Lemma 4.1.2 we deduced equation (4.1.1) which we restate here.

$$\tilde{\pi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v = inv.$$

Therefore,

$$\tilde{\pi}(H)v = \tilde{\pi}\left(-i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = nv.$$

By (5.1.2), we get

$$\begin{aligned}\tilde{\pi}(H)\tilde{\pi}(X)v &= \tilde{\pi}(HX)v \\ &= \tilde{\pi}(XH + 2X)v \\ &= \tilde{\pi}(XH)v + 2\tilde{\pi}(X)v \\ &= (n+2)\tilde{\pi}(X)v.\end{aligned}$$

Thus $\tilde{\pi}(X)v \in V_{n+2}$. Similarly,

$$\begin{aligned}\tilde{\pi}(H)\tilde{\pi}(Y)v &= \tilde{\pi}(HY)v \\ &= \tilde{\pi}(YH - 2Y)v \\ &= \tilde{\pi}(YH)v - 2\tilde{\pi}(Y)v \\ &= (n-2)\tilde{\pi}(Y)v.\end{aligned}$$

So $\tilde{\pi}(Y)v \in V_{n-2}$. We state these facts as the following Lemma.

Lemma 5.1.1. *Let (π, V) be an admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module with K-types V_n. Take $v \in V_n$. We have*

$$\begin{aligned}\tilde{\pi}(H)v &\in V_n \\ \tilde{\pi}(X)v &\in V_{n+2} \\ \tilde{\pi}(Y)v &\in V_{n-2}.\end{aligned}$$

It is now evident why this basis is so nice and why we consider the complexification. In particular we see that X acts by "raising" a vector in V_n to the subspace V_{n+2} and Y acts by "lowering" a vector in V_n to V_{n-2}.

In the last section we found the Casimir element with respect to the basis defined in 4.1.3 in equation (4.3.7). To find the Casimir element with respect to our new basis $\{H, X, Y\}$ we first write the basis from 4.1.3 as

$$\begin{aligned}\Xi &= \frac{1}{2}(X + Y) \\ \Gamma &= \frac{1}{2i}(X - Y - H) \\ \Pi &= \frac{-1}{2i}(X - Y + H).\end{aligned} \tag{5.1.3}$$

Then we expand out equation (4.3.7), to get

$$\begin{aligned}c &= 2\Xi^2 - \Gamma\Pi - \Pi\Gamma \\ &= \frac{1}{2}(X+Y)^2 - (X+Y) - \frac{1}{2}(X - Y - H)(X - Y + H) \\ &= \frac{1}{2}(X+Y)^2 - (X+Y) - \frac{1}{2}((X-Y)^2 - [H, X-Y] - H^2) \\ &= \frac{1}{2}(X+Y)^2 - (X+Y) - \frac{1}{2}((X-Y)^2 - 2X - 2Y - H^2) \\ &= \frac{1}{2}H^2 + YX + XY.\end{aligned}$$

Since c is in the center, $2c+1$ is also in the center. We define the element

$$\begin{aligned}\Omega := 2c+1 &= H^2+1+2XY+2YX\\ &= H^2+2H+1+4YX\\ &= (H-1)^2+4XY\\ &= (H+1)^2+4YX.\end{aligned}$$

From this we also get

$$XY = \frac{1}{4}(\Omega-(H-1)^2) \tag{5.1.4}$$

$$YX = \frac{1}{4}(\Omega-(H+1)^2) \tag{5.1.5}$$

By Theorem 4.1.1 we have that Ω acts on v via a scalar Λ.

Lemma 5.1.2. *Let (π, V) be an irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module with K-types V_n and $v \in V_n$. Set $\lambda \in \mathbb{C}$ to be a square root of Λ giving us*

$$\tilde{\pi}(\Omega)v = \lambda^2 v.$$

We then have

$$\begin{aligned}\tilde{\pi}(XY)v &= \frac{1}{4}(\lambda^2-(n-1)^2)v\\ \tilde{\pi}(YX)v &= \frac{1}{4}(\lambda^2-(n+1)^2)v.\end{aligned}$$

Proof. Using the equations 5.1.4 we have,

$$\begin{aligned}\tilde{\pi}(XY)v &= \tilde{\pi}(\frac{1}{4}(\Omega-(H-1)^2))v\\ &= \frac{1}{4}(\lambda^2-(n-1)^2)v\end{aligned}$$

and

$$\begin{aligned}\tilde{\pi}(YX)v &= \frac{1}{4}(\Omega-(H+1)^2)v\\ &= \frac{1}{4}(\lambda^2-(n+1)^2)v.\end{aligned}$$

QED

Now that we have established some basic facts about how this basis acts, we can classify irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-modules. First we introduce the following Lemma.

Lemma 5.1.3. *If (π, V) is an irreducible representation of $(\mathfrak{g}^{\mathbb{C}}, K)$ with K-types V_n and $v \in V_n$ is non-zero then*

- *If $\tilde{\pi}(YX)v = 0$, then $\tilde{\pi}(X)v = 0$*
- *If $\tilde{\pi}(XY)v = 0$, then $\tilde{\pi}(Y)v = 0$.*

Proof. Assume that $\tilde{\pi}(YX)v = 0$ but $\tilde{\pi}(X)v \neq 0$. Consider the non-trivial vector sub space

$$\text{span}\{\tilde{\pi}(X)^m v\}_{m=1}^{\infty}$$

This space is stable under H and X because H acts by a scalar and X acts by raising. It is also stable under Y which acts by lowering since $\tilde{\pi}(Y)\tilde{\pi}(X)v = 0$. Therefore this space is $\mathfrak{g}$-stable. However, this subspace does not include the non-zero vector $v \in V_n$. This contradicts the irreducibility of (π, V). Therefore we have proved the first claim. The second follows in the same way. *QED*

The next powerful Lemma we present gives us a very explicit description of what irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-modules look like.

Lemma 5.1.4. *Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K)$-module with K-types V_n, λ is defined as in Lemma 5.1.2 and $v \in V_n$ as non-zero. Define inductively for a non-negative integer m,*

$$\begin{aligned} w_n &:= v, \\ w_{n+2(m+1)} &:= \begin{cases} (\frac{2}{\lambda+(n+2m+1)})\tilde{\pi}(X)w_{n+2m} & \text{if } \lambda \neq -(n+2m+1) \\ 0 & \text{if } \lambda = -(n+2m+1) \end{cases}, \\ w_{n-2(m+1)} &:= \begin{cases} (\frac{2}{\lambda-(n-2m-1)})\tilde{\pi}(Y)w_{n-2m} & \text{if } \lambda \neq n-2m-1 \\ 0 & \text{if } \lambda = n-2m-1 \end{cases}. \end{aligned}$$

If $w_j \neq 0$ then,

$$\begin{aligned} \tilde{\pi}(H)w_j &= j \cdot w_j, \\ \tilde{\pi}(X)w_j &= \frac{1}{2}(\lambda + j + 1)w_{j+2}, \\ \tilde{\pi}(Y)w_j &= \frac{1}{2}(\lambda - (j-1))w_{j-2}. \end{aligned} \tag{5.1.6}$$

Clearly the w_j are stable under $\mathfrak{g}$ so the non-zero w_j form a basis for V.

Proof. For the first equation in 5.1.6, w_j is clearly either a multiple of

$$\tilde{\pi}(X)^{\frac{1}{2}(j-n)}v \quad \text{or} \quad \tilde{\pi}(Y)^{\frac{1}{2}(n-j)}v.$$

Since

$$\tilde{\pi}(X)^{\frac{1}{2}(j-n)}v \in V_{2\frac{1}{2}(j-n)+n} = V_j$$

and

$$\tilde{\pi}(Y)^{\frac{1}{2}(n-j)}v \in V_{n-2\frac{1}{2}(n-j)} = V_j$$

then

$$\tilde{\pi}(H)w_j = j \cdot w_j.$$

For the next equation in 5.1.6, first let $j \geq n$ so that $j = n + 2m$ for some $m \geq 0$. Now, assume that $\lambda \neq -(n + 2m + 1)$. By definition,

$$w_{j+2} = \frac{2}{\lambda + (j+1)} \tilde{\pi}(X) w_j.$$

So,

$$\frac{1}{2}(\lambda + (j+1)) w_{j+2} = \tilde{\pi}(X) w_j.$$

Now, if $\lambda = -(n+2m+1)$ then by definition $w_{j+2} = 0$. Then by Lemma 5.1.2

$$\begin{aligned} \tilde{\pi}(XY) w_{n+2m} &= \frac{1}{4}(\lambda^2 - (n+2m+1)^2) w_{n+2m} \\ &= 0. \end{aligned}$$

By Lemma 5.1.3, $\tilde{\pi}(X) w_{n+2m} = 0 = \tilde{\pi}(X) w_j$. So the equation still holds.

Now suppose j is negative, so for some $m \geq 0$, $j = n - 2m - 2$. If $\lambda \neq n - 2m - 1$ then by definition,

$$\tilde{\pi}(X) w_{n-2m-2} = \tilde{\pi}(X)\tilde{\pi}(Y) \frac{2}{\lambda - (n-2m-1)} w_{n-2m}..$$

By lemma 5.1.2 the right hand side is

$$\begin{aligned} \frac{1}{4}(\lambda^2 - (n-2m-1)^2) \frac{2}{\lambda - (n-2m-1)} w_{n-2m} \\ = \frac{1}{2}(\lambda + (n-2m-1)) w_{n-2m} \\ = \frac{1}{2}(\lambda + (j+1)) w_{j+2}. \end{aligned}$$

Finally if $\lambda = n - 2m - 1$ then by definition $w_j = w_{n-2m-2} = 0$ so the equation holds.

The third equation in 5.1.6 follows in the same way as the second equation *QED*

Lemma 5.1.4 almost completely describes the structure of an irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module. The piece that is missing is which of the w_j are zero. Once we know this, we will have a basis and a complete description of how $\mathfrak{g}^{\mathbb{C}}$ acts.

This missing piece is given by the following Lemma.

Lemma 5.1.5. *Take $m \geq 0$,*

1. *$w_{n+2(m+1)} = 0$ if and only if $w_{n+2m} = 0$ or $\lambda^2 = (n+2m+1)^2$.*
2. *$w_{n-2(m+1)} = 0$ if and only if $w_{n-2m} = 0$ or $\lambda^2 = (n-2m-1)^2$.*

Proof. If $w_{n+2m} = 0$ then $w_{n+2(m+1)} = 0$ by definition and if $\lambda = -(n+2m+1)$ then $w_{n+2(m+1)} = 0$ by definition. Now, if $w_{n+2m} \neq 0$ and $\lambda \neq -(n+2m+1)$ then $w_{n+2(m+1)}$ is a non-zero multiple of $\tilde{\pi}(X) w_{n+2m}$.

From Lemma 5.1.3 we have that $\tilde{\pi}(X)w_{n+2m}$ if and only if $\tilde{\pi}(YX)w_{n+2m} = 0$. But by Lemma 5.1.2 $\tilde{\pi}(YX)w_{n+2m} = 0$ if and only if $\frac{1}{4}(\lambda^2-(n+2m+1)^2)w_{n+2m} = 0$. The second claim follows in the same way. This proves the result. *QED*

Lemma 5.1.5 shows how the non-zero w_j as defined in 5.1.4 are completely determined by w_n and λ while Lemma 5.1.4 gives us a very explicit description of the structure of an irreducible admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module. These Lemmas combined gives us both a basis and a description of how $\mathfrak{g}^{\mathbb{C}}$ acts for a given $(\mathfrak{g}^{\mathbb{C}}, K)$-module in terms of a non-zero vector $v \in V_n$ for some $n \in \mathbb{Z}$ and a complex number λ. We now have the following important corollary.

Corollary 5.1.6. *Suppose (π, V) and (π', V') are irreducible admissible $(\mathfrak{g}, K)$-modules. They are isomorphic if and only if*

$$\tilde{\pi}(\Omega) = \tilde{\pi}'(\Omega)$$

and there exists an $n \in \mathbb{Z}$ such that V_n and V'_n are non-zero.

Proof. Under the conditions of Lemma 5.1.6 we can construct the corresponding sets of $\{w_j\}$ from Lemma 5.1.4 for both (π, V) and (π', V'). From Lemma 5.1.5, they will have the same basis since the non-zero w_j depend only on n and λ and similarly $\mathfrak{g}^{\mathbb{C}}$ will act on the basis vectors in the same way. The isomorphism between (π, V) and (π', V') is the obvious one which sends the corresponding basis vectors to each other so this is indeed a $(\mathfrak{g}, K)$-module isomorphism. *QED*

Furthermore Lemma 5.1.4 gives us a way of constructing an admissible $(\mathfrak{g}, K)$-module for a given λ and integer n.

Lemma 5.1.7. *For all $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$, there is an irreducible admissible $(\mathfrak{g}, K)$-module (π, V) such that*

$$\tilde{\pi}(\Omega) = \lambda^2$$

and

$$V_n \neq 0.$$

We will postpone the proof until next section where we will see that all of these $(\mathfrak{g}, K)$-modules do exist by looking at the Harish-Chandra module of the principal series.

Definition 5.1.8. *An integer $n \in \mathbb{Z}$ is called a* **lowest K-type** *of (π, V) if it is a K-type of (π, V) and $|n|$ is minimal.*

Lemma 5.1.9. *Suppose that (π, V) is an irreducible admissible $(\mathfrak{g}, K)$-module of lowest K-type n for $|n| > 1$. Then*

$$\tilde{\pi}(\Omega) = (|n| - 1)^2.$$

In particular, if $|n| > 1$ then (π, V) is uniquely determined by n.

Proof. Suppose first that n is positive. The minimality condition of n implies that $V_{n-2} = \{0\}$. Let $v \in V_n$. This means that

$$\tilde{\pi}(Y)v = 0$$

Therefore,

$$\begin{aligned}\tilde{\pi}(\Omega)v &= \tilde{\pi}((H-1)^2)v + 4\tilde{\pi}(XY)v \\ &= \tilde{\pi}((H-1)^2)v \\ &= (n-1)^2 v\end{aligned}$$

Therefore the lemma is true if n is positive. If n is negative the same argument works. *QED*

Notice that this lemma states that if an irreducible admissible $(\mathfrak{g}, K)$-module has lowest K-type $n \in \mathbb{Z}$ with $|n| > 1$, then it is uniquely determined by n.

Definition 5.1.10. *If (π, V) is an irreducible admissible $(\mathfrak{g}, K)$-module with lowest K-type n where $|n| > 1$, then we call it a* **discrete series representation**. *By Lemma 5.1.9, $\tilde{\pi}(\Omega) = (|n|-1)^2$. We use the notation*

$$X_d(n)$$

to denote this representation.

These are called discrete representations because they are parameterised by a discrete variable n. Clearly since each $X_d(n)$ has a unique lowest K-type, by Lemma 5.1.6, each $X_d(n)$ belongs to its own isomorphism class.

Definition 5.1.11. *If (π, V) is an irreducible admissible $(\mathfrak{g}, K)$-module with lowest K-type μ where $\mu \in \{0, 1, -1\}$ and $\tilde{\pi}(\Omega) = \lambda^2 I$ for some $\lambda \in \mathbb{C}$, then we call it a* **continuous series representation** *and denote this representation by $X_c(\lambda, \mu)$.*

Now, any $X_c(\lambda, 0)$ will not have any K-types in common with $X_c(\lambda, 1)$ or $X_c(\lambda, -1)$ because they have different parity and are irreducible. However, $X_c(\lambda, 1)$ may have K-types in common with $X_c(\lambda, -1)$.

Indeed, consider $X_c(\lambda, 1)$. This means that $w_1 \in V_1$ is non-zero. If $\lambda \neq 0$, then

$$\begin{aligned}\tilde{\pi}(Y)v &= \frac{1}{2}(\lambda - (1-1))w_{-1} \\ &= \frac{1}{2}(\lambda)w_{-1}.\end{aligned}$$

Therefore if $\lambda \neq 0$, V_{-1} is non-empty and by Lemma 5.1.6,

$$X_C(\lambda, 1) \cong X_C(\lambda, -1). \tag{5.1.7}$$

It is easy to see that we get the isomorphism (5.1.7) if and only if $\lambda \neq 0$. Therefore the only isomorphisms between the continuous series representations are

$$X_C(\lambda, 1) \cong X_C(\lambda, -1)$$

if $\lambda \neq 0$.

The two representations $X_c(0,1)$ and $X_c(0,-1)$ are referred to as the **Limits of the discrete series**.

Before we continue we remark that trying to classify irreducible admissible $(\mathfrak{g}, K)$ modules using such an explicit description as given in Lemma 5.1.4 is only a good strategy since $SL_2(\mathbb{R})$ is a basic example. In general describing the structure explicitly this way is far more difficult. The more general strategy to classify $(\mathfrak{g}, K)$-modules is to look inside the principal series representations. In the next section we demonstrate how all irreducible admissible $(\mathfrak{g}, K)$-module embed into the principal series in the case of $SL_2(\mathbb{R})$.

5.2 Principal series embeddings

Now we turn to finding realisations of these $(\mathfrak{g}, K)$-modules. Not only will we prove Lemma 5.1.7 but we will demonstrate the embedding theorem of Casselman, Corollary 5.7 from [Cas89] in the case of $SL_2(\mathbb{R})$. In particular, we will see how all $(\mathfrak{g}, K)$-modules embed into the Harish-Chandra module of the Principal series. This is a phenomenon which is present in the general theory and strengthens the sub-quotient theorem of Harish-Chandra.

Recall from 3.2.10 that the Harish-Chandra module of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ is just the space of K-finite vectors $V^{(K)}$ with the corresponding action of $\mathfrak{g}$ and K denoted by $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$. We have shown in Theorem 2.3.1 that the space of K-finite vectors $V^{(K)}$ of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ is of the form

$$V^{(K)} = \bigoplus_n V_n$$

where if $\epsilon = 1$, n runs over the even integers and $\epsilon = -1$, n runs over the odd integers. We complexify this Harish-Chandra module $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ to get an action of $\mathfrak{g}^{\mathbb{C}}$. Since $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is admissible by 2.2.5, Lemma 5.1.1 holds.

Recall that for an irreducible admissible $(\mathfrak{g}, K)$-module V, we denote by λ the complex number such that $\tilde{\pi}(\Omega) = \lambda^2 I$ (see Lemma 5.1.2). We also used a parameter we referred to as λ when defining the principal series representation in Definition 2.2.6. It is no coincidence that we used the same symbol, we clarify in the following Lemma.

Lemma 5.2.1. *For all* $f \in \mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$,

$$\tilde{R}(\Omega) f = \lambda^2 f.$$

Proof. By Theorem 4.2.6 we have that

$$\begin{aligned} \tilde{R}(\Omega) f &= \tilde{R}(2c+1) f \\ &= \tilde{\sigma}_{\epsilon,\lambda}(\gamma(2c+1)) f. \end{aligned}$$

By (4.3.8),

$$\begin{aligned} \gamma(2c+1) &= 2\gamma(c) + 1 \\ &= 2(2\Xi^2 - 2\Xi) + 1 \end{aligned}$$

So,

$$\begin{aligned}\tilde{\sigma}_{\epsilon,\lambda}(\gamma(2c+1)) &= \tilde{\sigma}_{\epsilon,\lambda}(2(2\Xi^2-2\Xi)+1)\\ &= 2(2\tilde{\sigma}_{\epsilon,\lambda}(\Xi)^2-2\tilde{\sigma}_{\epsilon,\lambda}(\Xi))+1.\end{aligned}$$

Now, for all $v \in V^{(K)}$,

$$\begin{aligned}\tilde{\sigma}_{\epsilon,\lambda}(\Xi)v &= \frac{d}{dt}\sigma_{\epsilon,\lambda}\left(\exp\left(t\Xi\right)\right)v|_{t=0}\\ &= \frac{d}{dt}\sigma_{\epsilon,\lambda}\left(\exp\left(t\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\right)v|_{t=0}\\ &= \frac{d}{dt}\sigma_{\epsilon,\lambda}\begin{pmatrix}e^{\frac{1}{2}t} & 0\\ 0 & e^{\frac{-1}{2}t}\end{pmatrix}v|_{t=0}\\ &= \frac{d}{dt}e^{\frac{1}{2}t(\lambda+1)}v|_{t=0}\\ &= \frac{1}{2}(\lambda+1)e^{\frac{1}{2}t(\lambda+1)}v|_{t=0}\\ &= \frac{1}{2}(\lambda+1)v.\end{aligned}$$

So,

$$\tilde{\sigma}_{\epsilon,\lambda}(\Xi) = \frac{1}{2}(\lambda+1).$$

Therefore

$$\begin{aligned}\tilde{\sigma}_{\epsilon,\lambda}(\gamma(2c+1)) &= 2(2(\frac{1}{2}(\lambda+1))^2-2\frac{1}{2}(\lambda+1)+1)\\ &= \lambda^2\end{aligned}$$

and

$$\tilde{R}(\Omega)f = \lambda^2 f.$$

QED

Now we would like to pick a basis for $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ for which equations 5.1.6 hold. We will then have a $(\mathfrak{g}, K)$-module for which Ω acts by the scalar λ^2. To find the discrete and continuous series representations we just need to find the irreducible subspaces of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ and Lemma 5.1.6 will tell us which irreducible representations they are.

From 1.4.13, each V_n is a one-dimensional span of a function. So we make a basis $\{f_n\}$ of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ by picking each $f_n \in V_n$ such that $f_n(1) = 1$.

Lemma 5.2.2. *The basis $\{f_n\}$ satisfies*

$$\begin{aligned}\tilde{\pi}(H)f_j &= j \cdot f_j, \\ \tilde{\pi}(X)f_j &= \frac{1}{2}(\lambda + j + 1)f_{j+2}, \\ \tilde{\pi}(Y)f_j &= \frac{1}{2}(\lambda - (j-1))f_{j-2}.\end{aligned} \tag{5.2.1}$$

Notice that the equations (5.2.1) are the same as 5.1.6.

Proof. Take some $f_n \in \{f_n\}$. Now,

$$\begin{aligned}X &= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{i}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}H + i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.\end{aligned}$$

First, by Lemma 4.2.5 we have that

$$\begin{aligned}\tilde{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} f_n(1) &= \tilde{\sigma}_{\epsilon,\lambda}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(f_n(1)) \\ &= \frac{d}{dt}\sigma_{\epsilon,\lambda}\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix} f_n(1)|_{t=0} \\ &= \frac{d}{dt}(e^t)^{\lambda+1} f_n(1)|_{t=0} \\ &= (\lambda+1)f_n(1) \\ &= (\lambda+1).\end{aligned}$$

It is clear from 2.2.3 that N acts by the identity. Since $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in \mathfrak{n}$ where $\mathfrak{n}$ is the Lie algebra of N,

$$\begin{aligned}\tilde{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} f_n(1) &= \tilde{\sigma}_{\epsilon,\lambda}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}(f_n(1)) \\ &= \frac{d}{dt}\sigma_{\epsilon,\lambda}\exp\left(t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right) f_n(1)|_{t=0} \\ &= \frac{d}{dt}\sigma_{\epsilon,\lambda} f_n(1)|_{t=0} \\ &= 0.\end{aligned}$$

Thus, since

$$X = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}H + i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

we have

$$\tilde{R}(X)f_n = \frac{1}{2}(\lambda + 1 + n)$$

and since

$$\begin{aligned} Y &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}H - i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} \tilde{R}(Y)f_n &= \frac{1}{2}(\lambda + 1) - \frac{n}{2} - 2) \\ &= \frac{1}{2}(\lambda - (n-1)) \end{aligned}$$

QED

We now have a basis for $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ and a description of how $\mathfrak{g}^{\mathbb{C}}$ acts on this basis. We are left to study the irreducible subspaces of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$. It is easier to do this by looking at diagrams. Consider Figure 5.1 which shows 5.1.6 pictorially when $\epsilon = 1$.

We can see the basis vectors and how H acts by a scalar, X acts by raising and Y acts by lowering. The equations 5.1.6 describe the action of X and Y and we can see that X and Y will act by zero on a vector $f_j \in V_j$ if and only if the equations

$$\lambda - (j-1) = 0 \tag{5.2.2}$$

$$\lambda + j + 1 = 0 \tag{5.2.3}$$

are satisfied. As an example if λ is not an integer then 5.2.2 and 5.2.3 are never satisfied and Y and X never act by zero. So, if λ is not an integer we can see in Figure 5.1 how every basis vector is reachable from every other basis vector through acting by X or Y. In this instance, $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is irreducible.

Since $\epsilon = 1$, the basis of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is given by even indices, so it is clear that equations 5.2.2 and 5.2.3 are satisfied for some even j if and only if λ is an odd integer.

We have the following Lemma.

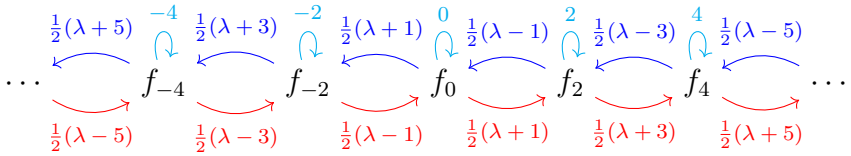

Figure 5.1: Action of X, Y, and H on $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ where $\epsilon = 1$

Lemma 5.2.3. *If $\epsilon = 1$ and λ is not an odd integer, then $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is irreducible and isomorphic to $X_C(\lambda, 0)$. If $\epsilon = -1$ and λ is not an even integer, then $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is irreducible and isomorphic to $X_C(\lambda, 1) \cong X_C(\lambda, -1)$.*

Proof. We have just proven that $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ is irreducible in the case that $\epsilon = 1$ and the irreducibility of $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ if $\epsilon = -1$ follows in the exact same way. From Lemma 5.1.6 we conclude. *QED*

Next, we consider the case when $\epsilon = 1$ and λ is an odd integer. We split into two cases, either λ is positive or λ is negative and Figure 5.3 and Figure 5.2 demonstrate what happens in either case.

In Figure 5.2 we see that since $-\lambda - 1 \geq 0$ and $\lambda + 1 \leq 0$, the space formed by the basis

$$\{f_n : \lambda + 1 \leq n \leq -\lambda - 1\}$$

is irreducible. Therefore we have the following Lemma.

Lemma 5.2.4. *If $\epsilon = 1$ and λ is a negative odd integer, then $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ contains $X_C(\lambda, 0)$ as an irreducible sub-module. If $\epsilon = -1$ and λ is a negative even integer, then $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ contains $X_C(\lambda, 1)$ as an irreducible sub-module.*

Proof. If $\epsilon = 1$, we can see from Figure 5.2 that the space

$$\{f_n : \lambda + 1 \leq n \leq -\lambda - 1\}$$

is irreducible so by 5.1.6 we have the result. The case for $\epsilon = -1$ follows in the same way. *QED*

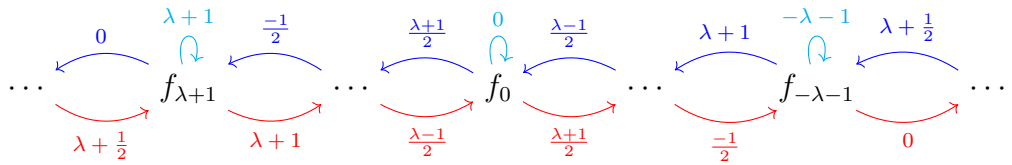

Figure 5.2: Action of X, Y, and H on $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ when $\epsilon = 1$ and λ is odd and negative

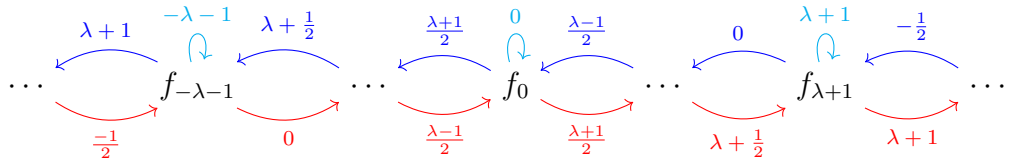

Figure 5.3: Action of X, Y, and H on $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ when $\epsilon = 1$ and λ is odd and positive

Now we turn to when λ is positive. We can see in Figure 5.3 that since Y acts by zero on $f_{\lambda+1}$, the space formed by the basis

$$\{f_n : n \geq \lambda + 1\}$$

is irreducible. Similarly X acts by zero on $f_{-\lambda-1}$ so the space formed by the basis

$$\{f_n : n \leq -\lambda - 1\}$$

is irreducible. Therefore we have the following Lemma

Lemma 5.2.5. *If $\epsilon = 1$ and λ is a positive odd integer, then $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ contains $X_d(\lambda)$ and $X_d(-\lambda)$ as irreducible sub-modules. If $\epsilon = -1$ and λ is a negative even integer, the $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ contains $X_d(\lambda)$ and $X_d(-\lambda)$ as irreducible sub-modules.*

5.3 Unitarity

We conclude this thesis with a brief discussion on unitarity. In general, determining the unitarity of $(\mathfrak{g}, K)$-modules is a difficult problem and so far we only know how to do so through an algorithm. As mentioned in the introduction, there is no simple way of relating the parameters of irreducible admissible representations to whether or not they are unitarity. To put this in context, in 2007 the unitary dual of the Lie Group E_8 was calculated using a super computer. More details on this can be found in [Vog07].

Definition 5.3.1. *Suppose (π, V) is a $(\mathfrak{g}, K)$-module. An invariant Hermitian form on V is a sesquilinear pairing $\langle \cdot, \cdot \rangle$ from V to $\mathbb{C}$ satisfying*

$$\langle \pi(k)v, w \rangle = \langle v, \pi(k)^{-1}w \rangle \tag{5.3.1}$$
$$\langle \tilde{\pi}(X)v, w \rangle = -\langle v, \tilde{\pi}(X)w \rangle \tag{5.3.2}$$

for all $k \in K, X \in \mathfrak{g}$ and $v, w \in V$. The form $\langle \cdot, \cdot \rangle$ is called **positive definite** *if $\langle v, v \rangle$ is a positive real number for every vector $v \in V$.*

If a positive definite invariant Hermitian form exists on V, we say that (π, V) is **infinitesimally unitary**

It is easy to prove that the Harish-Chandra module of every unitary representation is infinitesimally unitary. The inner product from the unitary representation will naturally satisfy both equations (5.3.1) and (5.3.2). The reason why is that equation (5.3.1) just specifies that K acts unitarily. Equation (5.3.2) can be thought of as a "differentiated" unitarity condition for the Lie algebra. We have the following proposition.

Proposition 5.3.2. *([Vog87] Proposition 2.18) Let V be an irreducible admissible $(\mathfrak{g}, K)$-module which is infinitesimally unitary. Then V is the Harish-Chandra module of a unique irreducible unitary representation of G.*

This important result shows how we can test unitarity on the level of $(\mathfrak{g}, K)$-modules. Constructing a Hermitian form is not too difficult and there is more or less a formula to do this. The difficultly part of the problem is to determine whether or not this form is positive definite. Since $SL_2(\mathbb{R})$ is a basic group, its unitarity is well known.

Theorem 5.3.3. *The irreducible unitary $(\mathfrak{g}, K)$-modules of $SL_2(\mathbb{R})$ are*

- *The two "limits of discrete series" representations $X_c(0,1)$, $X_c(0,-1)$.*
- *The irreducible principal series representations $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ where λ is purely imaginary and if $\epsilon = -1$ then $\lambda \neq 0$.*
- *The discrete series representations $X_d(n)$ for $n \in \mathbb{Z}$ with $|n| > 1$.*
- *The "complementary series" $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)}$ with $0 < |\lambda| < 1$ where λ is real and $\epsilon = 1$.*
- *The trivial representation*

The only equivalent $(\mathfrak{g}, K)$-modules in this list are $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})^{(K)} \cong \mathrm{Ind}_P^G(\sigma_{\epsilon,-\lambda})^{(K)}$.

Here the trivial representation is the one-dimensional vector space on which $\mathfrak{g}$ element acts by zero and K acts by the identity.

We can also visualize these results in Figures 5.4 and 5.5.

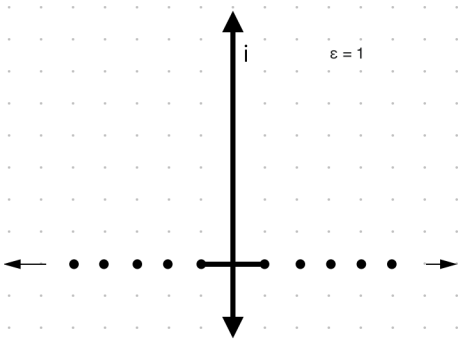

Figure 5.4: Graph of the values λ such that unitary representations embed into $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ when $\epsilon = 1$

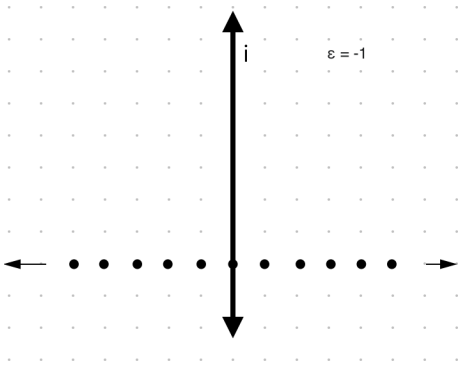

Figure 5.5: Graph of the values λ such that unitary representations embed into $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ when $\epsilon = -1$

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