# The Diagrammatic Spherical Category

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**Abstract**

The diagrammatic Hecke category, constructed by Libedinsky, Khovanov, Elias and Williamson, provides a categorification of the Hecke algebra. In this thesis we present another diagrammatic category which categorifies a certain module over the Hecke algebra, called the spherical module. We call this category the diagrammatic spherical category. Further, we construct a basis for the morphism spaces of this category, called the double-leaves basis.

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# Contents

# Chapter 1

# Introduction

Given a Coxeter system $(W, S)$, we can construct a $\mathbb{Z}[v, v^{-1}]$-algebra called the Hecke algebra. The Hecke algebra is central to many questions in representation theory, including the study of representations of finite and $p$-adic reductive groups. A Hecke category $\mathcal{H}$ is any category which decategorifies to a Hecke algebra. Hecke categories are ubiquitous throughout representation theory. One example is the category $\mathbb{S}$Bim of Soergel bimodules, whose structure captures information about the regular block of category $\mathcal{O}$ [Soe90]. While $\mathbb{S}$Bim is defined algebraically, there are also geometric Hecke categories. For example, take a reductive group $G$ and and Borel subgroup $B \subset G$. There are Hecke categories given by $B$-equivariant semi-simple complexes or parity sheaves on the flag variety $G/B$.

There are two module categories over the Hecke category called the spherical and anti-spherical categories, denoted $\mathcal{M}$ and $\mathcal{N}$. They decategorify to modules over the Hecke algebra, called the spherical and anti-spherical modules $M$ and $N$. Whereas the Hecke category depends only on a Coxeter system $(W, S)$, the spherical and anti-spherical modules and categories depend additionally on a subset $J \subset S$ of generators. The results of this paper require that this $J$ is *finitary*, meaning that $J$ generates a finite subgroup $W_J \leq W$. Like the Hecke category, these two module categories capture crucial representation-theoretic information. Understanding how objects decompose in $\mathcal{N}$ gives us characters of tilting modules for a reductive group $G$ in characteristic $p$ (see Section 1.5 of [LW22]). Simi-

larly, understanding object decompositions in $\mathcal{M}$ gives us characters of simple modules for $G$ [RW21]. This is analogous to how object decompositions in $\mathcal{H}$ give us the decomposition of objects in category $\mathcal{O}$.

We can compute such object decompositions by looking at the morphisms in each category. However, in their algebraic and geometric incarnations, these morphisms are not easy to work with. For this reason, a diagrammatic Hecke category has been constructed by Elias and Williamson [EW16], whose morphisms are drawn diagrammatically and are easier to compute with.

The purpose of this thesis is to construct a diagrammatic spherical category. More precisely, we construct a diagrammatic category $\mathcal{M}_{BS}$, whose Karoubi envelope $\mathcal{M} = Kar(\mathcal{M}_{BS})$ is a spherical category. This was done in type A by Elias [Eli16a], and this thesis extends his work to all types.

The thesis presents two main results. Our first main result is a basis for the morphism spaces of $\mathcal{M}_{BS}$, called the *double-leaves basis*. To be precise, the objects of $\mathcal{M}_{BS}$ are indexed by expressions $\underline{x}$ in the Coxeter system. Given two objects $\underline{x}, \underline{y} \in \mathcal{M}_{BS}$, we present an algorithm which constructs a collection $\mathcal{SDL}_{\underline{x},\underline{y}}$ of *double-leaf maps* in $\mathrm{Hom}_{\mathcal{M}_{BS}}(\underline{x}, \underline{y})$. Let $R := \mathrm{Sym}(\mathfrak{h}^*)$, where $\mathfrak{h}$ is a realization[1] of $(W, S)$. Then we have the following.

**Theorem 1.0.1.** *The collection $\mathcal{SDL}_{\underline{x},\underline{y}}$ is a basis of $Hom_{\mathcal{M}_{BS}}(\underline{x}, \underline{y})$ as a right $R$-module.*

The name "double-leaves" follows the basis of the Hecke category of the same name, constructed by Libedinsky [Lib15] in the algebraic setting, and by Elias and Williamson [EW16] in the diagrammatic setting.

The second result of this thesis is that this diagrammatic spherical category is equivalent to the *algebraic spherical category*. To explain what this algebraic spherical category is, it is necessary to introduce *singular Soergel bimodules*.

[1] We require that our realization is *reflection faithful*, see Section 3.4.

## 1.1 Singular Soergel bimodules

Singular Soergel bimodules were studied by Williamson [Wil11] in his thesis, following earlier work in the direction by Soergel, Stroppel and others. The category $\mathcal{S}\mathbb{S}\mathrm{Bim}$ of singular Soergel bimodules is a 2-category associated to a Coxeter system $(W, S)$. To define it, we first construct the 2-category $\mathcal{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$ of *singular Bott-Samelson bimodules*. Its objects are finitary[2] subsets $J \subset S$. The 1-morphisms in $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(I, J)$ are a certain class of $(J, I)$-bimodules. The 2-morphisms are bimodule morphisms. Then, the category $\mathcal{S}\mathbb{S}\mathrm{Bim}$ has the same objects as $\mathcal{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$, and we set

$$\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(I, J) := Kar(\mathrm{Hom}_{\mathcal{S}\mathbb{B}\mathbb{S}\mathrm{Bim}}(I, J)),$$

where '$Kar$' denotes the Karoubian closure.

Just as the category $\mathbb{S}\mathrm{Bim}$ categorifies the Hecke algebra, the 2-category $\mathcal{S}\mathbb{S}\mathrm{Bim}$ categorifies the *Schur algebroid* $\mathcal{S}$, which can be viewed as a 1-category. The fact that $\mathbb{S}\mathrm{Bim}$ categorifies the Hecke algebra $H$ can be viewed as a special case of this higher categorification. That is, there is a Hom set $\mathrm{Hom}_{\mathcal{S}}(\varnothing, \varnothing) \cong H$ in $\mathcal{S}$, with corresponding Hom category $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, \varnothing) \cong \mathbb{S}\mathrm{Bim}$ in $\mathcal{S}\mathbb{S}\mathrm{Bim}$.

Similarly, there is a Hom set $\mathrm{Hom}_{\mathcal{S}}(\varnothing, J)$ in $\mathcal{S}$ which is isomorphic to the spherical module $M$. Therefore, the corresponding Hom category $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$ in $\mathcal{S}\mathbb{S}\mathrm{Bim}$ categorifies $M$, and is thus a spherical category. We call $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$ the *algebraic spherical category*. The second main result of this thesis is that our diagrammatic and algebraic spherical categories are equivalent as module categories. More precisely, the diagrammatic spherical category $\mathcal{M}$ is a right module category over the diagrammatic Hecke category $\mathcal{H}$. Similarly, the algebraic spherical category $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$ is a right module category over $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, \varnothing) \cong \mathbb{S}\mathrm{Bim}$. Work of Elias and Williamson [EW16] showed that we have an equivalence $\mathcal{H} \cong \mathbb{S}\mathrm{Bim}$. Then we have the following:

**Theorem 1.1.1.** *We have an equivalence of categories $\mathcal{M}_{BS} \xrightarrow{\sim} Hom_{\mathcal{S}\mathbb{B}\mathbb{S}Bim}(\varnothing, J)$. After taking Karoubian closures, this gives us an equivalence between the diagrammatic and*

[2] I.e. the subgroup $W_J \leq W$ generated by $J$ is finite.

*algebraic spherical categories, $\mathcal{M}$ and $Hom_{\mathcal{S}\mathbb{S}Bim}(\varnothing, J)$, as module categories over $\mathcal{H}$ and $\mathbb{S}Bim$ respectively. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{H} & \xrightarrow{-\otimes-} & \mathcal{M} \\ \downarrow & \circlearrowleft & \downarrow \\ Hom_{\mathcal{S}\mathbb{S}Bim}(\varnothing, J) \times \mathbb{S}Bim & \xrightarrow{-\otimes-} & Hom_{\mathcal{S}\mathbb{S}Bim}(\varnothing, J) \end{array}$$

There has been recent major progress by Elias, Ko, Libedinsky and Patimo in constructing a diagrammatic category to model $\mathcal{S}\mathbb{B}\mathbb{S}$Bim. In [EKLP24] the authors define a diagrammatic 2-category **Frob** intended to model $\mathcal{S}\mathbb{B}\mathbb{S}$Bim. In particular, they define an evaluation functor $\mathcal{F} : \mathbf{Frob} \to \mathcal{S}\mathbb{B}\mathbb{S}$Bim. Further, they construct a basis for the 2-morphism spaces of $\mathcal{S}\mathbb{B}\mathbb{S}$Bim. This basis is the image of a collection of double-leaf morphisms in **Frob**, under the functor $\mathcal{F} : \mathbf{Frob} \to \mathcal{S}\mathbb{B}\mathbb{S}$Bim.

When we restrict this result to the Hom category $\mathrm{Hom}_{\mathbf{S}\mathbb{B}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$, we end up with a double-leaves basis for $\mathrm{Hom}_{\mathbf{S}\mathbb{B}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$. This is similar to the main results of this paper. In particular, combining Theorems 1.0.1 and 1.1.1 also gives us a double-leaves basis for $\mathrm{Hom}_{\mathbf{S}\mathbb{B}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$. The addition made in this thesis is that this is also a basis of the diagrammatic category $\mathcal{M}_{BS}$, not just the algebraic category $\mathrm{Hom}_{\mathbf{S}\mathbb{B}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$. It is this fact which is necessary to prove the equivalence in Theorem 1.1.1.

## 1.2 Structure of this thesis

We now describe the structure of the thesis. Chapters 2-5 introduce the necessary background material.

- Chapter 2: We describe Coxeter systems and the Hecke algebra.
- Chapter 3: We describe the categories of Bott-Samelson bimodules and Soergel bimodules associated to a Coxeter system.

- Chapter 4: We describe diagrammatic categories $\mathcal{H}_{BS}$ and $\mathcal{H}$, which are equivalent to the categories of Bott-Samelson bimodules and Soergel bimodules respectively.
- Chapter 5: We give the construction of the double-leaves basis of $\mathcal{H}_{BS}$.

Then, in Chapters 6-11 we move into the world of spherical modules and categories.

- Chapter 6: We describe the spherical module $M$ as well as the category $J\mathbb{BS}\text{Bim}$ of *J-singular Bott-Samelson bimodules*. The idempotent completion $Kar(J\mathbb{BS}\text{Bim})$ of $J\mathbb{BS}\text{Bim}$ is called the *(algebraic) spherical category*, since it categorifies the spherical module.
- Chapter 7: We construct diagrammatic categories $\mathcal{M}_{BS}$ and $\mathcal{M}$ which are equivalent to $J\mathbb{BS}\text{Bim}$ and $Kar(J\mathbb{BS}\text{Bim})$ respectively (though this equivalence will not be proven until the final chapter).
- Chapter 8: We construct various categories of *standard diagrammatics*, and prove equivalences between them. This will assist us in proving linear independence in the following chapter.
- Chapter 9: We construct a double-leaves basis of $\mathcal{M}_{BS}$, and prove that it is linearly independent.
- Chapter 10: We prove that these double-leaves also span, so that they are indeed a basis.
- Chapter 11: We use this basis to prove that the diagrammatic categories $\mathcal{M}_{BS}$ and $\mathcal{M}$ are equivalent to $J\mathbb{BS}\text{Bim}$ and $Kar(J\mathbb{BS}\text{Bim})$ respectively as module categories.

# Chapter 2

# Coxeter Systems and the Hecke Algebra

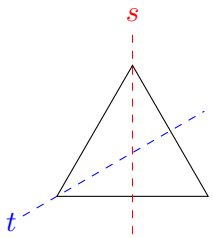
To construct a Hecke algebra, or a category of Soergel bimodules, we need to specify the Coxeter system to which it is associated. Thus Coxeter systems form the ground of all that follows in this thesis.

**Definition 2.0.1.** A *Coxeter system* is a pair $(W, S)$, where $W$ is a group — called a *Coxeter group* — generated by a set $S$, subject to relations

- $s^2 = 1$ for all $s \in S$ (quadratic relation)
- for each pair $s, t \in S$, there is an $m_{st} \in \{2, 3, ...\} \cup \{\infty\}$ such that $(st)^{m_{st}} = id$ (if $m_{st} \neq \infty$). This can also be written $sts\cdots = tst\cdots$, with $m_{st}$ terms on each side (braid relation)

Thus a Coxeter system is specified by a set $S$ of generators — called *simple reflections* — and a choice of $m_{st}$ for each pair $s, t \in S$. We can represent this information as a graph, called a *Coxeter graph*: The vertices are labelled by the generators $s \in S$. Given a pair $s, t \in S$, there is an edge between them if $m_{st} > 2$. If $m_{st} > 3$, this edge is labelled by $m_{st}$.

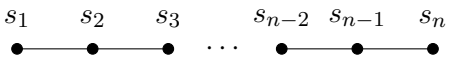
**Example 2.0.2.** (Type $A_2$). Let $W = S_3$, the symmetry group of a triangle. This is generated by a pair of reflections $S = \{s, t\}$, as depicted below:




We can check that $sts = tst$, so that $m_{st} = 3$. Therefore, the Coxeter system $(W, S)$ is represented by the following Coxeter graph.

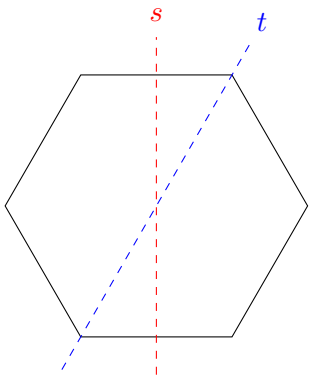

The group $S_3$ generalizes in two ways: to the family of symmetric groups, and the family of dihedral groups. We call the corresponding Coxeter systems type $A$ and type $I_2$, respectively.

**Example 2.0.3** (Type $A$)**.** The Coxeter system of type $A_n$ is the symmetric group $W = S_{n+1}$, generated by $S = \{s_1, ..., s_n\}$, where $s_i$ is the transposition $(i,\ i+1)$. One can check that this is given by the Coxeter graph

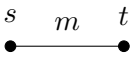



Note that the $n$ in $A_n$ refers to the number of generators, which generate the group $S_{n+1}$.

**Example 2.0.4.** (Type $I_2$) The Coxeter system of type $I_2(m)$ is the dihedral group $Dih_m$ — the symmetries of an $m$-gon — generated by a set $S = \{s, t\}$ of two adjacent reflections. Here is a picture for $m = 6$.




One can check that $st$ gives an anti-clockwise rotation by $\frac{2\pi}{m}$, and thus $(st)^m = id$, so that $m_{st} = m$. Thus the Coxeter graph is




Again, the 2 in $I_2$ refers to the number of generators. We can also set $m = \infty$. In this case there is no relation between $s$ and $t$.

## 2.1 Expressions

Given a Coxeter system $(W, S)$, any element $w \in W$ can be expressed as $w = s_1 s_2 \cdots s_n$, for some sequence of generators $s_i \in S$. In this case we say that the sequence $\underline{w} = (s_1, ..., s_n)$ is an *expression* of $w$. Note that for a given $w \in W$, there is more than one expression $\underline{w}$ of $w$. For example, in type $A_2$ (Example 2.0.2), we have $sts = tst$, and so the element $w = sts$ has two different expressions $(s, t, s)$ and $(t, s, t)$. In fact, $w$ has more than two expressions: $(s, s, s, t, s)$ is also an expression for $w$ (using the quadratic relation). But as this expression is longer than the others, we say it is not *reduced*.

**Definition 2.1.1.** We say an expression $\underline{w} = (s_1, ..., s_n)$ of $w$ is *reduced* if there are no shorter expressions of $w$.

**Definition 2.1.2.** We define a length function $l : W \to \mathbb{N} \cup \{0\}$, where $l(w)$ is the length of a reduced expression of $w$.

We will often abbreviate 'reduced expression' to 'rex'. We have this useful fact about reduced expressions due to Matsumoto.

**Theorem 2.1.3.** *(Matsumoto's theorem) Any two reduced expressions $\underline{w}$, $\underline{w}'$ of some element $w \in W$ are related by a series of applications of the braid relation.*

*Proof.* See [Mat64]. □

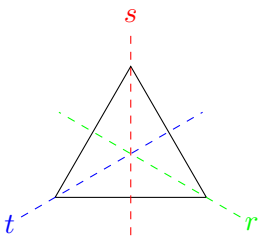
For a simple example, in type $A_2$ the reduced expressions $(s, t, s)$ and $(t, s, t)$ are related by a single application of the braid relation.

This notion of expressions will be necessary in constructing *Bott-Samelson bimodules* in Section 3.6. A Bott-Samelson bimodule will be determined by an expression $\underline{w}$.

## 2.2 The Bruhat order

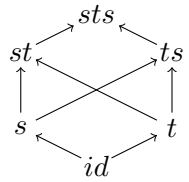
Given $(W, S)$, we will put a partial order on the group $W$, called the *Bruhat order*. Recall that the generators $s \in S$ are called simple reflections. Conjugates of simple reflections are just called *reflections*. They are called reflections because they are of order 2: for $s \in S$, $x \in W$, we have the reflection $xsx^{-1}$. Then $(xsx^{-1})^2 = xsx^{-1}xsx^{-1} = xs^2x^{-1} = xx^{-1} = id$, so $xsx^{-1}$ is of order 2.

Let $T$ denote the set of reflections in $W$. For example, for $W = S_3 = \langle s, t \rangle$, we have two simple reflections $s$ and $t$. But geometrically, there ought to be one more reflection, labelled here as $r$:



This $r$ is in fact the reflection $sts^{-1} = sts$ (using $s = s^{-1}$). So here, $T = \{s, t, sts\}$.

We can define a relation '$\to$' on $W$ as follows: Given $x, y \in W$, we write $x \to y$ if $l(x) < l(y)$, and there exists some reflection $t$ such that $xt = y$.[1] The *Bruhat graph* of $(W, S)$ is the directed graph with vertices the elements of $W$, and an arrow from $x$ to $y$ if $x \to y$. For example, here is the Bruhat graph of $W = S_3 = \langle s, t \rangle$.




**Definition 2.2.1.** The *Bruhat order* is a partial order '$<$' on $W$, defined to be the transitive closure of the relation $\to$.

## 2.3 The descent set

Given a Coxeter system $(W, S)$, and given some $w \in W$ and $s \in S$, we have $l(ws) = l(w) \pm 1$ (see [EMTW20], Proposition 1.41). That is, multiplying by a simple reflection either increases or decreases the length by 1. This means that in the Bruhat order, either $ws < w$ or $ws > w$.

**Definition 2.3.1.** We define the *right descent set* $\mathcal{R}(w) := \{s \in S \mid ws < w\}$. The *left descent set* $\mathcal{L}(w)$ is defined analogously: $\mathcal{L}(w) := \{s \in S \mid sw < w\}$.

**Theorem 2.3.2.** *(The Exchange Condition): Let $(s_1, ..., s_n)$ be a reduced expression for $w \in W$, and suppose $t \in \mathcal{R}(w)$. Then there is some $s_i$ such that $wt = s_1 \cdots \hat{s_i} \cdots s_n$.*

*Proof.* See Exercise 2.19 of [EMTW20]. □

[1] Putting $t$ on the right here is arbitrary; that is, one can check that $xt = t'x$, for some other $t' \in T$

**Corollary 2.3.3.** *For $w \in W$, the descent set $\mathcal{R}(w)$ is equal to the set*

$$\{t \in S \mid w \text{ admits a reduced expression ending in } t\}.$$

*Proof.* See Corollary 1.49 of [EMTW20]. □

## 2.4 The Hecke algebra

Given a Coxeter system $(W, S)$, we can construct a $\mathbb{Z}[v, v^{-1}]$-algebra $H(W, S)$, called the Hecke algebra.

**Definition 2.4.1.** The *Hecke algebra* $H(W, S)$ (or just $H$, when $(W, S)$ is clear from context) is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $\delta_s$ for $s \in S$, and relations

$$\delta_s^2 = 1 + (v^{-1} - v)\delta_s \text{ for all } s \in S, \tag{2.1}$$

$$\delta_s\delta_t\delta_s... = \delta_t\delta_s\delta_t... \text{ for all } s, t \in S, \tag{2.2}$$

where there are $m_{st}$ terms on each side of equation (2.2).

Notice that if we replaced $v$ with 1, this would become the group algebra $\mathbb{Z}[W]$. For this reason, $H(W, S)$ is called a *deformation* of the group algebra.

There is a very simple basis for $H$ called the *standard basis*. It contains a basis element $\delta_w$ for each $w \in W$. When $w$ is a simple reflection, $\delta_w$ is one of the generators introduced above. For a general $w \in W$, choose a reduced expression $(s_1, ..., s_k)$ for $w$, and set $\delta_w := \delta_{s_1}\delta_{s_2} \cdots \delta_{s_k}$. We need to check that this is well-defined, i.e. that it does not depend on our choice of reduced expression for $w$. Let $(t_1, ..., t_k)$ be a different reduced expression for $w$. By Matsumoto's Theorem 2.1.3, we can get from $(s_1, ..., s_k)$ to $(t_1, ..., t_k)$ by a series of braid relations. Equation (2.2) tells us we are allowed to apply these braid relations in the Hecke algebra. Therefore we can also get from $\delta_{s_1} \cdots \delta_{s_k}$ to $\delta_{t_1} \cdots \delta_{t_k}$ using the same series of braid relations. So $\delta_w$ is well-defined for every $w \in W$.

**Proposition 2.4.2.** *The set $\{\delta_w \mid w \in W\}$ is a basis of $H$ as a $\mathbb{Z}[v, v^{-1}]$-module.*

*Proof.* See Theorem 3.5 of [EMTW20]. □

We can check that every basis element $\delta_w$ is invertible. First, for a simple reflection $s \in S$, we have that

$$\begin{aligned}\delta_s \cdot (\delta_s + (v - v^{-1})) &= \delta_s^2 + (v - v^{-1})\delta_s \\ &= 1 + (v^{-1} - v)\delta_s + (v - v^{-1})\delta_s \\ &= 1,\end{aligned}$$

so that $\delta_s$ is invertible, with $\delta_s^{-1} = \delta_s + (v - v^{-1})$. This tells us that any basis element $\delta_w$ is invertible. Writing $\delta_w = \delta_{s_1} \cdots \delta_{s_k}$ for $s_i \in S$, we have $\delta_w^{-1} = \delta_{s_k}^{-1} \cdots \delta_{s_1}^{-1}$.

## 2.5 The Kazhdan-Lusztig basis

There is another basis of $H(W, S)$ of great significance in representation theory, called the *Kazhdan-Lusztig (KL) basis*. Again, we have a basis element $b_w$ for each $w \in W$. In order to define these $b_w$'s, we will need to introduce an involution on $H$.

**Definition 2.5.1.** The *Kazhdan-Lusztig involution* (or *bar involution*) is the $\mathbb{Z}[v, v^{-1}]$-linear map $H \to H : h \mapsto \overline{h}$ which sends $\delta_s \mapsto \delta_s^{-1}$ for $s \in S$, and sends $v \mapsto v^{-1}$. It extends to a map on all of $H$ as a ring automorphism, i.e. $\overline{hk} = \overline{h} \cdot \overline{k}$ for all $h, k \in H$.

**Definition 2.5.2.** Given $x \in W$, we define the Kazhdan-Lusztig basis element $b_x$ to be the unique element of $H$ satisfying:

1. (Self-duality): $\overline{b_x} = b_x$.
2. (Degree bound): $b_x = \delta_x + \sum_{y<x} h_{y,x}\delta_y$ (where $<$ denotes the Bruhat order), for some polynomials $h_{y,x} \in v\mathbb{Z}[v]$.

It is not clear that such elements necessarily exist, or if they do, that they are unique. In fact, for each $x \in W$, the KL-basis element $b_x$ exists and is unique, and moreover (as one

might expect), the collection $\{b_x \mid x \in W\}$ forms a basis of $H$ [KL79]. For a proof of these statements aligned with our notation see Sections 3.2 and 3.3 of [EMTW20].

The above definition of KL-basis elements defines them implicitly; can we see what they look like more explicitly? For simple reflections $s \in S$, it is easy to find $b_s$. We can see that $\delta_s + v$ satisfies the degree bound condition, and we have that

$$\begin{aligned}\overline{\delta_s + v} &= \overline{\delta_s} + \overline{v} \\ &= \delta_s^{-1} + v^{-1} \\ &= \delta_s + (v - v^{-1}) + v^{-1} \\ &= \delta_s + v.\end{aligned}$$

So $\delta_s + v$ is also self-dual, and thus $b_s = \delta_s + v$.

How would we find $b_w$ for general $w$? This is done using an inductive algorithm. For the details of this algorithm, see section 3.3.2 of [EMTW20]. Here we will only show an example to give a sense of how the algorithm works.

**Example 2.5.3.** Take $(W, S)$ to be type $A_2$ (Example 2.0.2), so that $W = S_3$, $S = \{s, t\}$, $m_{st} = 3$. We have six $b_w$'s to find, but we already have three of them: $b_{id} = 1$, $b_s = \delta_s + v$, $b_t = \delta_t + v$. So we need to find $b_{st}$, $b_{ts}$, and $b_{sts}$. To find $b_{st}$, we take the product $b_s b_t$. This equals

$$b_s b_t = (\delta_s + v)(\delta_t + v) = \delta_{st} + v\delta_s + v\delta_t + v^2.$$

This satisfies the degree bound condition, and it is self-dual, since both its factors $b_s$ and $b_t$ are self-dual. So we have found $b_{st} = b_s b_t$. Similarly, $b_{ts} = b_t b_s$. The only difficult one is $b_{sts}$. We start by taking the product $b_s b_t b_s = b_{st} b_s$. This equals

$$\begin{aligned}b_{st} b_s &= (\delta_{st} + v\delta_s + v\delta_t + v^2)(\delta_s + v) \\ &= \delta_{sts} + v\delta_{st} + v + \delta_s + v\delta_{ts} + v^2\delta_t + v^2\delta_s + v^3 \\ &= \delta_{sts} + v\delta_{st} + v\delta_{ts} + v^2\delta_t + (1 + v^2)\delta_s + v + v^3.\end{aligned}$$

This is once again self-dual, but does not quite satisfy the degree bound condition: The coefficients (other than that of $\delta_{sts}$) need to be in $v\mathbb{Z}[v]$, whereas the coefficient $(1 + v^2)$

of $\delta_s$ is not. To rectify this, we subtract off $b_s = \delta_s + v$, leaving

$$b_{sts} = b_s b_t b_s - b_s = \delta_{sts} + v\delta_{st} + v\delta_{ts} + v^2\delta_s + v^2\delta_t + v^3$$

which now satisfies both conditions.

That is the essence of the algorithm: To find $b_w$, where $w = s_1 \cdots s_k$ is a reduced expression, we take $b_{s_1 \cdots s_{k-1}} b_{s_k}$, and then possibly subtract off some $b_x$'s to put all the coefficients in $v\mathbb{Z}[v]$.

## 2.6 Trace and bilinear form on $H$

Here we introduce a form on $H$. This will be important later in giving the dimensions of Hom spaces between Soergel bimodules.

**Definition 2.6.1.** The *standard trace* $\epsilon : H \to \mathbb{Z}[v, v^{-1}]$ is the $\mathbb{Z}[v, v^{-1}]$-linear map on $H$ given by taking the coefficient of $\delta_{id}$. That is, $\epsilon(\sum_{x \in W} c_x \delta_x) := c_{id}$.

**Definition 2.6.2.** Let $i : H \to H$ be the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution defined by $i(\delta_x) := \delta_{x^{-1}}$. We define a $\mathbb{Z}[v, v^{-1}]$-bilinear form $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{Z}[v, v^{-1}]$ by

$$\langle x, y \rangle := \epsilon(i(x)y). \tag{2.3}$$

The standard basis is orthonormal with respect to this form. That is,

$$\langle \delta_x, \delta_y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \tag{2.4}$$

(See Lemma 3.15 of [EMTW20]).

**Remark 2.6.3.** The anti-involution $i$ does not in general fix KL-basis elements. For example, in type $A_2$, $i(b_{st}) = b_{ts}$. However, $i$ does fix the *longest* KL-basis element. Let $w_0$ be the longest element of a finite Coxeter group $W$. Then

$$b_{w_0} = \sum_{x \in W} v^{l(w_0) - l(x)} \delta_x. \tag{2.5}$$

(See Proposition 2.9 of [Soe97]). Then we can see that

$$i(b_{w_0}) = b_{w_0}. \tag{2.6}$$

**Remark 2.6.4.** We will use results about this $\langle \cdot, \cdot \rangle$ from [Wil11]. The form is defined in [Wil11] as $\langle x, y \rangle = \epsilon(x \cdot i(y))$, rather than $\epsilon(i(x) \cdot y)$. However, since both are bilinear forms under which the standard basis is orthogonal, they must be the same. We use this altered definition to give us the most convenient bilinear form on the spherical module (see Section 11).

# Chapter 3

# Soergel Bimodules

We will now describe the category $\mathbb{S}\mathrm{Bim}$ of Soergel bimodules. This gives a categorification of the Hecke algebra $H$.

## 3.1 Graded rings and modules

Before we launch into the machinery involved in Soergel bimodules, let us briefly set out notation for graded rings and modules.

**Definition 3.1.1.** A *graded ring* is a ring $A$ with a decomposition $A = \bigoplus_{i \in \mathbb{Z}} A^i$ into additive subgroups $A^i$, where $A^i A^j \subseteq A^{i+j}$. An element $a \in A^i$ is called *homogeneous of degree* $i$.

A familiar example of a graded ring is a polynomial ring like $A = \mathbb{R}[x_1, ..., x_n]$, where the graded piece $A^i$ is the space of homogeneous degree $i$ polynomials.

**Definition 3.1.2.** Given a graded ring $A$, a *graded module* $M$ over $A$ is an $A$-module which also has a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$ into additive subgroups $M^i$, where $A^i M^j \subseteq M^{i+j}$.

We can easily generalize the above definition to *graded bimodules*. For example, $A$ is a graded bimodule over itself.

We can take *grading shifts* of graded objects: given some graded object $B$ (ring or module), we set $B(n)$ to be the graded object with graded pieces $B(n)^i := B^{n+i}$. So all we are doing here is relabeling the graded pieces: degree $n$ elements of $B$ are considered degree 0 elements of $B(n)$. Given a polynomial $p = \sum_i p_i v^i \in \mathbb{Z}[v, v^{-1}]$, we set

$$B^{\oplus p} := \bigoplus_i B(i)^{\oplus p_i}.$$

In this way, powers of $v$ can encode grading shifts.

**Definition 3.1.3.** A morphism $f : B \to C$ of graded objects is *homogeneous of degree $k$* if $f$ sends $B^i$ to $C^{i+k}$ for all $i$.

Notice that this is exactly the same data as a morphism $f : B \to C(k)$ of degree 0. For this reason, some texts use the convention that morphisms between graded objects are required to be homogeneous of degree 0; we can talk about morphisms $B \to C$ of degree $k$ by talking about morphisms $B \to C(k)$ of degree 0. However, we will not impose that convention here, i.e. in general we will allow morphisms to be homogeneous of degree $k$. This is because not all of our categories allow grading shifts (for example, the category of Bott-Samelson bimodules). We will, however, impose the degree 0 requirement in certain categories (for example, the category of Soergel bimodules).

This leads to the notion of a graded category.

**Definition 3.1.4.** A *graded category* $\mathcal{C}$ is a category whose morphism spaces are graded abelian groups, and moreover composition of morphisms respects this grading; that is, for any objects $A, B, C \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(B, C)^i \circ \mathrm{Hom}_{\mathcal{C}}(A, B)^j \subseteq \mathrm{Hom}_{\mathcal{C}}(A, C)^{i+j}.$$

Here is an example of a graded category:

**Definition 3.1.5.** Given a commutative ring $A$, we let $A$-gBim denote the category of graded $A$-bimodules. We set

$$\mathrm{Hom}_{\text{A-gBim}}(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\text{A-gBim}}(X, Y)^n,$$

where $\mathrm{Hom}_{\text{A-gBim}}(X,Y)^n$ is the space of homogeneous A-bimodule morphisms of degree $n$.

Finally, let us introduce the notion of *graded rank*.

**Definition 3.1.6.** By a *graded free* $A$-module, we mean a graded $A$-module $M$ with a basis of homogeneous elements. If $\{m_i\}$ is a basis of $M$ of homogeneous elements, then we have

$$M \cong \bigoplus_i Am_i \cong \bigoplus_i A(-\deg m_i) \cong A^{\oplus p}$$

for some polynomial $p$. We call this $p$ the *graded rank* of $M$, and denote it by $\mathrm{rk}M$.

One potentially confusing aspect of this graded rank is that the degrees in the basis are the negatives of those in $p$. For example, If $M$ is a graded $A$-module generated by a single element of degree 2, then the graded rank of $M$ is $v^{-2}$. For this reason, many of our later formulae for the graded ranks of certain spaces (in Section 11) have bar involutions showing up in them, since the bar involution takes $v^d$ to $v^{-d}$.

Now let us introduce a particular graded ring.

**Definition 3.1.7.** Given a Coxeter system $(W,S)$, set $R := \mathbb{R}[\alpha_s \mid s \in S]$. That is, $R$ is a polynomial ring with indeterminates $\alpha_s$, one for each simple reflection $s \in S$.

Note that $R$ is a graded ring: it is graded by the degrees of its polynomials. We will use the (possibly surprising) convention that $\deg \alpha_s = 2$; i.e. we double all the degrees[1]. This grading will play a fundamental role in what is to come.

Soergel bimodules are bimodules over this ring $R$. In fact, $R$ itself is our first example of a Soergel bimodule. This $R$ will also be the basic building block in constructing Soergel bimodules: we will be taking certain tensor products of $R$ with itself. But in order to describe these tensor products, we first need to describe an action of $W$ on $R$.

[1]The reason for this is that $R$ manifests as a cohomology ring which is non-zero only in even degrees. See section 1.3 of [EW16]

## 3.2 The geometric representation

We want to describe an action of $W$ on $R$. To do this, it will in fact be enough to describe an action on the monomials of $R$ — that is, on the space

$$V := \mathrm{Span}_{\mathbb{R}}\{\alpha_s \mid s \in S\}$$

— and then extrapolate this action to all of $R$.

**Definition 3.2.1.** Given $(W, S)$, the *geometric representation* of $W$ is the vector space $V := \mathrm{Span}_{\mathbb{R}}\{\alpha_s \mid s \in S\}$, under the following action of $W$. Given $v \in V$ and $s \in S$, we set

$$s(v) := v - 2(v, \alpha_s)\alpha_s,$$

where $(\cdot, \cdot)$ is an inner product determined by

$$(\alpha_s, \alpha_t) := -\cos\frac{\pi}{m_{st}}$$

and extended bilinearly to all of $V$.

Thus, the reflection $s$ acts on $V$ by reflecting across the hyperplane orthogonal to $\alpha_s$. In particular, $s(\alpha_s) = -\alpha_s$.

**Example 3.2.2.** Let $(W, S)$ be of type $A_2$ (example 2.0.2). In this case, $V$ is a 2-dimensional real vector space spanned by $\alpha_s$ and $\alpha_t$. Our inner product gives $(\alpha_s, \alpha_t) = -\cos\frac{\pi}{3} = -\frac{1}{2}$. The reader can verify that the action of $s$ is given by $s(\alpha_s) = -\alpha_s,\ s(\alpha_t) = \alpha_s + \alpha_t$. Similarly, $t(\alpha_s) = \alpha_s + \alpha_t,\ t(\alpha_t) = -\alpha_t$.

What is 'geometric' about this representation? Note that, with respect to our inner product $(\cdot, \cdot)$, $\alpha_s$ and $\alpha_t$ are not orthogonal. In fact they are at an angle of $\frac{2\pi}{3}$ from one another. Therefore, the reflections $s$ and $t$ are also at an angle of $\frac{2\pi}{3}$ from one-another. Thus this represents $S_3$ as the symmetry group of an equilateral triangle. Similarly, one can check that the geometric representation of $Dih_n$ gives the symmetries of an $n$-gon.

Now that we have defined an action of $W$ on the monomials $\alpha_s$, we can easily extend the action to all of $R$ by

$$w(\prod_{s \in S} \alpha_s^{k_s}) := \prod_{s \in S} w(\alpha_s)^{k_s}$$

and extending linearly.

**Definition 3.2.3.** Given $s \in S$, let $R^s \subset R$ denote the ring of invariants of $R$ under the action of $s$. Given $J \subset S$, let $R^J \subset R$ denote the ring of invariants under all of $J$.

For example, $s(\alpha_s^2) = (-\alpha_s)^2 = \alpha_s^2$, so that $\alpha_s^2 \in R^s$. These rings of invariants $R^s$ will be crucial in the construction of Soergel bimodules.

## 3.3 Realizations

The category of Soergel bimodules (as well as the diagrammatic categories we will introduce later) depends on a choice of representation of $W$. We have given the example of the geometric representation above. We will also include some extra data with this representation, so that we have a *realization*.

**Definition 3.3.1.** Let $\Bbbk$ be a commutative integral domain. A *realization* of a Coxeter system $(W, S)$ over $\Bbbk$ is a free finite-rank $\Bbbk$-module $\mathfrak{h}$, together with subsets $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^* := \mathrm{Hom}_{\Bbbk}(\mathfrak{h}, \Bbbk)$, satisfying

1. $\alpha_s(\alpha_s^\vee) = 2$ for all $s \in S$.
2. For $v \in \mathfrak{h}$, the assignment $s(v) := v - \alpha_s(v)\alpha_s^\vee$ gives a representation of $W$.
3. A minor technical condition (see Definitions 3.1 and 3.6 of [EW16]).

Given a realization as above, we can also construct a representation of $W$ on $\mathfrak{h}^*$. For $\gamma \in \mathfrak{h}^*$, set

$$s(\gamma) := \gamma - \gamma(\alpha_s^\vee)\alpha_s.$$

We can recover the geometric representation in exactly this way from the *geometric realization* $\mathfrak{h}_{\mathrm{geom}}$. This is defined as follows: let $\mathfrak{h}_{\mathrm{geom}}$ be the free $\mathbb{R}$-module with basis $\{\alpha_s^\vee\}_{s\in S}$, and with elements $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}_{\mathrm{geom}}^*$ defined by

$$\alpha_s(\alpha_t^\vee) = -2\cos\frac{\pi}{m_{st}}.$$

Then, given a general realization $\mathfrak{h}$, we can set

$$R := \bigoplus_{m \geq 0} S^m(\mathfrak{h}^*),$$

the symmetric algebra on $\mathfrak{h}^*$. We view this as a graded ring with $\mathfrak{h}^*$ in degree 2. When $\mathfrak{h} = \mathfrak{h}_{\text{geom}}$, we get $R$ as defined in Section 3.2.

## 3.4 Reflection faithfulness

Here we address a technical detail: some of our later results require our realization to be *reflection faithful*.

**Definition 3.4.1.** A realization $\mathfrak{h}$ of $(W, S)$ is *reflection faithful* if $\mathfrak{h}$ is a faithful representation, and if the set of reflections in $W$ is in bijection with the codimension 1 hyperplanes of $\mathfrak{h}$ which are fixed by some element of $W$.

Unfortunately, the geometric realization $\mathfrak{h}_{\text{geom}}$ constructed above is not always reflection faithful. For example, let $W$ be the infinite dihedral group $I_2(\infty)$ generated by $s, t$. Then both $s$ and $t$ fix the same hyperplane in $\mathfrak{h}$. For this reason we will sometimes need to use a realization other than the geometric realization $\mathfrak{h}_{\text{geom}}$. In Section 2 of [Soe07], Soergel constructs a reflection faithful representation by mimicking the action of an affine Weyl group on a Cartan subalgebra of an affine Kac-Moody group. We will refer to this reflection faithful realization as $\mathfrak{h}_{KM}$.

Soergel relied on the reflection faithfulness of $\mathfrak{h}_{KM}$ to prove various results in [Soe07] (in particular, Soergel's categorification theorem and Soergel's conjecture). It was later proved by Libedisnky [Lib08] that these results hold for $\mathfrak{h}_{\text{geom}}$ if and only if they hold for $\mathfrak{h}_{KM}$.

We finally require that $\Bbbk$ is infinite.

## 3.5 Demazure operators

We take this opportunity to introduce a collection of operators called *Demazure operators*. These will appear in the construction of our diagrammatic categories in Chapter 4. Take a Coxeter system $(W, S)$ and some $s \in S$. First, we say that $f \in R$ is an *$s$-anti-invariant* if $s(f) = -f$. Note that for any $f \in R$, $f - s(f)$ is an anti-invariant. It is a fact (see Example 4.12 of [EMTW20]) that any $s$-anti-invariant is divisible by $\alpha_s$.[2] Now we can define a Demazure operator $\partial_s : R \to R$ by

$$\partial_s(f) := \frac{f - s(f)}{\alpha_s}. \tag{3.1}$$

Since both the numerator and demoninator are $s$-anti-invariants, we have that $\partial_s(f)$ is an $s$-invariant for any $f \in R$, i.e. the image of $\partial_s$ lies is $R^s$.

## 3.6 Bott-Samelson bimodules and Soergel bimodules

Now we can introduce our first non-trivial Soergel bimodules, known as Bott-Samelson bimodules. The simplest one is

$$B_s := R \otimes_{R^s} R(1)$$

where $R(1)$ is a grading shift of $R$ by 1. This Bott-Samelson bimodule $B_s$ is an $R$-bimodule, and we can put a grading on it in a natural way. Given a simple tensor $p \otimes q \in B_s$, we have $\deg(p \otimes q) = \deg p + \deg q$. For example, $\deg(\alpha_s \otimes \alpha_t) = 2 + 1 = 3$, since $\alpha_t \in R(1)$ has degree $2 - 1 = 1$.

Now, why would we define such a bimodule as $B_s = R \otimes_{R^s} R(1)$? One reason is that $B_s$ behaves very much like our KL-basis element $b_s$ in the Hecke algebra. In particular, note that $b_s^2 = vb_s + v^{-1}b_s$. Similarly:

**Proposition 3.6.1.** *We have that $B_s \otimes_R B_s = B_s(1) \oplus B_s(-1)$.*

[2] This is a generalization of the fact that odd functions in $x$ are divisible by $x$.

Thus the grading shifts behave like powers of $v$; i.e. $B_s(n)$ corresponds to $v^n b_s$ (This "corresponds to" will be made precise in Theorem 3.6.5). To prove Proposition 3.6.1, we will need the following lemma:

**Lemma 3.6.2.** *We have that $R \cong R^s \oplus R^s(-2)$ as $R^s$-bimodules.*

*Proof.* See Examples 4.11 and 4.12 in [EMTW20].

*Proof of Proposition 3.6.1.* We have that

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R(1) \otimes_{R^s} R(1) \\ &\cong R \otimes_{R^s} (R^s(1) \oplus R^s(-1)) \otimes_{R^s} R(1) \\ &\cong (R \otimes_{R^s} R^s(1) \otimes_{R^s} R(1)) \oplus (R \otimes_{R^s} R^s(-1) \otimes_{R^s} R(1)) \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &\cong B_s(1) \oplus B_s(-1) \end{aligned}$$ □

So we have constructed an $R$-bimodule $B_s$ which behaves like the algebra element $b_s$. In fact, we are going to do the same for all KL-basis elements $b_w$; that is, for each $w \in W$, we will construct an $R$-bimodule $B_w$ which "behaves like" $b_w$ (again, this will be made precise later).

For example, let us look again at the case of type $A_2$ in Example 2.5.3. Here we had $b_{st} = b_s b_t$, $b_{ts} = b_t b_s$. So if we are looking for some bimodule $B_{st}$ which corresponds to $b_{st}$, the natural candidate would be $B_s \otimes_R B_t$ (which we denote by $B_s B_t$). Modules of this form — i.e. products of $B_{s_i}$'s — are called *Bott-Samelson bimodules*.

**Definition 3.6.3.** Given an expression $\underline{w} = (s_1, ..., s_n)$, the *Bott-Samelson bimodule* corresponding to $\underline{w}$ is $BS(\underline{w}) := B_{s_1} B_{s_2} \cdots B_{s_n}$. We set $BS(\varnothing) = R$. Given a Coxeter system $(W, S)$, we associate the *category of Bott-Samelson bimodules* $\mathbb{BS}$Bim. This is the full subcategory of $R$-gBim (see definition 3.1.5 ) whose objects are Bott-Samelson bimodules.

Continuing our example of $W = S_3$, we saw that $W$ has six Kazhdan-Lusztig basis elements: $b_{id}, b_s, b_t, b_{st}, b_{ts}$, and $b_{sts}$. We now have $R$-bimodules corresponding to five of these six:

$$\begin{aligned} b_{id} &\sim R \\ b_s &\sim B_s \\ b_t &\sim B_t \\ b_{st} &\sim B_{st} = B_s B_t \\ b_{ts} &\sim B_{ts} = B_t B_s. \end{aligned}$$

All five of these modules are Bott-Samelson bimodules. For example, $B_s = BS((s))$, and $B_{st} = BS((s,t))$. We have one KL-basis element left, however: $b_{sts} = b_s b_t b_s - b_s$. It is unclear that we can model this using a Bott-Samelson bimodule, since we would have to take something like "$B_s B_t B_s - B_s$", where the subtraction doesn't make any sense. However, we can rewrite $b_{sts} = b_s b_t b_s - b_s$ as $b_s b_t b_s = b_{sts} + b_s$. This suggests (correctly) that the Bott-Samelson $B_s B_t B_s$ might decompose as

$$B_s B_t B_s \cong B_{sts} \oplus B_s \tag{3.2}$$

where $B_{sts}$ is some (as yet mysterious) $R$-bimodule corresponding to $b_{sts}$. Thus, if we want to model all KL-basis elements of $W$ using $R$-bimodules, we must take not only Bott-Samelson bimodules, but also direct summands of them. This motivates the following definition:

**Definition 3.6.4.** A *Soergel bimodule* is a direct summand of a finite direct sum of grading shifts of Bott-Samelson bimodules.

We can then consider the category $\mathbb{S}$Bim of Soergel bimodules. Its objects are (of course) Soergel bimodules. Its morphisms are homogeneous $R$-bimodule morphisms of degree 0.[3]

[3]As noted in section 3.1, we now require morphisms to be of degree 0. This is because we now have grading shifts in our category, so we can consider morphisms $B \to C$ of degree $k$ by considering morphisms $B \to C(k)$ of degree 0.

So far we have been speaking only vaguely about the correspondence between the Hecke algebra elements $b_w$ and $R$-bimodules $B_w$. We are now in a position to make this correspondence precise. Let us first define the *split Grothendieck group* $[\mathbb{S}\mathrm{Bim}]_\oplus$ of the category of Soergel bimodules: it is the free abelian group generated by elements $[B]$ for $B \in \mathbb{S}\mathrm{Bim}$, under the relations

$$[B] = [B'] \text{ if } B \cong B';$$
$$[B] = [B'] + [B''] \text{ if } B \cong B' \oplus B''.$$

We can turn $[\mathbb{S}\mathrm{Bim}]_\oplus$ into a ring by setting

$$[B] = [B'][B''] \text{ if } B \cong B' \otimes B''.$$

Finally, we can turn $[\mathbb{S}\mathrm{Bim}]_\oplus$ into a $\mathbb{Z}[v, v^{-1}]$-algebra by setting

$$v^n \cdot [B] := [B(n)].$$

**Theorem 3.6.5.** *(Soergel's Categorification Theorem, [Soe07])*
*Suppose our realization $\mathfrak{h}$ is a reflection faithful realization over an infinite field of characteristic $\neq 2$. Then:*

*(a) For each $w \in W$, there is an indecomposable Soergel bimodule $B_w \in \mathbb{S}Bim$. These $B_w$'s are precisely the indecomposables of $\mathbb{S}Bim$ (up to isomorphism and grading shift).*

*(b) The indecomposable $B_w$ appears as a direct summand of $BS(\underline{w})$ for any reduced expression $\underline{w}$ of $w$. Moreover, all other summands of $BS(\underline{w})$ are $B_x$ for some $x < w$.*

*(c) There is a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism $[\mathbb{S}Bim]_\oplus \xrightarrow{\sim} H$ sending $[B_s] \mapsto b_s$. That is, $\mathbb{S}Bim$ is a categorification of $H$.*

*Proof.* See [Soe07], Satz 1.10, 5.15 and 6.14. □

This map $[\mathbb{S}\mathrm{Bim}]_\oplus \xrightarrow{\sim} H$ is called the *character map*, and will be described explicitly at the end of the next section.

We have by Theorem 3.6.5 (c) above that the indecomposable $B_s$ gets sent to the KL-basis element $b_s$. We might hope that this extends to all elements of $W$; that is, that the character map sends $[B_w] \mapsto b_w$ for all $w \in W$. This is known as Soergel's conjecture.

**Conjecture 3.6.6.** The character map $[\mathbb{S}\text{Bim}]_\oplus \xrightarrow{\sim} H$ of Theorem 3.6.5 (c) sends $[B_w] \mapsto b_w$ for all $w \in W$.

This conjecture was proved by Elias and Williamson in [EW14] in the case that $\Bbbk$ is of characteristic 0. In the case that $\Bbbk$ is of characteristic $p > 0$, Soergel's conjecture fails, and the image of the $[B_w]$'s under the character map is known as the $p$-canonical basis.

## 3.7 Standard bimodules

We have just seen that the Hecke algebra $H$ is categorified by $\mathbb{S}$Bim. Under this correspondence, the KL-basis elements $b_w$ correspond to indecomposables $B_w$ (at least in characteristic 0). Which bimodules correspond to the *standard* basis elements $\delta_w$? The answer is *standard* bimodules, which are denoted $R_w$. These are *not* Soergel bimodules, only $R$-bimodules. The sense in which they "correspond to" the $\delta_w$'s will be made precise shortly. First, we need to define them: standard bimodules $R_w$ are the ring $R$, except with a "twisted" right action. To avoid confusing notation, we will first define this twist for a general $R$-bimodule $M$.

**Definition 3.7.1.** Given an $R$-bimodule $M$ and an element $w \in W$, we define the $R$-bimodule $M_w$ to be the same as $M$, but with a different right action of $R$: for $m \in M_w$, $r \in R$, we have $m \cdot_w r := mw(r)$. Left multiplication of $M_w$ is the same as for $M$.

Applying this to the $R$-bimodule $R$ gives us the standard bimodule $R_w$ (Note that $R_{id} = R$). It is easily seen that $R_x \otimes_R R_y \cong R_{xy}$ via $f \otimes g \mapsto fx(g)$, and with inverse map $f \mapsto f \otimes 1$. This allows us to make the following definition.

**Definition 3.7.2.** We define the monoidal category *Std*Bim as the full subcategory of $R$-gBim (see Definition 3.1.5) consisting of all direct sums of grading shifts of standard

bimodules $R_w$.

Why would we define such objects? Because they are the building blocks of Soergel bimodules. For example, we have the following short exact sequences of $R$-bimodules:

$$0 \to R_s(-1) \xrightarrow{1 \mapsto d_s} B_s \xrightarrow{f \otimes g \mapsto fg} R(1) \to 0, \tag{3.3}$$

$$0 \to R(-1) \xrightarrow{1 \mapsto c_s} B_s \xrightarrow{f \otimes g \mapsto fs(g)} R_s(1) \to 0, \tag{3.4}$$

where $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, $d_s := \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$. From (3.3) we have a filtration $0 \subset R_s(-1) \subset B_s$ of $B_s$, with subquotients $R_s(-1)$ and $R(1)$; i.e. we have a filtration of $B_s$ by standard bimodules (with grading shifts). It turns out (see Section 5.3 of [EMTW20]) that there is a filtration of *any* Soergel bimodule by standard bimodules. In fact, there are many such filtrations for a given Soergel bimodule, but we can choose a canonical one, called the $\Delta$-filtration.

**Definition 3.7.3.** (Definition 5.8 of [EMTW20]) Fix an enumeration $x_0, x_1, ...$ of all elements of $W$ which respects the Bruhat order, i.e. $x_i < x_j$ implies $i < j$. A $\Delta$-*filtration* of a Soergel bimodule $B$ is a filtration $0 = B^k \subset B^{k-1} \subset \cdots \subset B^0 = B$ such that $B^i/B^{i+1} \cong R_{x_i}^{\oplus h_{x_i}}$ for $h_{x_i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

Given a fixed enumeration, this filtration is unique. Moreover the multiplicities $h_x$ of $R_x$ do not depend on the choice of enumeration ( [Soe07], Lemma 6.3). The $\Delta$-filtration gives us the isomorphism $[\mathbb{S}\mathrm{Bim}]_{\oplus} \xrightarrow{\sim} H$: the standard module $R_w$ is sent to $\delta_w$. More precisely, let $B$ be a Soergel bimodule with $\Delta$-filtration $0 = B^k \subset B^{k-1} \subset \cdots \subset B^0 = B$, having subquotients $B^i/B^{i+1} \cong R_{x_i}^{\oplus h_i}$. We define the character map ch: $\mathbb{S}\mathrm{Bim} \to H$ by

$$ch(B) := \sum_{i=0}^{k} v^{l(x_i)} h_{x_i} \delta_{x_i}. \tag{3.5}$$

This is the isomorphism in Theorem 3.6.5 c).

For example, we saw above that we have a filtration $0 \subset R_s(-1) \subset B_s$ of $B_s$, with subquotients $R_s(-1)$ and $R(1)$. We can see that this is the $\Delta$-filtration, so that $h_{id} = v$, $h_s = v^{-1}$. Thus $\mathrm{ch}(B_s) = v \cdot v^{-1}\delta_s + v\delta_{id} = \delta_s + v = b_s$.

## 3.8 Defect and the Hom formula

Given two Soergel bimodules $B$ and $B'$, the Hom space $\mathrm{Hom}_{\mathbb{S}Bim}(B, B')$ is a left (or right) $R$-module. In [Soe07], Soergel showed that this Hom space is a free $R$-module, with graded rank given by the following formula.

**Theorem 3.8.1.** *(Soergel Hom formula) Suppose our realization $\mathfrak{h}$ is a reflection faithful realization over a field of characteristic $\neq 2$. Then for any $B, B' \in \mathbb{S}Bim$, the Hom space $Hom_{\mathbb{S}Bim}(B, B')$ is free as a left (or right) $R$-module, with graded rank given by*

$$rkHom_{\mathbb{S}Bim}(B, B') = \overline{\langle ch(B), ch(B')\rangle},$$

*where $\langle \cdot, \cdot \rangle$ is the bilinear form on $H$ defined in (2.3).*

To see more precisely what this formula is telling us, we need to introduce the notion of *defect*.

Take any expression $\underline{x} = (s_1, ..., s_n)$. A *subexpression* $\underline{e} \subset \underline{x}$ is a binary sequence $(e_1, ..., e_n)$ (so $e_i \in \{0, 1\}$). The idea is that $\underline{x} = (s_1, ..., s_n)$ expresses some element $x$, and that by leaving out some of these $s_i$'s, we express a different element $\underline{x}^{\underline{e}} := s_1^{e_1} \cdots s_n^{e_n}$, which we call the *target* of $\underline{e}$. The '1's in the binary sequence $\underline{e}$ correspond to the $s_i$'s that we leave in, and the '0's to those we leave out.

**Example 3.8.2.** Take $\underline{x} = (s, t, s)$ in $W = S_3$. Then the subexpression $\underline{e} = (0, 1, 1)$ has target $\underline{x}^{\underline{e}} = ts$, while $(1, 0, 1)$ has target $ss = \mathrm{id}$. Note that different subexpressions can express the same group element: the subexpressions $(1, 0, 0)$ and $(0, 0, 1)$ both express $s$.

Take some expression $\underline{x} = (s_1, ..., s_n)$ and a subexpression $\underline{e} \subset \underline{x}$. The *stroll* associated to $\underline{e}$ is the sequence $w_0, w_1, ..., w_n$, where $w_k := s_1^{e_1} \cdots s_k^{e_k}$. So for example, the stroll associated to $(1, 0, 1) \subset (s, t, s)$ would be $\mathrm{id}, s, s, \mathrm{id}$. We can take the stroll $w_0, ..., w_n$ associated to some $\underline{e} \subset \underline{x}$ and decorate it with a sequence $d_1, ..., d_n$, where $d_i \in \{U0, U1, D0, D1\}$. The $U$ and $D$ tell us whether we go up or down in the Bruhat order at that point in the stroll.

The 1 and 0 correspond to the 1 or 0 at that point in $\underline{e}$. More precisely, we have

$$d_i = \begin{cases} Ue_i & \text{if } w_{i-1}s_i > w_{i-1}, \\ De_i & \text{if } w_{i-1}s_i < w_{i-1}. \end{cases}$$

So for $\underline{e} = (1,0,1) \subset (s,t,s)$, we would have $d_1, d_2, d_3 = U1, U0, D1$.

Given a subexpression $\underline{e} \subset \underline{x}$, we define the *defect* of $\underline{e}$ as

$$def(\underline{e}) := \#U0 - \#D0,$$

that is, the number of $U0$'s in its decorated sequence, minus the number of $D0$'s.

Now, we define $b_{\underline{x}} := b_{s_1} b_{s_2} \cdots b_{s_n}$, where $b_{s_i}$ is the KL-basis element corresponding to $s_i$. We have the following lemma.

**Lemma 3.8.3.** *(Deodhar's lemma) For any expression $\underline{x} = (s_1, ..., s_n)$, we have the formula*

$$b_{\underline{x}} = \sum_{\underline{e} \subset \underline{x}} v^{def(\underline{e})} \delta_{\underline{x}^{\underline{e}}}.$$

*Proof.* See Lemma 3.36 in [EMTW20]. □

This means that the coefficient of $\delta_w$ in $b_{\underline{x}}$ is

$$\sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{x}^{\underline{e}} = w}} v^{def(\underline{e})}.$$

When we combine this with the above Hom formula, as well as the orthonormality of the standard basis (2.4), we get

$$\begin{aligned} \overline{\mathrm{rkHom}_{\mathbb{S}\mathrm{Bim}}(BS(\underline{x}), BS(\underline{y}))} &= \langle ch(BS(\underline{x})), ch(BS(\underline{y})) \rangle \\ &= \langle b_{\underline{x}}, b_{\underline{y}} \rangle \\ &= \sum_{w \in W} \Big( \sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{x}^{\underline{e}} = w}} v^{def(\underline{e})} \Big) \Big( \sum_{\substack{\underline{f} \subset \underline{y} \\ \underline{y}^{\underline{f}} = w}} v^{def(\underline{f})} \Big) \\ &= \sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{f} \subset \underline{y} \\ \underline{x}^{\underline{e}} = \underline{y}^{\underline{f}}}} v^{def(\underline{e}) + def(\underline{f})}. \end{aligned}$$

This tells us that there is a basis of $\mathrm{Hom}(BS(\underline{x}), BS(\underline{y}))$ consisting of one morphism for every pair of subexpressions $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $\underline{x}^{\underline{e}} = \underline{y}^{\underline{f}}$. This is exactly the form that our "double-leaves" basis will take.

## 3.9 Localization

The standard bimodules $R_w$ show up as subquotients of Soergel bimodules (see Section 3.7). When we localize everything, the situation becomes even simpler: these subquotients become direct summands. Let $Q$ be the field of fractions of $R$. We can localize the Bott-Samelson bimodule $B_s$ by taking

$$B_s \otimes_R Q = R \otimes_{R^s} R(1) \otimes_R Q \cong R \otimes_{R^s} Q.^4$$

One can show that we have an isomorphism

$$R \otimes_{R^s} Q \cong Q \otimes_{Q^s} Q$$

as $(R, Q)$-bimodules. In fact, we can show a more general isomorphism.

**Lemma 3.9.1.** *Given any expression $\underline{w} = (s_1, ..., s_n)$, we have an isomorphism of $(R, Q)$-bimodules*

$$BS(\underline{w}) \otimes_R Q \cong Q \otimes_{Q^{s_1}} Q \otimes_{Q^{s_2}} \cdots \otimes_{Q^{s_n}} Q.$$

*Proof.* See Lemma 3.20 of [EW16]. □

This means that we can in fact consider $BS(\underline{w}) \otimes_R Q$ to be a $Q$-bimodule. Thus we have a localization functor $\mathbb{BS}\mathrm{Bim} \to Q\text{-Bim}$.

**Definition 3.9.2.** We define $\mathbb{BS}\mathrm{Bim}_Q$ to be the essential image of this localization functor $\mathbb{BS}\mathrm{Bim} \to Q\text{-Bim}$. We define $\mathbb{S}\mathrm{Bim}_Q$ to be the full subcategory of $Q$-Bim consisting of direct summands of direct sums of objects in $\mathbb{BS}\mathrm{Bim}$[5].

[4] We do away with gradings when we localize since $Q$ is not graded.

[5] In the language of Section 4.5, we have $\mathbb{S}\mathrm{Bim}_Q := Kar(\mathbb{BS}\mathrm{Bim}_Q)$.

We can thus extend our localization functor to $\mathbb{S}\text{Bim} \to \mathbb{S}\text{Bim}_Q$. These localized Soergel bimodules are built out of standard $Q$-bimodules. That is, for $w \in W$ we can define the standard $Q$ bimodule $Q_w$ exactly analogously to Definition 3.7.1. The following proposition tells us that when we localize a Soergel bimodule, the standard sub-quotients become direct summands.

**Proposition 3.9.3.** *Let $B$ be a Soergel bimodule with $\Delta$-filtration $0 = B^k \subset B^{k-1} \subset \cdots \subset B^0 = B$, having subquotients $B^i/B^{i+1} \cong R_{x_i}^{h_{x_i}}$. Then*

$$B \otimes_R Q \cong \bigoplus_{i=1}^{n} Q_{x_i}^{h_{x_i}(1)}.$$

*Proof.* See Lemma 6.10 of [Soe07]. □

This is extremely helpful: if we know the image of some Soergel bimodule $B$ under the character map ch: $\mathbb{S}\text{Bim} \to H$, then by (3.5) we know these $h_{x_i}$'s and thus we know how $B \otimes_R Q$ decomposes into standard $Q$-bimodules. For example, $\text{ch}(B_s) = b_s = \delta_s + v$, so that $B_s \otimes_R Q \cong Q_s \oplus Q$. In fact, this lets us easily decompose $BS(\underline{w}) \otimes_R Q$ for any expression $\underline{w}$: Let $\underline{w} = (s_1, ..., s_n)$, and set $b_{\underline{w}} := b_{s_1} \cdots b_{s_n}$. Deodhar's lemma (3.8.3) tells us that

$$b_{\underline{w}} = \sum_{\underline{e} \subset \underline{w}} v^{def(\underline{e})} \delta_{\underline{w}^{\underline{e}}}.$$

We will let $Q_{\underline{e}}$ denote $Q_{\underline{w}^{\underline{e}}}$ throughout the rest of the thesis. Then this formula, combined with Proposition 3.9.3, gives us that

$$BS(\underline{w}) \otimes_R Q \cong \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{e}}. \tag{3.6}$$

For example,

$$B_s B_t B_s \otimes_R Q \cong Q_{sts} \oplus Q_{st} \oplus Q_{ts} \oplus Q_t \oplus Q_s \oplus Q_s \oplus Q \oplus Q.$$

Just as we have localized Bott-Samelson and Soergel bimodules, we can localize standard bimodules. We have an isomorphism of $(R, Q)$-bimodules

$$R_x \otimes_R Q \cong Q_x.$$

This means that, just as with $\mathbb{BS}$Bim and $\mathbb{S}$Bim, we have a localization functor $Std\text{Bim} \to Q\text{-Bim}$.

**Definition 3.9.4.** We define $Std\text{Bim}_Q$ to be essential image of this localization functor $Std\text{Bim} \to Q\text{-Bim}$. It is the full subcategory of $Q$-Bim consisting of direct sums of standard bimodules $Q_x$.

It is clear from the definition of $\mathbb{S}\text{Bim}_Q$ (Definition 3.9.2), as well as Proposition 3.9.3, that $Std\text{Bim}_Q \cong \mathbb{S}\text{Bim}_Q$.

Here we see how localization makes everything simpler: while the category $\mathbb{S}$Bim is quite complicated, the category $\mathbb{S}\text{Bim}_Q \cong Std\text{Bim}_Q$ is quite simple. For example, morphisms in $Std\text{Bim}_Q$ are easy to describe: they are just given by matrices. To see this, note that $\text{Hom}(\bigoplus_i Q_{x_i}, \bigoplus_j Q_{y_j}) \cong \bigoplus_{i,j} \text{Hom}(Q_{x_i}, Q_{y_j})$. If our realization is faithful, then

$$\text{Hom}(Q_x, Q_y) \cong \begin{cases} Q & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \tag{3.7}$$

(See Lemma 5.2 of [EMTW20]). Therefore, morphisms between localized Soergel bimodules are given by matrices with entries in $Q$, and furthermore, most of these entries will be 0 in general. This localization will be our primary tool when we prove the linear independence of the double-leaves basis in Section 9.

# Chapter 4

# Diagrammatics

In moving from the Hecke algebra $H$ to the category $\mathbb{S}$Bim, we now have morphisms between objects. Studying these morphisms can give us extra information. However, writing these morphisms algebraically can quickly get messy and unworkable. Luckily there is an elegant diagrammatic system for representing morphisms between Soergel bimodules: we can draw morphisms rather than writing them.

More precisely, in [EW16] Elias and Williamson construct a diagrammatic category which we will denote $\mathcal{H}$, and which will be equivalent to $\mathbb{S}$Bim[1]. This is constructed through a similar means to $\mathbb{S}$Bim; that is, we first construct a diagrammatic 'Bott-Samelson' category $\mathcal{H}_{BS}$, which is equivalent to $\mathbb{BS}$Bim. We then construct the category $\mathcal{H}$ by in some sense taking "direct summands" of objects in $\mathcal{H}_{BS}$. This process will be made precise in Section 4.5.

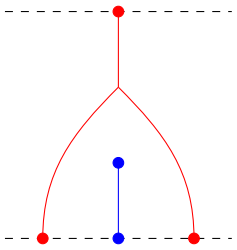
The objects in $\mathcal{H}_{BS}$, just as those of $\mathbb{BS}$Bim, are indexed by expressions $(s_1, ..., s_n)$ in $(W, S)$. To each simple reflection $s_i \in S$ we associate a different colour. Then, the object in $\mathcal{H}_{BS}$ associated to an expression $(s_1, ..., s_n)$ is a sequence of dots of the corresponding colours. We can take the tensor product of two sequences of dots by putting them next to one-another, so that $\mathcal{H}_{BS}$ is a monoidal category.

[1]More precisely, they will be equivalent if our realization is reflection faithful.

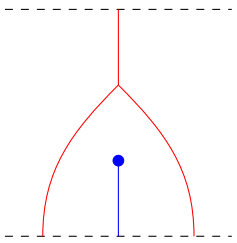
**Example 4.0.1.** We let $s \in S$ be red and $t \in S$ be blue. Then the object in $\mathcal{H}_{BS}$ indexed by the expression $(s, t, s)$ is

We think of this as representing the Bott-Samelson $B_s B_t B_s$. That is, the red dots are $B_s$, the blue dots are $B_t$. We also have an 'empty object' corresponding to the empty expression $\underline{\varnothing}$. We think of this as representing $BS(\underline{\varnothing}) = R$.

We abusively let expressions $\underline{x}$ denote the corresponding objects in $\mathcal{H}_{BS}$. Given two such objects $\underline{x}$ and $\underline{y}$, morphisms $\underline{x} \to \underline{y}$ are given by ($\Bbbk$-linear combinations of) certain coloured string diagrams, with bottom boundary $\underline{x}$ and top boundary $\underline{y}$. For example, here is a morphism $(s, t, s) \to (s)$:



In practice, we don't usually draw the dots on the top and bottom lines:



The blue dot in the above diagram does not represent the object $(t)$; rather, it represents a certain morphism $(t) \to \underline{\varnothing}$ to be described below.

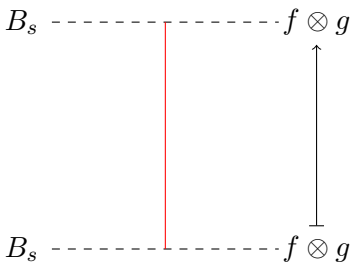
Such diagrammatic morphisms have an algebraic interpretation as morphisms of Bott-Samelson bimodules; that is, we have a monoidal functor $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}\mathrm{Bim}$[2], assigning every diagrammatic morphism to a morphism in $\mathbb{BS}\mathrm{Bim}$. This functor will be an equivalence of monoidal categories.

Now, to finish describing $\mathcal{H}_{BS}$, we need to describe what string diagrams are allowed and what their relations are. We do this first in the case of a single colour.

## 4.1 One-colour diagrammatics

We will describe morphisms in the category $\mathcal{H}_{BS}$ associated to the Coxeter system $(W, S)$ with a single simple reflection, so that $S = \{s\}$ and $W = S_2$. Since we only have one simple reflection $s$, our diagrams will only have one colour (say, red).

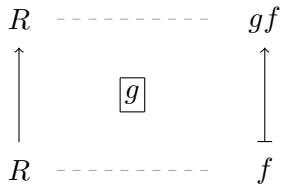
First, the identity on $(s)$ is just given by a single vertical strand:



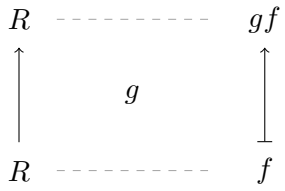

We include alongside these diagrams their algebraic interpretations; that is, their images under the functor $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}\mathrm{Bim}$. The identity on $\underline{\varnothing}$ is given by the empty diagram.


[2]We choose $\mathcal{A}$ for 'algebraic'.

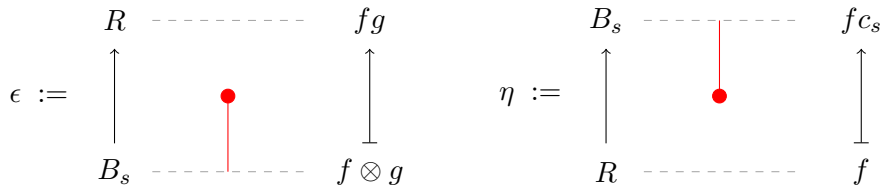
We have a morphism $R \to R$ given by multiplying by a polynomial $g \in R$. This is represented diagrammatically by placing $g$ inside a box.



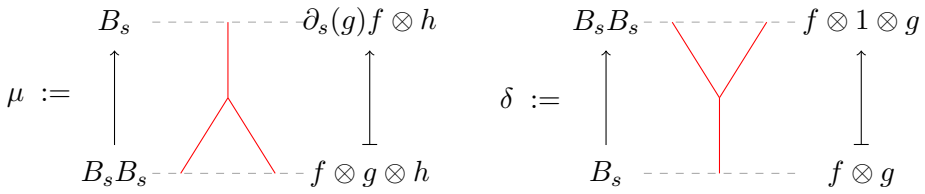

We will occasionally ignore this box to make diagrams less cluttered.




Then, the morphisms in $\mathcal{H}_{BS}$ for one colour are generated (in a way made precise below) by these identity morphisms and polynomials, together with the following four generators:

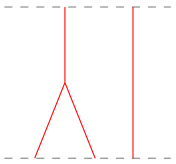



(4.1)

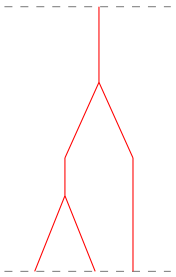



Here, $\partial_s$ is the Demazure operator (see Section 3.5), and $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$. One can check that $fc_s = c_s f$ for all $f \in R$. We call $\epsilon$ an 'up-dot' and $\eta$ a 'down-dot'. We call $\mu$ a 'merging trivalent vertex', and $\delta$ a 'splitting trivalent vertex'. We set the degrees of each of these generators to match that of their algebraic image. That is, $\epsilon$ and $\eta$ each have degree 1, and $\mu$ and $\delta$ each have degree $-1$. Thus, $\mathcal{H}_{BS}$ is a graded category (see Section 3.1). We find the degree of a diagrammatic morphism by adding the degrees of each generator that shows up, as well as any boxed polynomials.

We can combine these generators in a number of ways: First, we can combine them horizontally as tensor products: given two diagrams $f : B \to B'$ and $g : C \to C'$, we can represent the map $f \otimes g : B \otimes B' \to C \otimes C'$ by putting the two diagrams next to each other. For example, take $\mu : B_s \otimes B_s \to B_s$ and $id : B_s \to B_s$. We can combine these horizontally to get $\mu \otimes id : B_s \otimes B_s \otimes B_s \to B_s \otimes B_s$.



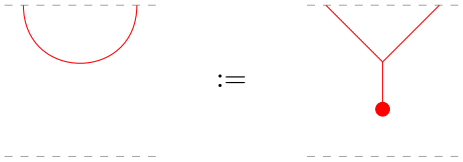
Second, we can combine diagrams vertically. This is just composition of maps. For example, we can compose $\mu \otimes id$ above with $\mu$ to get:



Third, we can add diagrams which have the same source and target. This is just addition of morphisms. For example, we can add $id : B_s \to B_s$ to $\eta \circ \epsilon$.

We represent the morphism $\epsilon \circ \mu$ as a *cap*.

Similarly, we represent $\delta \circ \eta$ as a *cup*.



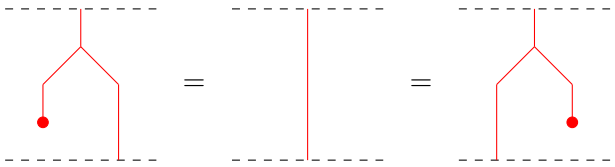
Now comes a crucial point: we consider diagrams to be equivalent up to isotopy. That is, we treat the diagrams as if they were made of elastic string that we can move and stretch. For example, we consider the following diagrams to represent the same morphism in $\mathcal{H}_{BS}$:

Of course, it is not at all trivial that isotopic diagrams represent the same morphism algebraically; that is, that the functor $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}\text{Bim}$ is well-defined. But we will see in Theorem 4.4.1 that it is.

The diagrams in $\mathcal{H}_{BS}$ satisfy certain *local relations*, which can be applied to any part of a diagram, leaving the rest of the diagram unaffected. These relations are what make the diagrammatics so powerful: they allow us to take complicated combinations of morphisms and simplify them easily.

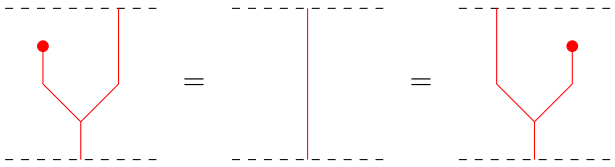
Here is the full set of (one-colour) relations:

**Unit:**



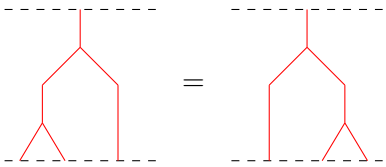
(4.2)

**Counit:**



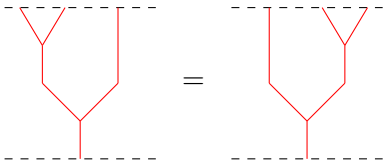
(4.3)

**Associativity:**



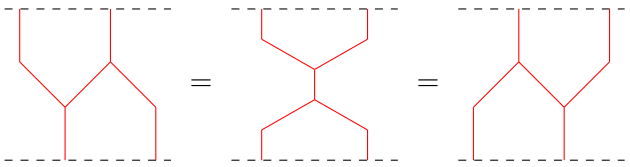
(4.4)

**Coassociativity:**



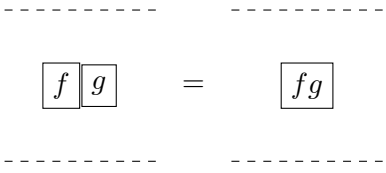
(4.5)

**Frobenius Reciprocity:**



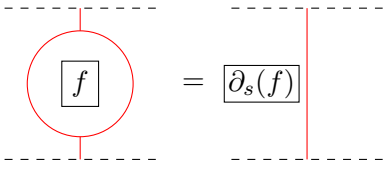
(4.6)

**Multiplication:**



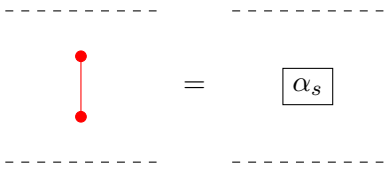

(4.7)

**Keyhole:**



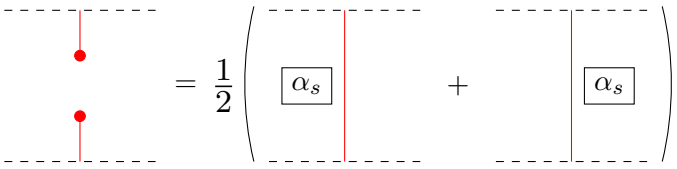

(4.8)

**Barbell:**



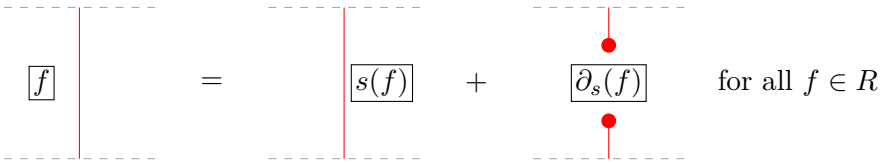

(4.9)

**Fusion:**




(4.10)

**Polynomial Forcing:**




for all $f \in R$

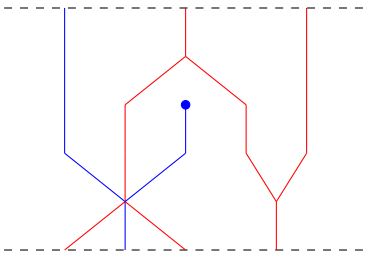
(4.11)

This concludes our presentation of $\mathcal{H}_{BS}$ in one-colour.

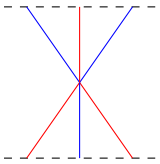
## 4.2 Two-colour diagrammatics

So far we have only been working with one-colour diagrammatics. This one colour stands for one simple reflection $s \in S$, in a Coxeter system $(W, S)$. If we want to work with Coxeter systems with more than one generator (i.e. anything except type $A_1$), we will need to extend our diagrammatic machinery to include these extra generators. In this section we will describe the category $\mathcal{H}_{BS}$ associated to Coxeter systems with two generators $s$ and $t$ (labelled red and blue respectively). These are the Coxeter systems of type $I_2(m)$, with $m = m_{st}$ (see Example 2.0.4). For example, $m_{st} = 3$ gives us our favourite Coxeter group $W = S_3$.

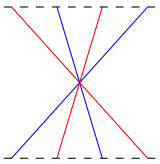
The objects in $\mathcal{H}_{BS}$ will now be sequences of red and blue dots. The morphisms will be string diagrams in two colours, like the one below:



Every part of the above diagram will be familiar from the one-colour diagrammatics, except one: the 6-valent vertex in the lower left:



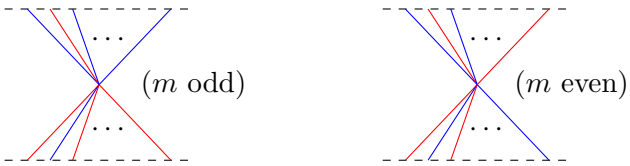
This is called the $2m_{st}$-*valent vertex*. In our example, $m_{st} = 3$, so this is the 6-valent vertex shown above. For $m_{st} = 4$, the 8-valent vertex would look like



(Note that we also have versions of these with the colours switched). In fact, this $2m_{st}$-valent vertex is the only additional generator we need for two-colour diagrammatics. Of course, it brings with it a host of new relations. But before we describe those relations, we will describe the image of the $2m$-valent vertex under $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}\mathrm{Bim}$.

Recall that for each $w \in W$, we have an indecomposable Soergel bimodule $B_w$ (see Theorem 3.6.5). When $W = S_3$, we had in equation (3.2) the decomposition $B_s B_t B_s \cong B_{sts} \oplus B_s$. Similarly, we can swap the $s$'s and $t$'s here to get the decomposition $B_t B_s B_t \cong B_{tst} \oplus B_t$. We see that $B_{sts} = B_{tst}$ is a summand of both $B_s B_t B_s$ and $B_t B_s B_t$. Therefore, we can take the following morphism from $B_s B_t B_s$ to $B_t B_s B_t$: Project onto $B_{sts}$ and then include into $B_t B_s B_t$. That is precisely what the 6-valent vertex represents.

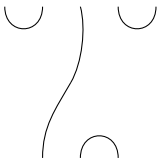
How do we generalize this to other dihedral groups, where $m_{st} \neq 3$? For a given dihedral group $W = Dih_n$, the longest word $w_0$ has two different reduced expressions: $\underline{x} = (s, t, s, ...)$ and $\underline{y} = (t, s, t, ...)$. Theorem 3.6.5 b) tells us that the indecomposable $B_{sts...} = B_{tst...}$ appears as a unique direct summand of both $BS(\underline{x})$ and $BS(\underline{y})$. Therefore we can construct a morphism from $BS(\underline{x})$ to $BS(\underline{y})$ that is projection onto $B_{w_0}$, followed by inclusion into $BS(\underline{y})$. This is the morphism represented by the $2m_{st}$-valent vertex:



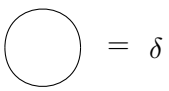

Note that this morphism is not easy to describe explicitly.

This new generator bring new relations. To describe these, we need to introduce the Temperley-Lieb category and Jones-Wenzl projectors.

Let $TL_\delta$ denote the *(generic) Temperley-Lieb category*. Its objects are positive integers $n$, representing a row of $n$ dots. Morphisms $m \to n$ in $TL_\delta$ are crossing-less matches up to isotopy, with $m$ dots on the bottom and $n$ on the top. For example, here is a morphism from 3 to 5.



Composition of morphisms is given by vertical concatenation. Finally, whenever there is a closed circle (a "bubble"), this evaluates to a formal variable $\delta$.

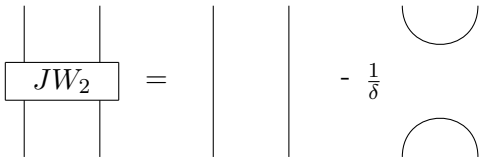


So the category is defined over the base ring $\mathbb{Z}[\delta]$. We can form different variants of the Temperley-Lieb category by specializing $\delta$. For example, specializing $\delta = -2$ gives a category $TL_{-2}$ which controls the representation theory of $\mathfrak{sl}_2$.[3]

There are certain idempotents in $TL_\delta$ called *Jones-Wenzl projectors*. To define them, we need to extend scalars from $\mathbb{Z}[\delta]$ to $\mathbb{Q}(\delta)$. Then, there is one Jones-Wenzl projector $JW_n : n \to n$ for each $n$. It is the unique morphism satisfying the following properties:

- $JW_n$ is sent to zero by post-composition with any cap.
- $JW_n$ is sent to zero by pre-composition with any cup.
- The coefficient of the identity morphism in $JW_n$ is 1.
- $JW_n^2 = JW_n$.

We picture $JW_2$ below.




(4.12)

It is straight-forward to check that this morphism satisfies the four required properties.

We can elaborate $TL_\delta$ into a 2-category $2TL_{\delta,\gamma}$, the *two-colour Temperley Lieb category*. Intuitively we should think of this as $TL_\delta$, with regions in our string diagrams coloured in alternating red and blue.

(4.13)

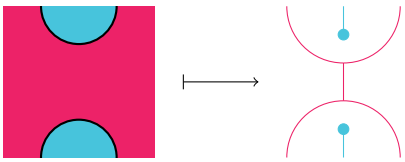
[3]In particular, $TL_{-2}$ is equivalent to the full subcategory of representations of $\mathfrak{sl}_2$ consisting of tensor products of the natural representation $V = \mathbb{C}^2$.

(We have drawn our red and blue lighter here to provide contrast to the black lines. We should think of them as the same red and blue as before.) More precisely, we define the two-colour Temperley-Lieb category $2TL_{\delta,\gamma}$ as follows. The objects are the two colours red and blue, which we think of as representing $s$ and $t$. We can picture them as coloured line intervals. The 1-morphisms are tensor generated by a morphism $s \to t$ and a morphism $t \to s$. We picture these morphisms as going from right to left. For instance, the morphism $s \to t \to s \to t$ can be pictured as follows.



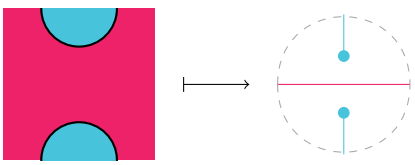
The 2-morphisms $2TL_{\delta,\gamma}$ are Temperley-Lieb diagrams coloured in alternating red and blue, as in (4.13). We think of these as morphisms going from bottom to top. Finally, a red bubble surrounded by a blue region evaluates to $\delta$, whereas a blue bubble surrounded by a red region evaluates to $\gamma$.

Now, let us specialize $\delta \mapsto a_{ts} := \partial_t(\alpha_s)$, and $\gamma \mapsto a_{st} := \partial_s(\alpha_t)$. We can take a 2-morphism in $2TL_{\delta,\gamma}$ and send it to a morphism in $\mathcal{H}_{BS}$ by taking deformation retracts of the coloured regions. Here is an example:



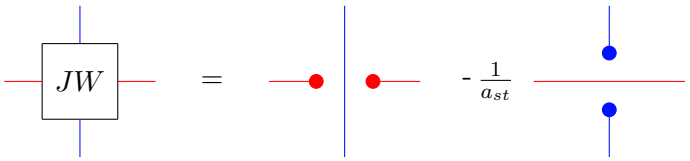
We have specialized $\delta$ and $\gamma$ so as to make this retract well-defined.

We will perform this retract slightly differently to above, however. Following Section 5.2 of [EW16], we will take the deformation retract onto a disk, with alternating red and blue boundary points.



Of course, this diagram in a disk is not a well-defined morphism in $\mathcal{H}_{BS}$, since it has no well-defined source and target. We will rectify this shortly.

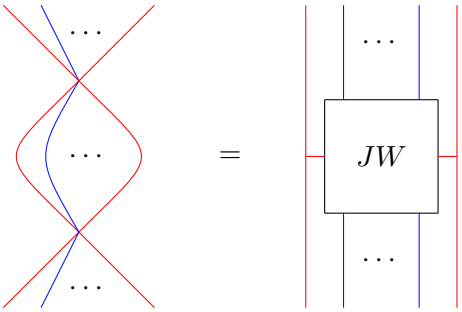
We are interested in the images of the Jones-Wenzl projectors $JW_n$ under this deformation retract onto a disk. We will call the image of $JW_n$ under deformation retract to a disk a *JW-box*, where the $n$ is implied by the number of strands coming out of the box. For example, here is the image of $JW_2$.




(4.14)

We can now state the two-colour relations in $\mathcal{H}_{BS}$.

**Elias-Jones-Wenzl:**



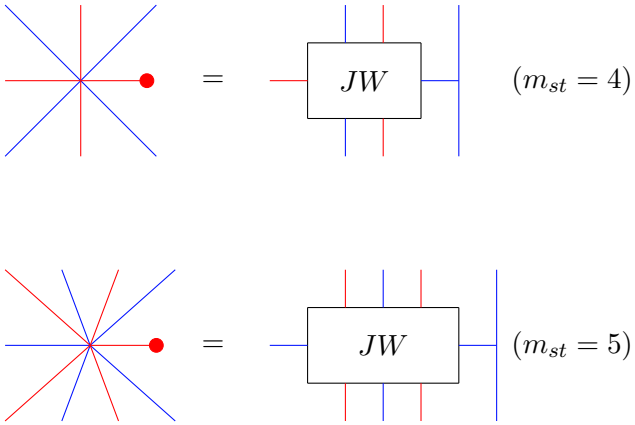

(4.15)

The above is the relation when $m_{st}$ is odd. There is of course the analogous relation for $m_{st}$ even.

**Two-colour dot contraction:**

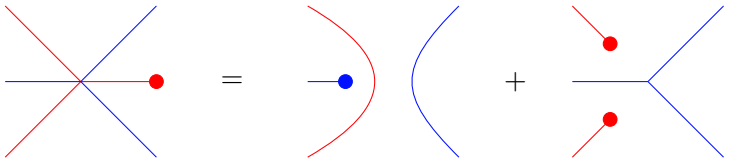
We present two-colour dot contraction for $m_{st} = 4$ and 5, but it is true for any value

of $m_{st}$.



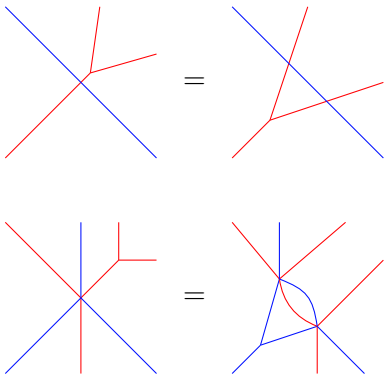

$(m_{st} = 4)$ $(m_{st} = 5)$ (4.16)

Using the $JW$-box in figure (4.14), we can see what this looks like explicitly for $m_{st} = 3$.

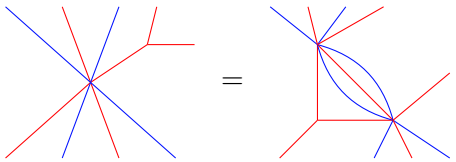


**Two-colour associativity:**

We present two-colour associativity for $m_{st} = 2, 3$ and 4, but it is true for any value of $m_{st}$.



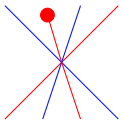
$(m_{st} = 2)$ $(m_{st} = 3)$ (4.17)



$(m_{st} = 4)$

So, the morphisms in $\mathcal{H}_{BS}$ in two-colours are string diagrams constructed from the one-colour generators and the $2m_{st}$-valent vertex, up to isotopy, and modulo the one-colour and two-colour relations given.

Before moving on to three colours, let us dwell for a moment on the 'two-colour dot contraction' relation (4.16), since this will be of fundamental importance in later proofs. In particular, we will often encounter $2m$-valent vertices with an up-dot somewhere along the top, like this:

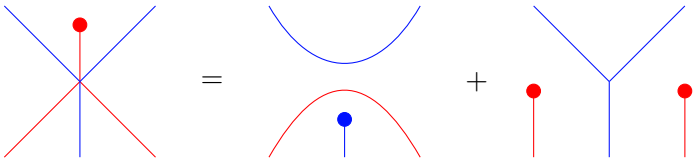


(4.18)

In this situation, we call pull the dot through the $2m$-valent vertex. More precisely:

**Proposition 4.2.1.** *A $2m$-valent vertex with an up-dot anywhere along the top can be written as a sum of diagrams, each with an up-dot somewhere along the bottom.*

Here is an example (for $m_{st} = 3$), which is a result of two-colour dot contraction in figure (4.2).



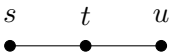
*Proof.* Recall that the JW-box is defined by Jones-Wenzl projectors in $TL_\delta$, in the manner described earlier in this section. These Jones-Wenzl projectors are sums of crossingless matches from $m$ to $m$. It is clear that such crossingless matches must either have an identity strand on the left, or a cap on the bottom and a cup on top. After colouring the

regions and deformation-retracting to a disk, this imples that the JW-box must either have a dot on the left, or an up-dot on the bottom and a down-dot on the top. For example in (4.14), the first diagram has a dot on the left, while the second has an up-dot on the bottom and a down-dot on the top. Combining this fact with two-colour dot contraction proves the proposition. □

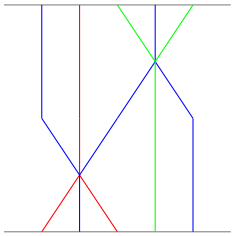
We now want to describe $\mathcal{H}_{BS}$ associated to a general Coxeter system, with potentially more than two generators. We might expect that when we move beyond two-colours, we will need to introduce a host of extra generators and relations. However, it turns out that there are no extra generators, and only one extra relation. This last relation involves three colours. To motivate it, we introduce *rex moves*.

## 4.3 Rex moves

Given an element $w \in W$, and two reduced expressions $\underline{w}, \underline{w'}$ of $w$, by Matsumoto's theorem (Theorem 2.1.3) these two reduced expressions differ by a sequence of applications of the braid relation. Given such a sequence, we can construct a corresponding diagrammatic morphism $\underline{w} \to \underline{w'}$, where we have a $2m$-valent vertex for every application of the braid relation. Such a morphism is called a *rex move*. For example, consider $W = S_4 = \langle s, t, u \rangle$, with Coxeter graph



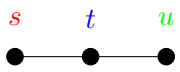

The expression $\underline{w} = (s, t, s, u, t)$ can be transformed into $\underline{w'} = (t, s, u, t, u)$ via braid relations: $(s, t, s, u, t) \mapsto (t, s, t, u, t) \mapsto (t, s, u, t, u)$. This gives us a rex move $\underline{w} \to \underline{w'}$.

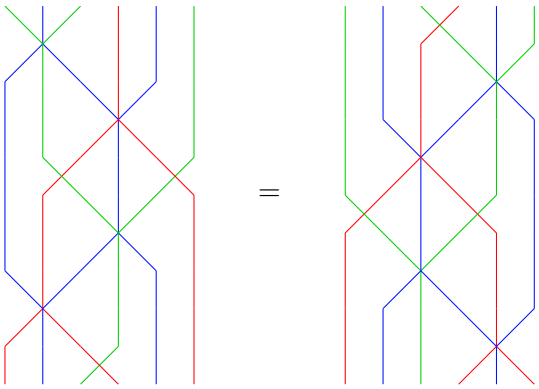


## 4.4 Three-colour diagrammatics

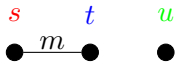
As mentioned above, we have only one relation left to introduce. It applies to any finitary rank 3 subset of $S$, i.e. a three element subset $I \subset S$ which generates a finite subgroup $W_I \leq W$. The specific form of the three-colour relation depends on the type of $W_I$. Precisely, $W_I$ must be one of the following types: $A_3$, $A_1 \times I_2(m)$, $B_3$ or $H_3$ (Theorem 1.34 of [EMTW20]). Here we present the new three-colour relations when $W_I$ is of type $A_3$ and $A_2 \times I_2(m)$. For the others see [EW16]. We include the Coxeter graph to indicate which colours are which.

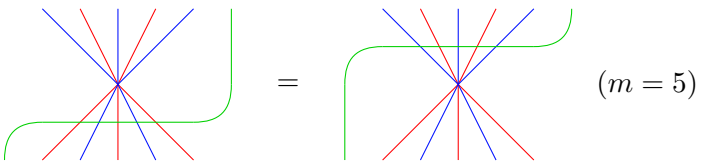
**Type $A_3$:**



**Type $A_1 \times I_2(m)$**



The relation for type $A_2 \times I_2(m)$ is fairly intuitive: the element $u$ moves past $s$ and $t$ in the Coxeter group, and in the diagrammatics the $u$-strand moves past the $2m_{st}$-valent vertex.

The relation for type $A_3$ looks more intimidating. Notice, however, that both sides are a rex move $(s,t,u,s,t,s) \to (u,t,u,s,t,u)$. It is not true in general that two different rex moves are always the same. One has to choose particular reduced expressions, and particular rex moves between them[4].

We have thus concluded our presentation of $\mathcal{H}_{BS}$. We have also defined a functor $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}\mathrm{Bim}$.

**Theorem 4.4.1.** *Suppose our realization is such that Soergel's categorification theorem (3.6.5) holds. Then $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BS}Bim$ is well-defined and is an equivalence of categories.*

*Proof.* See Claim 5.14 and Theorem 6.30 of [EW16]. □

## 4.5 Diagrammatics of Soergel bimodules and the Karoubian closure

Thus far we have constructed a diagrammatic category $\mathcal{H}_{BS}$ which is equivalent to $\mathbb{BS}\mathrm{Bim}$. We will now extend this to a category $\mathcal{H}$ which is equivalent to $\mathbb{S}\mathrm{Bim}$. Recall that we extended $\mathbb{BS}\mathrm{Bim}$ into $\mathbb{S}\mathrm{Bim}$ by including direct summands of objects in $\mathbb{BS}\mathrm{Bim}$[5]. It is not immediately clear how to do this to $\mathcal{H}_{BS}$; objects in $\mathcal{H}_{BS}$ are sequences of coloured dots, and there is no clear notion of what a direct summand of these would be. There is, however, a formal way of expanding a category $\mathcal{C}$ to include direct summands of objects, known as taking the *Karoubian closure* $Kar(\mathcal{C})$. It is also called *idempotent completion*. Our motivating example will be $\mathbb{S}\mathrm{Bim} = Kar(\mathbb{BS}\mathrm{Bim})$. In particular, the objects of $\mathbb{S}\mathrm{Bim}$

[4]In type $H_3$, there is no choice of reduced expressions such that two different rex moves between them are equal. The best that can be said is that two different rex moves are the same modulo "lower terms". See Remark 5.4 in [EW16].

[5]More precisely, we included direct summands of direct sums of grading shifts of objects in $\mathbb{BS}\mathrm{Bim}$.

are *direct summands* of *direct sums* of *grading shifts* of objects in $\mathbb{BS}$Bim. We will treat each of these three elements one by one.

First, let $\mathcal{A}$ be a *pre-additive* category, that is, a category whose Hom spaces are abelian groups. We can construct an additive category $\mathcal{A}^{\oplus}$, called the *additive closure*. The objects of $\mathcal{A}^{\oplus}$ are formal finite direct sums of objects of $\mathcal{A}$, and the morphism spaces are what one would expect:

$$\mathrm{Hom}_{\mathcal{A}^{\oplus}}(\bigoplus_i A_i, \bigoplus_j B_j) := \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(A_i, B_j).$$

Thus we have added direct sums to our category.

Now we want to know how to formally include grading shifts in a category. We will require our category to be a *graded category*, a notion we introduced in Section 3.1. It is a category whose morphism spaces are graded abelian groups, such that composition of morphisms respects this grading. Given a graded category $\mathcal{C}$ and two objects $A, B \in \mathcal{C}$, we denote the Hom space in $\mathcal{C}$ as

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}}(A, B) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n_{\mathcal{C}}(A, B).$$

For example, $\mathbb{BS}$Bim is a graded category: Morphisms in $\mathrm{Hom}^n_{\mathbb{BS}\mathrm{Bim}}(A, B)$ are homogeneous morphisms of degree $n$.

There is a related notion of a *category with shift functor*. A *shift functor* on a pre-additive category $\mathcal{C}$ is an automorphism $(1) : \mathcal{C} \to \mathcal{C}$ which respects additivity. Of course, one should think of the grading shift described in Section 3.1 as the canonical example of such a functor.

Given a graded category $\mathcal{C}$, we can construct a category $\mathcal{C}^{sh}$ with shift functor as follows: The objects of $\mathcal{C}^{sh}$ are $\{X(n) \mid X \in Ob(\mathcal{C}),\ n \in \mathbb{Z}\}$. The morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}^{sh}}(X(n), Y(m)) := \mathrm{Hom}^{m-n}_{\mathcal{C}}(X, Y).$$

Speaking intuitively, we are including grading shifts of objects, and mandating that morphisms between them are homogeneous of degree 0.

Now we describe how to formally include direct summands in a category. The idea is that given a direct sum decomposition $X \cong Y \oplus Z$, we can identify the direct summand $Y$ with the idempotent $X \to X$ which projects onto $Y$. Given an additive category $\mathcal{C}$, let $Kar'(\mathcal{C})$ denote the category described as follows[6]: the objects of $Kar'(\mathcal{C})$ are pairs $(X, e)$, where $X \in Ob(\mathcal{C})$ and $e : X \to X$ is an idempotent. The morphisms in $Kar'(\mathcal{C})$ are those of $\mathcal{C}$, pre- and post-composed with the appropriate idempotents. That is, given two objects $(X, e)$ and $(Y, f)$ in $Kar'(\mathcal{C})$, the morphisms are

$$\mathrm{Hom}_{Kar'(\mathcal{C})}((X, e), (Y, f)) := \{f\varphi e \mid \varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)\}.$$

**Definition 4.5.1.** Given a pre-additive graded category $\mathcal{C}$, the *Karoubi envelope* of $\mathcal{C}$ is

$$Kar(\mathcal{C}) := Kar'(\mathcal{C}^{\oplus, sh})$$

As stated before, we have that $\mathbb{S}\mathrm{Bim} = Kar(\mathbb{B}\mathbb{S}\mathrm{Bim})$. Recall that $\mathcal{H}_{BS}$ is a graded category. It is also pre-additive, since we are allowed to take sums of diagrams. Therefore we can use the Karoubian closure to define $\mathcal{H}$.

**Definition 4.5.2.** The *diagrammatic Hecke category* is $\mathcal{H} := Kar(\mathcal{H}_{BS})$.

Suppose we have a functor $F : \mathcal{C} \to \mathcal{D}$ between two pre-additive graded categories $\mathcal{C}$ and $\mathcal{D}$. We can naturally extend this to a functor $Kar(F) : Kar(\mathcal{C}) \to Kar(\mathcal{D})$ via $(X, e) \mapsto (F(X), F(e))$. It is straight-forward that if $F$ is an equivalence of categories, then so is $Kar(F)$. This yields the following corollary to Theorem 4.4.1.

**Corollary 4.5.3.** *When our realization is such that Soergel's categorification theorem (3.6.5) holds, then we have an equivalence of monoidal categories* $Kar(\mathcal{A}) : \mathcal{H} \xrightarrow{\sim} \mathbb{S}Bim$.

**Remark 4.5.4.** A category $\mathcal{C}$ is called *idempotent complete* if every idempotent splits. That is, for every idempotent $e : X \to X$ for $X \in \mathcal{C}$, we have $X \cong Y \oplus Z$ and $e$ is projection onto one of these summands $Y$ or $Z$. The category $Kar(\mathcal{C})$ is idempotent complete by construction.

[6]We are reserving the term *Karoubian closure* and the notation $Kar(\mathcal{C})$ for when we also include direct sums and grading shifts. See Definition 4.5.1.

# Chapter 5

# Light-Leaves and Double-Leaves

Given two Bott-Samelson bimodules $B, B'$, the set $\mathrm{Hom}_{\mathbb{BS}\mathrm{Bim}}(B, B')$ is a left (or right) $R$-module. A basis of this Hom set was constructed by Libedinsky in [Lib15], and is called the *double-leaves basis*, since it consists of so-called *double-leaves*. More precisely, Libedinsky gave an algorithm for constructing a basis: Given any two Bott-Samelson bimodules $B, B'$, the algorithm produces a basis for $\mathrm{Hom}_{\mathbb{BS}\mathrm{Bim}}(B, B')$. The diagrammatic counterpart to this double-leaves basis was constructed in [EW16]. Thus, given any two elements $\underline{x}, \underline{y} \in \mathcal{H}_{BS}$, we have a double-leaves basis of $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{x}, \underline{y})$ as a left (or right) $R$-module. We will use the terms 'light-leaves' and 'double-leaves' to refer to these diagrammatic incarnations. To describe these diagrammatic double-leaves, we have to describe what they are built from: *light-leaves*.

## 5.1 The light-leaves algorithm

Light-leaves are certain morphisms between objects in $\mathcal{H}_{BS}$, and will form the building blocks of the above mentioned double-leaves. Take some expression $\underline{x} = (s_1, ..., s_n)$ and a subexpression $\underline{e} \subset \underline{x}$. Recall from Section 3.8 that to $\underline{e}$ we can associate a stroll $w_0, ..., w_n$, where $w_k = s_1^{e_1} \cdots s_k^{e_k}$. Recall also that to this stroll, we can associate a decoration

$d_1, ..., d_n$, where $d_i \in \{U0, U1, D0, D1\}$, and

$$d_i = \begin{cases} Ue_i & \text{if } w_{i-1}s_i > w_{i-1}, \\ De_i & \text{if } w_{i-1}s_i < w_{i-1}. \end{cases}$$

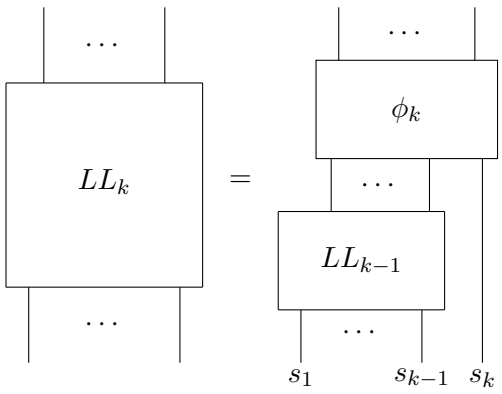
Now, for any $\underline{x}$ and any $\underline{e} \subset \underline{x}$ we will form a *light-leaf* map $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$, where $\underline{w}$ is a reduced expression for $w = \underline{x}^{\underline{e}}$. So for an object $\underline{x} \in \mathcal{H}_{BS}$ of length $n$, we will have $2^n$ light-leaf maps from $\underline{x}$, corresponding to the $2^n$ subexpressions $\underline{e} \subset \underline{x}$. These $LL_{\underline{x},\underline{e}}$'s will be constructed by an algorithm, and each step of the algorithm will depend on these labels $d_i$.

Here we give the algorithm for constructing light-leaves. We will follow the description in Chapter 10.4.2 of [EMTW20]. To describe it precisely is somewhat intricate, but the examples given afterwards should make the procedure more intuitive.

We take an expression $\underline{x} = (s_1, .., s_n)$, and a subexpression $\underline{e} = (e_1, ..., e_n) \subset \underline{x}$. We will construct a morphism $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$, where $\underline{w}$ is a reduced expression for $w = \underline{x}^{\underline{e}}$. One important point is that we will have choices in how we construct this light-leaf: there is not a notion of *the* light-leaf $LL_{\underline{x},\underline{e}}$. Rather, the label $LL_{\underline{x},\underline{e}}$ could refer to any one of a number of possible constructions. In particular, we will have choices of reduced expressions and rex moves at each step in the construction (for example, there may be many choices for the rex $\underline{w}$). When it comes time to construct a basis out of these light-leaves, we will fix a choice of the construction of $LL_{\underline{x},\underline{e}}$, for each $\underline{e} \subset \underline{x}$. But in fact, this choice will not matter: we end up with a basis no matter which constructions of the light-leaves we choose. This flexibility in the construction of light-leaves will be helpful to us in our proofs in the later sections.

We use the notation $\underline{x}_{\leq k} := (s_1, ..., s_k)$, and $\underline{e}_{\leq k} := (e_1, ..., e_k)$. Then set $w_k := \underline{x}_{\leq k}^{\underline{e}_{\leq k}} = s_1^{e_1} \cdots s_k^{e_k}$ (so $w_0, w_1, ..., w_n$ is the stroll associated to $\underline{e}$). Now, the light-leaf map $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$ will be constructed using a recursive algorithm. At each step $i$ we will construct a map $LL_k : \underline{x}_{\leq k} \to \underline{w}_k$, where $\underline{w}_k$ is some reduced expression for $w_k$ . We will refer to these $\underline{w}_k$'s as *intermediate expressions*. We get to choose each reduced expression $\underline{w}_k$, so this gives us our first degree of freedom in constructing the light-leaf. We build $LL_k$ from

$LL_{k-1}$ by tacking on a map $\phi_k : (\underline{w}_{k-1}, s_k) \to \underline{w}_k$ in the way pictured:




(5.1)

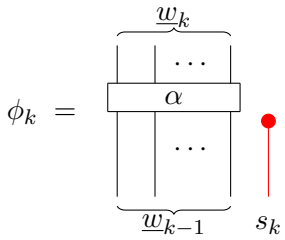
So each $LL_k$ incorporates a new $s_k$ on the bottom right, until we have $LL_{\underline{x},\underline{e}} := LL_n : \underline{x} = (s_1, ..., s_n) \to \underline{w}$.

Our light leaf is built out of these maps $\phi_k$: We start by setting $LL_1 = \phi_1$, and then successively tack on $\phi_k$'s to get our $LL_k$'s. These $\phi_k$'s can each be one of four different kinds of map, depending on four different options $(U0, U1, D0, D1)$ for the decoration $d_k$ at that step. These $\phi_k$'s contain two different kinds of rex moves: "necessary" rex moves $\beta$, and "optional" rex moves $\alpha$. These optional rex moves $\alpha$ give us our second degree of freedom in the construction of the light-leaf.

**U0**

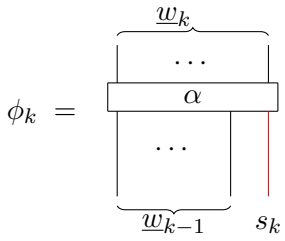
If $d_k = U0$, then $\phi_k$ is some rex move $\alpha : \underline{w}_{k-1} \to \underline{w}_k$, with an up-dot on the right[1].

[1]We have called $\alpha$ an 'optional' rex move, but if $\underline{w}_k$ is chosen to be different from $\underline{w}_{k-1}$ then $\alpha$ is not optional. However we still get the freedom to choose the specific form of $\alpha$. The precise thing to say is that we get to choose either the expressions $\underline{w}_k$, or the rex moves $\alpha$. The choice of $\alpha$'s determines the choice of $\underline{w}_k$'s, whereas the choice of $\underline{w}_k$'s does not entirely determine the choice of $\alpha$'s.



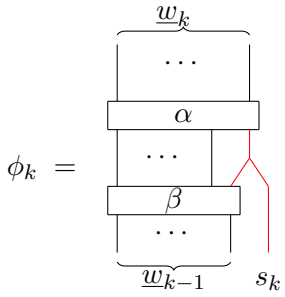

**U1**

If $d_k = U1$, then $\phi_k$ is some rex move $\alpha : (\underline{w}_{k-1}, s_k) \to \underline{w}_k$




**D0**

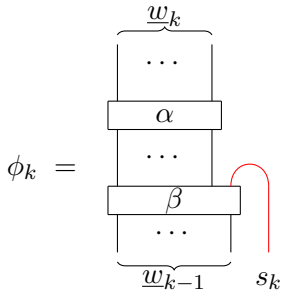
If $d_k = D0$, we have:




Since we have $d_i = D0$, that means that $w_{k-1}s_k < w_{k-1}$, so that by Corollary 2.3.3, there is a reduced expression of $w_{k-1}$ with $s_k$ on the right. So we apply a rex move $\beta$ to put $s_k$ on the right. Then, we apply a merging trivalent vertex to this $s_k$ strand, and finally apply a rex move $\alpha$ to give the target $\underline{w}_k$.

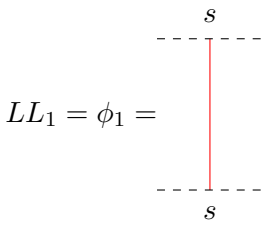
**D1**

The case for $d_k = D1$ is very similar to $D0$:

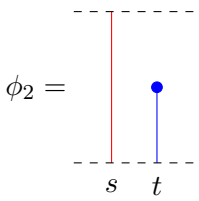



Let's now see how this works in an example.

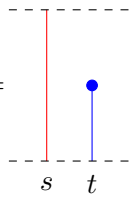
**Example 5.1.1.** Take $W = S_3 = \langle s, t \rangle$, $\underline{x} = (s, t, s)$, $\underline{e} = (1, 0, 0)$. So $w = \underline{x}^{\underline{e}} = s$, and $\underline{w} = (s)$. We will construct the light leaf $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$, i.e. from $(s, t, s) \to (s)$. Our associated stroll is $w_0, w_1, w_2, w_3 = \mathrm{id}, s, s, s$, and our decorations are $d_1, d_2, d_3 = U1, U0, D0$. Note that each of these $w_k$'s only has one reduced expression, so we don't get any choices here in how to construct the light-leaf. We start by constructing $LL_1 = \phi_1$. We have that $d_1 = U1$, so $\phi_1$ is just the identity on $(s)$.



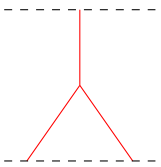

Then we construct $LL_2 : (s, t) \to (s)$ by tacking $\phi_2$ onto this $LL_1$. Since $d_2 = U0$, we have that $\phi_2 : (s, t) \to (s)$ is the following.



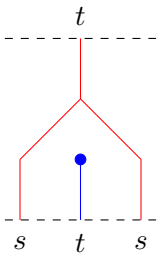
Tacking this onto $LL_2$ in the way pictured in the diagram (5.1) is just putting id: $(s) \to (s)$ on the lower left of $\phi_2$, which won't change anything:

$LL_2 =$ 


The last stage is more interesting. We want to construct $LL_3$ via $\phi_3 : (s, s) \to (s)$. Since $d_3 = D0$, we have:

$\phi_3 =$ 

Tacking this onto $LL_2$ gives us $LL_3 = LL_{\underline{x},\underline{e}}$:

$LL_{\underline{x},\underline{e}} = LL_3 =$ 


Note that the degree of a light-leaf map $LL_{\underline{x},\underline{e}}$ is exactly the defect $def(\underline{e}) = \#U0 - \#D0$. This is because the morphisms $\phi_k$ corresponding to $U0$ and $D0$ have degree 1 and -1 respectively, while those corresponding to $U1$ and $D1$ are neutral. In the example above, the decoration of $\underline{e}$ was $U1, U0, D0$, so that the defect is $1 - 1 = 0$, and the morphism $LL_{\underline{x},\underline{e}}$ pictured above is indeed a degree 0 morphism.

## 5.2 The double-leaves basis

Now that we have constructed our light-leaves, we will use them to construct bases of morphism spaces. Take any two expressions $\underline{x}$ and $\underline{y}$. We will construct a diagrammatic basis of $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{x}, \underline{y})$. The morphisms of this basis are called double-leaves, and they will each be made of two light-leaves stuck together.

We first need to fix a choice of the construction of the light-leaves. We write that $w \leq \underline{x}$ if there is some subexpression $\underline{e} \subset \underline{x}$ such that $\underline{x}^{\underline{e}} = w$. For each $w \leq \underline{x}$, we fix a choice of reduced expression $\underline{w}$ of $w$. Similarly, for each $w' \leq \underline{y}$, we fix a choice of rex $\underline{w}'$ of $w'$. These choices must be compatible, in the sense that if $w \leq \underline{x}$ and $w \leq \underline{y}$, then we make the same choice of rex $\underline{w}$ of $w$ in both cases. Then, for each $\underline{e} \subset \underline{x}$, we fix a choice of the construction of $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$, where $w = \underline{x}^{\underline{e}}$. Similarly, for each $\underline{f} \subset \underline{y}$, we fix a choice of the construction of $LL_{\underline{y},\underline{f}} : \underline{y} \to \underline{w}'$, where $w' = \underline{y}^{\underline{f}}$.

Now, suppose we have subexpressions $\underline{e} \subset \underline{x}$ and $\underline{f} \subset \underline{y}$ such that $\underline{x}^{\underline{e}} = \underline{y}^{\underline{f}} = w$. Then by the way we have fixed our rex's, we will have two light-leaves $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$ and $LL_{\underline{y},\underline{f}} : \underline{y} \to \underline{w}$ with the same image. Let $\overline{LL_{\underline{y},\underline{f}}}$ denote $LL_{\underline{y},\underline{f}}$ flipped upside down, so that $\overline{LL_{\underline{y},\underline{f}}}$ is a morphism $\underline{w} \to \underline{y}$. Then we can take the composition $\mathbb{DL}_{\underline{f},\underline{e}} := \overline{LL_{\underline{y},\underline{f}}} \circ LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w} \to \underline{y}$. This $\mathbb{DL}_{\underline{f},\underline{e}}$ is called a *double-leaf*, and such double-leaves form a basis of $\mathrm{Hom}_{\mathcal{H}_{BS}}(\underline{x}, \underline{y})$:

**Theorem 5.2.1.** *Given two expressions $\underline{x}, \underline{y}$, let $\mathcal{DL}_{\underline{x},\underline{y}}$ be a collection of maps containing one double-leaf map $\mathbb{DL}_{\underline{f},\underline{e}}$ for each pair $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ expressing the same element. Then $\mathcal{DL}_{\underline{x},\underline{y}}$ is a basis of $Hom_{\mathcal{H}_{BS}}(\underline{x}, \underline{y})$ as a left (or right) $R$-module.*

*Proof.* See [EW16], Sections 6 and 7. □

**Remark 5.2.2.** Note that this basis agrees with the Soergel Hom formula. We saw at the end of Section 3.8 that this Hom formula could be expressed as

$$\overline{\mathrm{rkHom}_{\mathbb{S}\mathrm{Bim}}(BS(\underline{x}), BS(\underline{y}))} = \sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{f} \subset \underline{y} \\ \underline{x}^{\underline{e}} = \underline{y}^{\underline{f}}}} v^{def(\underline{e}) + def(\underline{f})}. \tag{5.2}$$

That is, in our basis for $\mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(BS(\underline{x}), BS(\underline{y}))$ we should have one morphism of degree $def(\underline{e}) + def(\underline{f})$ for each pair $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $\underline{x}^{\underline{e}} = \underline{y}^{\underline{f}}$. When we pass across the equivalence $\mathbb{BS}\mathrm{Bim} \cong \mathcal{H}_{BS}$, these are exactly our double-leaves, and the degrees match up since the degree of a light-leaf $LL_{\underline{x},\underline{e}}$ is exactly $def(\underline{e})$.

# Chapter 6

# The Spherical Category

Thus far we have been investigating two categories which each categorify the Hecke algebra: the category $\mathbb{S}$Bim and its diagrammatic counterpart $\mathcal{H}$. We will now move on to study a certain module over the Hecke algebra, called the spherical module $M$. We will do so using methods exactly analogous to those we have used to study the Hecke algebra. That is, we will construct a "Bott-Samelson-esque" category $J\mathbb{BS}$Bim, and take its Karoubian closure, giving us a categorification of $M$. Then, we will construct a diagrammatic category $\mathcal{M}_{BS}$, equivalent to $J\mathbb{BS}$Bim (though this equivalence will not be shown until the final chapter). Its Karoubian closure $\mathcal{M}$ will then be a second categorification of $M$. The first central result of this thesis will be to construct a 'light-leaves' basis of $\mathcal{M}_{BS}$, and to prove that it is indeed a basis. In Chapter 11 we use this basis to prove that $\mathcal{M}_{BS}$ is equivalent to $J\mathbb{BS}$Bim, and thus so are their Karoubian closures.

## 6.1 The spherical and anti-spherical modules

Given a Coxeter system $(W, S)$, choose a subset $J \subset S$ of the generators, and let $W_J$ be the subgroup of $W$ generated by $J$. Such a subgroup is called a *parabolic subgroup*. We will require that $J$ is *finitary*, meaning that $W_J$ is a finite subgroup. Each coset of $W_J \backslash W$

has a unique minimal coset representative (or *m.c.r.*) (see p. 31 of [Bou02]). We let ${}^JW$ be the set of minimal coset representatives of $W_J\backslash W$. Then

$$ {}^JW = \{w \in W \mid sw > w \text{ for all } s \in J\}. $$

**Example 6.1.1.** Let us take $W = S_3 = \langle s, t\rangle$, and $J = \{s\}$. Then $W_J\backslash W$ has three cosets: $\{id, s\}, \{t, st\}, \{ts, sts\}$. Thus, the collection ${}^JW$ of minimal coset representatives is $\{id, t, ts\}$.

We will take this opportunity to state a pair of lemmas, which will be necessary later on:

**Lemma 6.1.2.** *Given $w \in W$, there is a unique decomposition $w = uz$, where $u \in W_J$ and $z \in {}^JW$. Moreover, $l(w) = l(u) + l(z)$.*

*Proof.* See p. 31 of [Bou02]. □

**Lemma 6.1.3.** *If $x \in {}^JW$ but $xs \notin {}^JW$ for some $s \in S$, then*

*(a) $xs > x$*

*(b) $xs = tx$ for some $t \in I$.*

*Proof.* See Lemmas 3.1 and 3.2 of [Deo77]. □

Now, let $H := H(W, S)$, $H_J := H(W_J, J)$ be the Hecke algebras associated to $(W, S)$ and $(W_J, J)$ respectively. Set $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$. We are going to construct a right $H$-module of the form $\mathcal{L} \otimes_{H_J} H$, but to do this, we need to make $\mathcal{L}$ into an $H_J$-module. One way to do this is to construct an algebra morphism $\varphi : H_J \to \mathcal{L}$, since then we can set $p \cdot h := p\varphi(h)$ for $h \in H_J$, $p \in \mathcal{L}$. What are our possible choices of $\varphi$? Revisiting the relations which define the Hecke algebra, we had the quadratic relation

$$ \delta_s^2 = 1 + (v^{-1} - v)\delta_s, \text{ for all } s \in J. $$

We can rewrite this as

$$ (\delta_s + v)(\delta_s - v^{-1}) = 0, \text{ for all } s \in J. $$

Then $\varphi[(\delta_s+v)(\delta_s-v^{-1})] = (\varphi(\delta_s)+v)(\varphi(\delta_s)-v^{-1}) = 0$. This tells us that $\varphi(\delta_s)$ is either $-v$ or $v^{-1}$ for all $s \in J$. Let $\varphi_u(\delta_s) = u$, for $u \in \{-v, v^{-1}\}$, and let $\mathcal{L}(u)$ be $\mathcal{L}$ as a right $H_J$-module, with action defined by $\varphi_u$. This gives us two right $H$-modules:

$$N = N(J) := \mathcal{L}(-v) \otimes_{H_J} H, \text{ the } \textit{anti-spherical module};$$

$$M = M(J) := \mathcal{L}(v^{-1}) \otimes_{H_J} H, \text{ the } \textit{spherical module};$$

Note that we have one of each for each finitary $J \subset S$.

Let us focus on the spherical module[1]. For $x \in {}^J W$, set $m_x := 1 \otimes \delta_x \in M$. Then the set $\{m_x \mid x \in {}^J W\}$ forms a basis of $M$ as a right (or left) $\mathcal{L}$-module. This is because for any $w \notin {}^J W$, we have $w = sy$ for some $s \in J$, $y < w$. Then

$$m_w = 1 \otimes \delta_w = 1 \otimes \delta_{sy} = 1 \otimes \delta_s\delta_y = v^{-1} \otimes \delta_y = v^{-1}m_y.$$

Continuing in this way, we can write $m_w$ as a polynomial times $m_x$ for some $x \in {}^J W$. We call the set $\{m_x \mid x \in {}^J W\}$ the *standard basis* of $M$. We have the following formula for how to multiply this standard basis by the KL-basis of $H$ on the right. For $s \in S$ and $x \in {}^J W$ we have

$$m_x b_s = \begin{cases} m_{xs} + v^{-1}m_x & \text{if } xs < x,\ xs \in {}^J W \\ m_{xs} + vm_x & \text{if } xs > x, xs \in {}^J W \\ (v+v^{-1})m_x & \text{if } xs \notin {}^J W. \end{cases} \tag{6.1}$$

The spherical module $M$ can be realized as a right ideal of the Hecke algebra.

**Proposition 6.1.4.** *Given a finitary subset $J \subset S$, let $w_J$ be the longest element of $W_J$. Then we have an isomorphism of right $H$-modules $\phi : M(J) \xrightarrow{\sim} b_{w_J}H$ defined by $\phi(p \otimes f) := b_{w_J}pf$ for $p \in \mathcal{L}$, $f \in H$.*

*Proof.* It is immediate that $\phi$ is a right $H$-module morphism (if it is well-defined). So first, we need to check that it is well-defined. That is, for $f \in H_J$, we need to check that $\phi(1 \otimes f) = \phi(\varphi(f) \otimes 1)$, where $\varphi = \varphi_{v^{-1}} : H_J \to \mathcal{L}$ is defined above. It is enough to check

[1]The anti-spherical module was categorified by Libedinsky and Williamson in [LW22].

this for the elements $b_s,\ s \in J$. For this we use the fact ( [Lus14], Theorem 6.6) that for $s \in J$, $b_{w_J} b_s = (v + v^{-1}) b_{w_J}$. This means that, for $s \in J$,

$$\phi(1 \otimes b_s) = b_{w_J} b_s = (v + v^{-1}) b_{w_J} = \phi(\varphi(b_s) \otimes 1),$$

since $\varphi(b_s) = (v + v^{-1})$.

Then, we need to check that $\phi$ is an isomorphism. It is clear that it is surjective, since for any $h \in H$, $\phi(1 \otimes h) = b_{w_J} h$.

Now we want to see that it is injective. Take any element $\sum_{x \in {}^J W} p_x m_x = \sum_{x \in {}^J W} p_x \otimes \delta_x \in M$, and suppose

$$\phi(\sum_{x \in {}^J W} p_x \otimes \delta_x) = \sum_{x \in {}^J W} p_x b_{w_J} \delta_x = 0.$$

We claim that for each $x \in {}^J W$,

$$b_{w_J} \delta_x = \delta_{w_J x} + \sum_{\substack{y \in W \\ y < w_J x}} q_{y,x} \delta_y$$

for some $q_{y,x} \in \mathcal{L}$, so that the elements $b_{w_J} \delta_x,\ x \in {}^J W$ are upper-triangular and thus linearly independent. To see this, let $b_{w_J} = \sum_{w \in W_J} r_w \delta_w$ for some $r_w \in \mathcal{L}$, so that

$$b_{w_J} \delta_x = \sum_{w \in W_J} r_w \delta_w \delta_x = \sum_{w \in W_J} r_w \delta_{wx}.$$

This last equality follows from Lemma 6.1.2, since for each $w \in W_J$, $wx$ is the unique decomposition into $w \in W_J$, $x \in {}^J W$, which means $l(wx) = l(w) + l(x)$ and $\delta_w \delta_x = \delta_{wx}$. Furthermore, for each $w \in W_J$ with $w \neq w_J$, we have

$$l(wx) = l(w) + l(x) < l(w_J) + l(x) = l(w_J x),$$

so that

$$b_{w_J} \delta_x = \sum_{w \in W_J} r_w \delta_{wx} = \delta_{w_J x} + \sum_{\substack{w \in W_J \\ w \neq w_J}} r_w \delta_{wx} = \delta_{w_J x} + \sum_{y < w_J x} q_{y,x} \delta_y$$

as required. Therefore the elements $b_{w_J} \delta_x$ for $x \in {}^J W$ are linearly independent. Thus,

$$\phi(\sum_{x \in {}^J W} p_x \otimes \delta_x) = \sum_{x \in {}^J W} p_x b_{w_J} \delta_x = 0$$

implies that $p_x = 0$ for all $x \in {}^J W$, and therefore $\sum_{x \in {}^J W} p_x \otimes \delta_x = 0$, so that $\phi$ is injective. Thus $\phi$ is an isomorphism of right $H$-modules. $\square$

We state a fact about this $b_{w_J}$ that will be useful shortly.

**Definition 6.1.5.** Given a finitary subset $J \subset S$, we define the *Hilbert polynomial* $\pi(J)$ to be

$$\pi(J) := v^{-d_J} \sum_{x \in J} v^{2l(x)},$$

where $d_J$ is the length of the longest element of $W_J$.

Then we have

$$b_{w_J}^2 = \pi(J) b_{w_J} \tag{6.2}$$

(see equation 2.9 in [Wil11]).

Finally, we introduce a bar involution on $M$.

**Definition 6.1.6.** The bar involution $\bar{\cdot} : M \to M$ is the abelian group automorphism defined by $\overline{p \otimes h} := \overline{p} \otimes \overline{h}$ for $p \in \mathcal{L},\ h \in H$.

Just like the Hecke algebra, the spherical module has a Kazhdan-Lusztig basis.

**Theorem 6.1.7.** *( [Deo87]) The spherical module $M(J)$ has a Kazhdan-Lusztig basis $\{c_x \mid x \in {}^J W\}$. It is defined to be the unique basis satisfying*

1. $\overline{c_x} = c_x$ *for all* $x \in {}^J W$,
2. *For each* $x \in {}^J W$ *we have*

$$c_x = m_x + \sum_{\substack{y \in {}^J W \\ y < x}} k_{y,x} m_y$$

*for some polynomials* $k_{y,x} \in v\mathbb{Z}[v]$.

Just as the Hecke algebra is categorified by $\mathbb{S}\mathrm{Bim} = Kar(\mathbb{BS}\mathrm{Bim})$, we are going to categorify this spherical module with the category $Kar(J\mathbb{BS}\mathrm{Bim})$, where $J\mathbb{BS}\mathrm{Bim}$ is the category of *$J$-singular Bott-Samelson bimodules*. To motivate this, we will introduce *singular Soergel bimodules*.

## 6.2 Singular Soergel bimodules

Singular Soergel bimodules are a generalization of Soergel bimodules. More precisely, singular Soergel bimodules form a 2-category, and $\mathbb{S}\mathrm{Bim}$ appears as a Hom category inside this 2-category.

We will now define the 2-category $\mathcal{S}\mathbb{S}\mathrm{Bim}$ of *singular Soergel bimodules*. In a procedure that will seem familiar by now, we first define the 2-category $\mathcal{S}\mathbb{BS}\mathrm{Bim}$ of *singular Bott-Samelson bimodules*. These are defined as follows. The objects of $\mathcal{S}\mathbb{BS}\mathrm{Bim}$ are finitary subsets $J \subset S$. The 1-morphisms $J_1 \to J_2$ are the $(R^{J_2}, R^{J_1})$-bimodules (note the order) which are tensor-generated by $Ind_J^I := {}_{R^I}R_{R^J}$ and $Res_J^I := {}_{R^J}R_{R^I}(l(J) - l(I))$, where $I \subsetneq J$ are any two finitary subsets of $S$ which differ by a single simple reflection[2]. The 2-morphisms are bimodule morphisms.

We can think of a 1-morphism in $\mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(J, I)$ as given by a sequence $(I = I_0, I_1, ..., I_n = J)$, where $I_i$ and $I_{i+1}$ always differ by a single element. For example, in the case where $I_1 \supset I_2 \subset I_3 \supset \cdots \supset I_{n-1} \subset I_n$, then the corresponding bimodule is

$$ {}_{R^{I_1}}R^{I_2} \otimes_{R^{I_3}} \cdots \otimes_{R^{I_{n-2}}} R^{I_{n-1}}_{R^{I_n}}(d), $$

where $d$ is the number of restriction bimodules used in the product. For a simpler example, take the sequence $(\varnothing, s, \varnothing)$. This yields the $R$-bimodule

$$ R \otimes_{R^s} R(1), $$

[2]The notation ${}_{R^I}R_{R^J}$ means $R$ considered as an $(R^I, R^J)$-bimodule. The notation $l(J)$ means the size of $J$.

which is the Bott-Samelson bimodule $B_s$. Indeed, we can see that Bott-Samelson bimodules are all contained in the Hom category $\mathrm{Hom}_{\mathcal{SBBSBim}}(\varnothing, \varnothing)$. However, not all bimodules in $\mathrm{Hom}_{\mathcal{SBBSBim}}(\varnothing, \varnothing)$ are Bott-Samelsons. For example, the sequence $(\varnothing, \{s\}, \{s,t\}, \{s\}, \varnothing)$ gives the bimodule

$$R(1) \otimes_{R^s} R^s(1) \otimes_{R^{s,t}} R^s \otimes_{R^s} R \cong R \otimes_{R^{s,t}} R(2),$$

which is not a Bott-Samelson bimodule.

We then define the 2-category $\mathcal{SS}$Bim of *singular Soergel bimodules*: it has the same objects as $\mathcal{SBBS}$Bim, and

$$\mathrm{Hom}_{\mathcal{SS}\mathrm{Bim}}(I, J) := Kar(\mathrm{Hom}_{\mathcal{SBBS}\mathrm{Bim}}(I, J)).$$

Just as $\mathbb{S}$Bim categorifies the Hecke algebra, the 2-category $\mathcal{SS}$Bim categorifies the *Schur algebroid*.

**Definition 6.2.1.** The *Schur algebroid* $\mathcal{S}$ is the $\mathbb{Z}[v, v^{-1}]$-linear category defined as follows. Its objects are finitary subsets $J \subset S$. The morphisms in $\mathrm{Hom}_{\mathcal{S}}(I, J)$ are right $H$-module morphisms $M(I) \to M(J)$, where $M(I)$ and $M(J)$ are the spherical modules associated to $I$ and $J$.

We saw in Proposition 6.1.4 that $M(J) \cong b_{w_J} H$. Therefore one right $H$-module morphism $M(I) \to M(J)$ is given by left multiplication by $b_{w_J} h b_{w_I}$. In fact, this gives all morphisms from $M(I)$ to $M(J)$; that is, $\mathrm{Hom}_{\mathcal{S}}(I, J) \cong b_{w_J} H b_{w_I}$ (see [Wil11] Remark 2.1). As a slight technical detail, we associate the element $b_{w_J} h b_{w_I} \in b_{w_J} H b_{w_I}$ with the morphism in $\mathrm{Hom}_{\mathcal{S}}(I, J)$ given by left multiplication by $b_{w_J} h b_{w_I}/\pi(I)$, where $\pi(I)$ is the Hilbert polynomial (Definition 6.1.5). This is because, due to (6.2) we have

$$b_{w_J} h b_{w_I} \cdot b_{w_I} h' = b_{w_J} h b_{w_I}^2 h' = \pi(I) b_{w_J} h b_{w_I} h',$$

so we want to divide out by $\pi(I)$. Then, composition of morphisms in $\mathcal{S}$ is given by the product $- *_J - : b_{w_K} H b_{w_J} \times b_{w_J} H b_{w_I} \to b_{w_K} H b_{w_I}$ defined by

$$h_1 *_J h_2 := \frac{h_1 h_2}{\pi(J)}. \tag{6.3}$$

The gives the Schur algebroid its 'algebroid' structure. Note in particular that $\mathrm{Hom}_{\mathcal{S}}(\varnothing, J) \cong b_{w_J} H \cong M(J)$, so that the spherical module $M(J)$ shows up as a Hom set in $\mathcal{S}$. Moreover, the composition of morphisms

$$\mathrm{Hom}_{\mathcal{S}}(\varnothing, J) \times \mathrm{Hom}_{\mathcal{S}}(\varnothing, \varnothing) \to \mathrm{Hom}_{\mathcal{S}}(\varnothing, J)$$

is exactly the right module structure

$$M(J) \times H \to M(J).$$

Now, a main theorem of [Wil11] is that the split Grothendieck group of $\mathcal{S}\mathbb{S}\mathrm{Bim}$ is isomorphic to the Schur algebroid $\mathcal{S}$. This is made precise in the theorem below (see Theorems 1.1, 1.2 and 1.4 of [Wil11]). We let ${}^I\mathcal{S}^J := \mathrm{Hom}_{\mathcal{S}}(J, I) \cong b_{w_I} H b_{w_J}$, and let ${}^I\mathcal{B}^J := \mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(J, I)$. We will write $\mathcal{B} := {}^{\varnothing}\mathcal{B}^{\varnothing}$, and ${}^I\mathcal{B} := {}^I\mathcal{B}^{\varnothing}$.

**Theorem 6.2.2.** *(a) Each ${}^I\mathcal{S}^J$ has a basis $\{{}^I b^J_p \mid p \in W_I \backslash W / W_J\}$ indexed by double cosets, which we call the Kazhdan-Lusztig basis.*

*(b) The indecomposables of ${}^I\mathcal{B}^J$ are also indexed by double cosets $W_I \backslash W / W_J$. They are denoted ${}^I B^J_p$ for $p \in W_I \backslash W / W_J$.*

*(c) We have a character map $ch : {}^I\mathcal{B}^J \to {}^I\mathcal{S}^J$ such that $ch : [{}^I\mathcal{B}^J]_{\oplus} \xrightarrow{\sim} {}^I\mathcal{S}^J$ is an isomorphism of abelian groups, and such that the following diagram commutes.*

$$\begin{array}{ccc} {}^I\mathcal{B}^J \times {}^J\mathcal{B}^K & \xrightarrow{- \otimes -} & {}^I\mathcal{B}^K \\ \downarrow{\scriptstyle ch \times ch} & & \downarrow{\scriptstyle ch} \\ {}^I\mathcal{S}^J \times {}^J\mathcal{S}^K & \xrightarrow{- *_J -} & {}^I\mathcal{S}^K \end{array} \tag{6.4}$$

*Moreover, in the case where Soergel's conjecture (Conjecture 3.6.6) holds, the map $ch$ sends ${}^I B^J_p$ to ${}^I b^J_p$.*

As a particular case of this, we have that $\mathcal{B} = \mathbb{S}\mathrm{Bim}$, and this categorifies ${}^{\varnothing}\mathcal{S}^{\varnothing} = H$. Another case is that ${}^J\mathcal{B}$ categorifies ${}^I\mathcal{S}^{\varnothing} \cong b_{w_J} H \cong M(J)$. Recall that ${}^J\mathcal{B} = \mathrm{Hom}_{\mathcal{S}\mathbb{S}\mathrm{Bim}}(\varnothing, J)$

was defined as the Karoubian closure of $\mathrm{Hom}_{\mathcal{SBBSBim}}(\varnothing, J)$. This in turn was some subcategory of $(R^J, R)$-mod which contained all Bott-Samelson bimodules, as well as other bimodules. It turns out that it is enough to consider only Bott-Samelsons. That is, we make the following definition.

**Definition 6.2.3.** Let $J\mathbb{BBS}\mathrm{Bim}$ be the full subcategory of $(R^J, R)$-mod, whose objects are Bott-Samelson bimodules, restricted on the left to be left $R^J$-modules. We call this the category of *$J$-singular Bott-Samelson bimodules.*

**Proposition 6.2.4.** *We have that* ${}^J\mathcal{B} \cong Kar(J\mathbb{BBS}Bim)$.

*Proof.* See Claim 2.4[3] of [Eli16a]. □

This category $J\mathbb{BBS}\mathrm{Bim}$ is much easier to work with than $\mathrm{Hom}_{\mathcal{SBBSBim}}(R, R^J)$. In terms of sequences of subsets $I \subset S$, a Bott-Samelson $BS(s_1, ..., s_n)$ is indexed by a sequence $(\varnothing, \{s_1\}, \varnothing, ..., \varnothing, \{s_n\}, \varnothing)$. A $J$-singular Bott-Samelson bimodule is given by precomposing such a sequence with a 'decreasing' sequence $(J = J_d, J_{d-1}, ..., J_1 = \{s\})$ for some $s \in J$, where $J_i$ contains one less element than $J_{i+1}$. Thus we see that the bimodules contained in $J\mathbb{BBS}\mathrm{Bim}$ are far more constrained than those in $\mathrm{Hom}_{\mathcal{SBBSBim}}(R, R^J)$. However, the Karoubian closures of the two categories are the same.

**Definition 6.2.5.** We refer to this category $Kar(J\mathbb{BBS}\mathrm{Bim}) \cong \mathrm{Hom}_{\mathcal{SBSBim}}(\varnothing, J)$ as the *algebraic spherical category*, since it categorifies the spherical module.

The remainder of the thesis will be highly analogous to what we have already presented, only we replace $H$ with $M$, and replace $\mathbb{BBS}\mathrm{Bim}$ with $J\mathbb{BBS}\mathrm{Bim}$. That is, we are going to construct a diagrammatic category $\mathcal{M}_{BS}$ equivalent to $J\mathbb{BBS}\mathrm{Bim}$[4], and then construct a "double-leaves" basis of $\mathcal{M}_{BS}$. The Karoubian closure $\mathcal{M} := Kar(\mathcal{M}_{BS})$ provides another categorification of the spherical module, which we call the *diagrammatic spherical category.*

[3]Technically this claim is made in type A, but the proof applies to all types.

[4]Again, this equivalence will not be proven until Chapter 11.

# Chapter 7

# Spherical Diagrammatics

## 7.1 The categories $\mathcal{M}_{BS}$ and $\mathcal{M}$

We now present a diagrammatic category $\mathcal{M}_{BS}$ which is equivalent to $J\mathbb{BS}$Bim. This category was introduced in type A by Elias in [Eli16a][1]. The definition extends to all types without significant change.

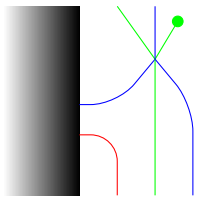
The only difference between $\mathbb{BS}$Bim and $J\mathbb{BS}$Bim is that objects in $J\mathbb{BS}$Bim are considered left $R^J$-modules. This gives us more freedom with the morphisms: since $R^J \subset R$, a left $R$-module morphism is certainly a left $R^J$-module morphism as well. But the converse is not always true: we have $R^J$-module morphisms which are not $R$-module morphisms.

For example, consider the morphism $\varphi : B_s \to R$ given by $\varphi(f \otimes g) := \partial_s(f)g$ for some $s \in J$. This is not a left $R$-module morphism: for $h \in R$, $\varphi(h \cdot f \otimes g) = \partial_s(hf) \otimes g$, and in general $\partial_s(hf) \neq h\partial_s(f)$. However, if $h \in R^J$, then we do have $\partial_s(hf) = h\partial_s(f)$ (this can be checked easily from the definition of $\partial_s$). Therefore $\varphi$ is a morphism of $(R^J, R)$-bimodules, but not of $(R, R)$-bimodules.

So we have more morphisms in $J\mathbb{BS}$Bim than in $\mathbb{BS}$Bim. This should be reflected in the

[1] It is denoted $\mathcal{T}_J$ in [Eli16a]

diagrammatic category $\mathcal{M}_{BS}$. The morphisms will look something like this:



That is, the diagrams will look the same as those in $\mathcal{H}_{BS}$, except that we have this black 'wall' on the left which our strands can plug into. This wall represents our parabolic subset $J \subset S$, and only colours in $J$ can plug into the wall.

We now present $\mathcal{M}_{BS}$ formally.

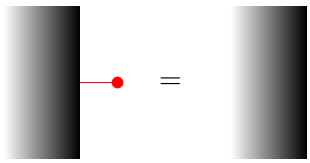
**Definition 7.1.1.** The category $\mathcal{M}_{BS}$ is defined as follows. Its objects are the same as those of $\mathcal{H}_{BS}$: they are sequences of coloured dots, with each colour corresponding to a simple reflection in $S$. Again we will use an expression $\underline{x}$ to denote the corresponding object in $\mathcal{M}_{BS}$. The morphisms $\underline{x} \to \underline{y}$ are certain string diagrams with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, with a wall on the left. Precisely, they are isotopy classes of diagrams generated by the generators of $\mathcal{H}_{BS}$, together with two new generating morphisms

(7.1)

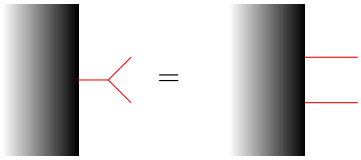
for each $s \in J$. Since diagrams are defined to be the same up to isotopy, we can consider this to be the addition of only a single generator

so long as we keep in mind that this is not a well-defined morphism in itself; it needs to be attached to other strands to give it a well-defined source and target.

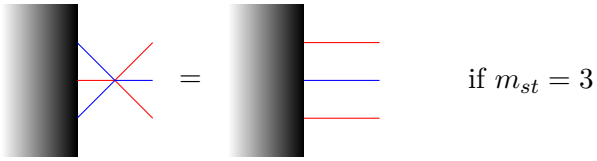
The diagrams in $\mathcal{M}_{BS}$ are subject to the same local relations as those in $\mathcal{H}_{BS}$ (see Chapter 4), together with new relations corresponding to the new generator:



(7.2)



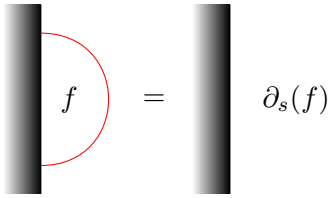
(7.3)



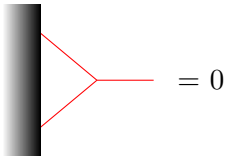

(7.4)

The relation (7.4) has been drawn in the case $m_{st} = 3$, but holds for general $m_{st}$.

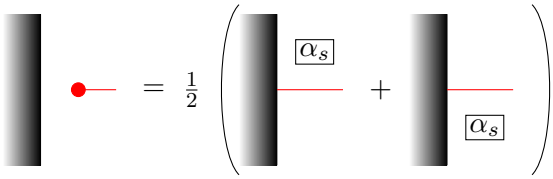
The relations (7.2) - (7.4) say that dots, trivalent vertices and $2m_{st}$-valent vertices can all be pulled into the wall. These give rise to some other relations:




(7.5)




(7.6)




(7.7)

Relations (7.5) and (7.6) are both a result of combining (7.3) with the keyhole relation (4.8). Relation (7.7) is a result of combining (7.2) with fusion (4.10).

**Definition 7.1.2.** We define the *diagrammatic spherical category* to be $\mathcal{M} := Kar(\mathcal{M}_{BS})$.

**Remark 7.1.3.** It is clear that $\mathcal{M}_{BS}$ is a right module category over $\mathcal{H}_{BS}$. That is, we have a map

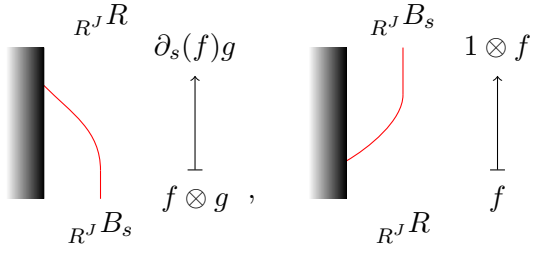
$$\mathcal{M}_{BS} \times \mathcal{H}_{BS} \to \mathcal{M}_{BS}$$

giving a right action of $\mathcal{H}_{BS}$ on $\mathcal{M}_{BS}$. This map concatenates objects and morphisms in the natural way. This action of $\mathcal{H}_{BS}$ on $\mathcal{M}_{BS}$ induces an action of $\mathcal{H}$ on $\mathcal{M}$ by taking Karoubi envelopes. Thus, $\mathcal{M}$ is a right module category over $\mathcal{H}$. Similarly, we know from the structure of $\mathcal{S}\mathbb{S}\mathrm{Bim}$ (see Theorem 6.2.2) that the algebraic spherical category $Kar(J\mathbb{BSS}\mathrm{Bim})$ is a right module category over $\mathbb{S}\mathrm{Bim}$. In Chapter 11, we will show that the diagrammatic and algebraic spherical categories $\mathcal{M}$ and $Kar(J\mathbb{BSS}\mathrm{Bim})$ are equivalent as module categories. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{H} & \xrightarrow{\quad - \otimes - \quad} & \mathcal{M} \\ \downarrow & \circlearrowleft & \downarrow \\ Kar(J\mathbb{BSS}\mathrm{Bim}) \times \mathbb{S}\mathrm{Bim} & \xrightarrow{\quad - \otimes - \quad} & Kar(J\mathbb{BSS}\mathrm{Bim}) \end{array}$$

The first step in this direction will be to construct a functor $\mathcal{M}_{BS} \to J\mathbb{BSS}\mathrm{Bim}$, which we will do presently. After taking Karoubi envelopes, this induces the functor $\mathcal{M} \to Kar(J\mathbb{BSS}\mathrm{Bim})$ in the above diagram. We prove in Chapter 11 that this is an equivalence.

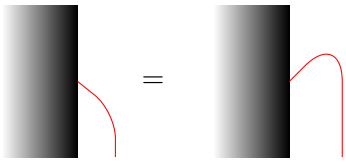
We will construct a functor $\tilde{\mathcal{A}} : \mathcal{M}_{BS} \to J\mathbb{BSS}\mathrm{Bim}$[2], analogous to the functor $\mathcal{A} : \mathcal{H}_{BS} \to \mathbb{BSS}\mathrm{Bim}$ presented in Chapter 4. We define $\tilde{\mathcal{A}}$ to be the same as $\mathcal{A}$ on generators of $\mathcal{H}_{BS}$. So we only need to define it on the new generators (7.1). The images of these maps in $J\mathbb{BSS}\mathrm{Bim}$ is given by the following.




(7.8)

[2]The category $J\mathbb{BSS}\mathrm{Bim}$ is defined in Definition 6.2.3.

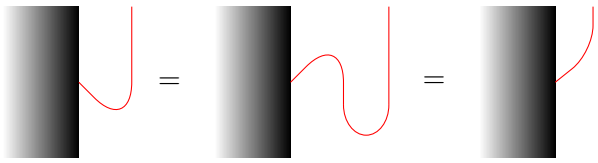
It can easily be checked that these are indeed $(R^J, R)$-module morphisms. To see that the functor $\tilde{\mathcal{A}}$ is well-defined up to isotopy, it suffices to check that it respects the following relation.



The functor $\tilde{\mathcal{A}}$ sends the right-hand diagram to the map

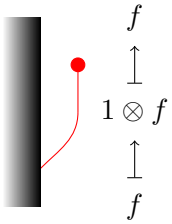
$$f \otimes g \mapsto 1 \otimes f \otimes g \mapsto \partial_s(f) \otimes g \mapsto \partial_s(f)g,$$

which is exactly the left hand side. From this, we also get

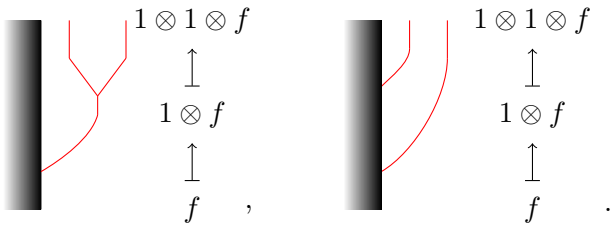


so that the functor $\tilde{\mathcal{A}}$ is well-defined up to isotopy.

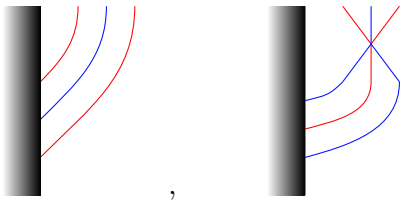
We also want to show that it respects the relations (7.2), (7.3) and (7.4). The fact that it respects (7.2) comes from the computation




so that this is the identity morphism. The relation (7.3) comes from the following computations.




Finally, the relation (7.4) follows because both of the following maps send 1 to the 1-tensor[3].




Thus we have shown that the functor $\tilde{\mathcal{A}}$ is well-defined. Once we have constructed the double-leaves basis of $\mathcal{M}_{BS}$, we will be able to show that the functor $\tilde{\mathcal{A}}$ is in fact an equivalence of categories (see Theorem 11.0.1).

## 7.2 Spherical light-leaves

Just as in the case of $\mathcal{H}_{BS}$, we would like a basis of the morphism spaces of $\mathcal{M}_{BS}$. Specifically, we want an algorithm that takes any two objects $\underline{x}, \underline{y} \in \mathcal{M}_{BS}$ and gives us a diagrammatic basis for $\mathrm{Hom}_{\mathcal{M}_{BS}}(\underline{x}, \underline{y})$. This algorithm will be much the same as the light-leaves algorithm described earlier, with slight variations. First, we construct the spherical light-leaves, and then put them together into a "double-leaves" basis.

Recall the construction of light-leaves in $\mathcal{H}_{BS}$: For any expression $\underline{x} = (s_1, ..., s_n)$ and any subexpression $\underline{e} = (e_1, ..., e_n) \subset \underline{x}$, we constructed a light leaf $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$, where $\underline{w}$ is a reduced expression for $w := \underline{x}^{\underline{e}}$. This was done by taking the stroll $w_0, ..., w_n$ associated to

[3]This is because the $2m$-valent vertex send the 1-tensor to the 1-tensor. See Definition 3.9 of [EK11].

$\underline{e}$ and decorating it with a sequence $d_1, ..., d_n$, with $d_i \in \{U0, U1, D0, D1\}$. There are some small differences in the spherical case: First, instead of taking the stroll $w_j = s_1^{e_1} \cdots s_j^{e_j}$, we take the *coset representative stroll* (or just 'coset stroll') $z_0, ..., z_n$, where $z_j$ is the minimal coset representative of $W_J w_j$. So $z_0, ..., z_n$ is a stroll in the set ${}^J W$ of minimal coset representatives. This set ${}^J W$ has a partial order inherited from the Bruhat order. We again decorate this coset stroll with a sequence $d'_1, ..., d'_n$. In the spherical case, there are two extra options for $d'_i$, which we call $X0$ and $X1$. The $d'_i$s are chosen by the rule

$$d'_i = \begin{cases} Ue_i & \text{if } z_{i-1}s_i > z_{i-1} \text{ and } z_{i-1}s_i \in {}^J W \\ De_i & \text{if } z_{i-1}s_i < z_{i-1} \text{ and } z_{i-1}s_i \in {}^J W \\ Xe_i & \text{if } z_{i-1}s_i \notin {}^J W \end{cases} \tag{7.9}$$

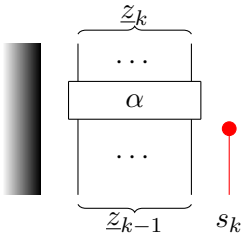
So for example, take $W = S_3 = \langle s, t \rangle$, $J = \{s\}$, $\underline{x} = (t, s, t)$, and $\underline{e} = (0, 1, 1)$. Then our usual (non-coset) stroll is $w_0, w_1, w_2, w_3 = id, id, s, st$, but $s$ and $st$ are not minimal coset representatives (since they have $s$ on the left). So our coset stroll is $z_0, z_1, z_2, z_3 = id, id, id, t$, and our decoration is $d'_1, d'_2, d'_3 = U0, X1, U1$.

## 7.3 Spherical light-leaves algorithm

Take an expression $\underline{x} = (s_1, ..., s_n)$ and a subexpression $\underline{e} \subset \underline{x}$. We will construct a morphism $SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{z}$ in $\mathcal{M}_{BS}$, where $\underline{z}$ is now a reduced expression for the minimal coset representative $z$ of $\underline{x}^{\underline{e}}$. Once again, this $SLL_{\underline{x},\underline{e}}$ will be constructed inductively: at each step $k$ we construct a map $SLL_k : \underline{x}_{\leq k} \to \underline{z}_k$, where $\underline{x}_{\leq k} := (s_1, ..., s_k)$ and $\underline{z}_k$ is a reduced expression for the minimal coset representative $z_k$ of $w_k := s_1^{e_1} \cdots s_k^{e_k}$ (so $z_0, ..., z_n$ is the coset stroll associated to $\underline{e}$). We again call these expressions $\underline{z}_k$ *intermediate expressions.* These $SLL_k$'s will be built out of maps $\phi_k : (\underline{z}_{k-1}, s_k) \to \underline{z}_k$ in exactly the same way as in the non-spherical case (see figure (5.1)). In the non-spherical case there were four options for $\phi_k$, corresponding to the four possibilities for $d_k : U0, U1, D0$ and $D1$. In the spherical case, $d'_k$ can also be $X0$ and $X1$, giving us two more possibilities for $\phi_k$:

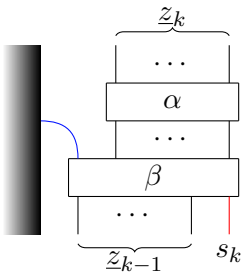
**X0**

If $d'_k = X0$ then we get the same $\phi_k$ as for $U0$:




So in fact, there is only one genuinely new possibility for $\phi_k$, and that is when $d'_k = X1$.

**X1**



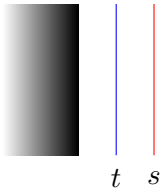

(7.10)

If $d'_k = X1$, then $z_{k-1}s_k$ is not a minimal coset representative. Therefore by Lemma 6.1.3 (b), we have $z_{k-1}s_k = tz_{k-1}$ for some $t \in J$. This $t$ is given in blue in the diagram. By Lemma 6.1.3 (a), $z_{k-1}s_k > z_{k-1}$, so that $(\underline{z}_{k-1}, s_k)$ is a reduced expression. Thus, $(t, \underline{z}_{k-1})$ is also a reduced expression (being of the same length), and so there is some rex move $\beta$ taking $(\underline{z}_{k-1}, s_k)$ to $(t, \underline{z}_{k-1})$. We apply this $\beta$, and then plug $t$ into the wall. Finally, we apply some rex move $\alpha$ to get the required target $\underline{z}_k$.

This will be made clearer with an example.

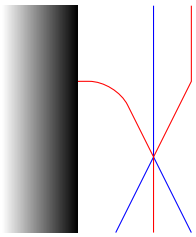
**Example 7.3.1.** We take $W = S_3 = \langle s, t \rangle$, $J = \{s\}$, $\underline{x} = (t, s, t)$ and $\underline{e} = (1, 1, 1) \subset \underline{x}$. Then $\underline{x}^{\underline{e}} = tst = sts$, and the minimal coset representative of this is $z = ts$. So we will construct a light leaf $SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{z}$, where $\underline{z} = (t, s)$. First we find that $d'_1, d'_2, d'_3 = U1, U1, X1$ (the $X1$ comes because $tst = sts$ is not a minimal coset representative). Thus the first two steps $SLL_1$ and $SLL_2$ proceed exactly as in the non-spherical algorithm:

$SLL_2 =$




The last step is more interesting. Since $tst$ is not a minimal coset representative, we have some reduced expression of $tst$ with $s$ on the left: of course this is $sts$. Thus we first apply a rex move $\beta$ to take $(t, s, t)$ to $(s, t, s)$: this is just the 6-valent vertex. Then we plug the $s$-strand into the wall.

$SLL_{\underline{x},\underline{e}} = SLL_3 =$



Just as with the non-spherical light-leaves, we have some flexibility in the construction of a light-leaf $SLL_{\underline{x},\underline{e}}$; that is, we get choices in the intermediate expressions $\underline{z}_k$ and the rex moves $\alpha$ and $\beta$. So once again, there is not a notion of *the* light-leaf $SLL_{\underline{x},\underline{e}}$. Rather, the label $SLL_{\underline{x},\underline{e}}$ could refer to any of a number of possible constructions, depending on our choices of the $\underline{z}_k$'s and rex moves. When it comes time to form a basis using these light-leaves, we fix a choice of the construction of $SLL_{\underline{x},\underline{e}}$ for each $\underline{e} \subset \underline{x}$.

## 7.4 The double-leaves basis

Just as in the non-spherical case, we can now use our light-leaves to construct bases of morphism spaces. Given any two expression $\underline{x}$ and $\underline{y}$, we can produce 'double-leaf' maps $\underline{x} \to \underline{y}$ by sticking together two light-leaves.

We do this just as in Section 5.2. We first fix our light-leaves: For each $z \in {}^J W$ such that $W_J z = W_J \underline{x}^{\underline{e}}$ for some $\underline{e} \subset \underline{x}$, we fix a reduced expression $\underline{z}$ of $z$. Similarly, for each

$z' \in {}^J W$ such that $W_J z' = W_J \underline{y}^{\underline{f}}$ for some $\underline{f} \subset \underline{y}$, we fix a rex $\underline{z}'$ of $z'$. We do this in a compatible way, i.e. if $z = z'$, we choose $\underline{z} = \underline{z}'$.

Then, take subexpressions $\underline{e} \subset \underline{x}$ and $\underline{f} \subset \underline{y}$, and let $z$ and $z'$ be minimal coset representatives of $\underline{x}^{\underline{e}}$ and $\underline{y}^{\underline{f}}$ respectively. We fix constructions of light-leaf maps $SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{z}$ and $SLL_{\underline{y},\underline{f}} : \underline{y} \to \underline{z}'$. If $z = z'$, then by the way we have fixed reduced expressions, these both have the same image $\underline{z} = \underline{z}'$. Thus we can construct a 'spherical double-leaf' $\mathbb{SDL}_{\underline{f},\underline{e}} := \overline{SLL}_{\underline{y},\underline{f}} \circ SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{z} \to \underline{y}$. We will spend the next few chapters proving that these double-leaves form a basis of Hom-spaces in $\mathcal{M}_{BS}$:

**Theorem 7.4.1.** *Suppose our realization $\mathfrak{h}$ is reflection faithful*[4]*. Given $\underline{x}, \underline{y} \in \mathcal{M}_{BS}$, let $\mathcal{SDL}_{\underline{x},\underline{y}}$ contain one double-leaf map $\mathbb{SDL}_{\underline{f},\underline{e}}$ for every pair of subexpressions $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $W_J \underline{x}^{\underline{e}} = W_J \underline{y}^{\underline{f}}$. Then $\mathcal{SDL}_{\underline{x},\underline{y}}$ is a basis for $Hom_{\mathcal{M}_{BS}}(\underline{x}, \underline{y})$ as a right $R$-module.*

[4]In fact, $\mathfrak{h}$ only needs to be faithful. In particular, if it is reflection faithful, it is also faithful. For the sake of simplicity, we always assume throughout the paper that $\mathfrak{h}$ is reflection faithful, since this is necessary for the results in Section 11.

# Chapter 8

# Standard Diagrammatics

In Chapter 9 we will prove the linear independence of the spherical double-leaves, using localization techniques. Let $Q$ once again be the field of fractions of $R$. Our aim in this chapter is to show that, after localization, we can express our double-leaf maps as matrices with entries in $Q$. This means we can use the tools of linear algebra to investigate the double-leaves.

More precisely, our double-leaves lie in $\mathcal{M}_{BS}$. When we localize them, they lie in $\mathcal{M}_{BS,Q} := \mathcal{M}_{BS} \otimes Q$. We will show that $Kar(\mathcal{M}_{BS,Q})$ is equivalent to $JStd\text{Bim}_Q$ (Proposition 8.4.6). The category $JStd\text{Bim}_Q$ is the same as $Std\text{Bim}_Q$ (see Definition 3.9.4), except we consider objects as $(Q^J, Q)$-bimodules rather than $Q$-bimodules. We will see in Lemma 8.3.5 that, like in $Std\text{Bim}_Q$, morphisms in $JStd\text{Bim}_Q$ are just matrices with entries in $Q$. Since we have a fully-faithful embedding

$$\mathcal{M}_{BS,Q} \hookrightarrow Kar(\mathcal{M}_{BS,Q}) \cong JStd\text{Bim}_Q,$$

we can consider our double-leaf maps to be matrices in $Q$.

It will be useful to first construct the analogous picture for our non-spherical categories. That is, we take the localization $\mathcal{H}_{BS,Q} := \mathcal{H}_{BS} \otimes Q$, and show that $Kar(\mathcal{H}_{BS,Q}) \cong Std\text{Bim}_Q$. This is the content of Sections 8.1 and 8.2, which largely reiterate material

from [EW16]. In Sections 8.3 and 8.4 we move on to our spherical categories, to show that $Kar(\mathcal{M}_{BS,Q}) \cong JStd\text{Bim}_Q$. In both cases, our strategy for proving the equivalences will be to relate them via a third category of standard diagrammatics $\mathcal{H}_Q^{\text{std}}$ or $\mathcal{M}_Q^{\text{std}}$, yielding the following commutative diagrams of equivalences.

$$\begin{array}{ccc} & Kar(\mathcal{H}_{BS,Q}) & \\ \mathcal{S}td \nearrow & \circlearrowleft & \searrow \mathcal{G} \\ \mathcal{H}_Q^{\text{std}} & \xrightarrow{\mathcal{F}_Q^{\text{std}}} & Std\text{Bim}_Q \end{array} \tag{8.1}$$

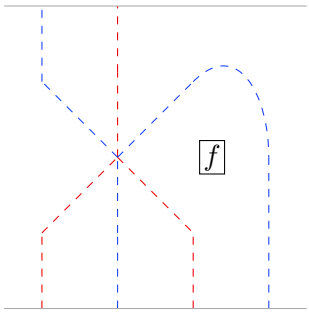
$$\begin{array}{ccc} & Kar(\mathcal{M}_{BS,Q}) & \\ \tilde{\mathcal{S}td} \nearrow & \circlearrowleft & \searrow \tilde{\mathcal{G}} \\ \mathcal{M}_Q^{\text{std}} & \xrightarrow{\tilde{\mathcal{F}}_Q^{\text{std}}} & JStd\text{Bim}_Q \end{array} \tag{8.2}$$

Again, it is proving the equivalence $\tilde{\mathcal{G}} : Kar(\mathcal{M}_{BS,Q}) \to JStd\text{Bim}_Q$ above which is the primary goal of this chapter.

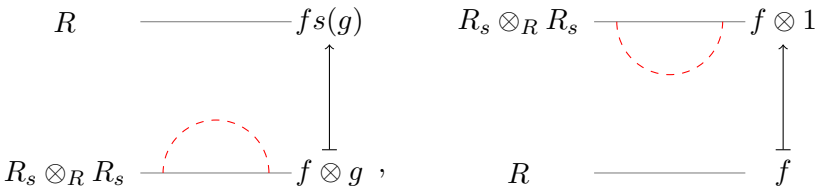
## 8.1 The category $\mathcal{H}^{\text{std}}$ and the functor $\mathcal{F}^{\text{std}} : \mathcal{H}^{\text{std}} \to Std\text{Bim}$

We will define a diagrammatic $\Bbbk$-linear additive category $\mathcal{H}^{\text{std}}$. It is the diagrammatic version of $Std\text{Bim}$. That is, we have a functor $\mathcal{F}^{\text{std}} : \mathcal{H}^{\text{std}} \to Std\text{Bim}$, and when our realization is faithful, this is an equivalence (see Theorem 8.1.1). The objects of $\mathcal{H}^{\text{std}}$ are (direct sums of grading shifts of) sequences of coloured dots, with each colour corresponding to a simple reflection in $S$. Whereas in $\mathcal{H}_{BS}$ the sequence $\underline{w} = (s_1, ..., s_n)$ could be thought of as representing the Bott-Samelson bimodule $BS(\underline{w}) = B_{s_1} \otimes \cdots \otimes B_{s_n}$, in $\mathcal{H}^{\text{std}}$ the sequence $\underline{w}$ represents the standard bimodule $R_{s_1} \otimes_R \cdots \otimes_R R_{s_n} \cong R_w$. More precisely, we have a functor $\mathcal{F}^{\text{std}} : \mathcal{H}^{\text{std}} \to Std\text{Bim}$ which takes $(s_1, ..., s_n)$ to $R_{s_1} \otimes_R \cdots \otimes_R R_{s_n}$. In particular, while the Bott-Samelson bimodule $BS(\underline{w})$ depends on the choice of reduced expression $\underline{w}$ of $w$, the standard bimodule $R_w \cong R_{s_1} \otimes \cdots \otimes R_{s_n}$ does not. Thus given two different expressions $\underline{w}$ and $\underline{w}'$ for $w$, the corresponding objects in $\mathcal{H}^{\text{std}}$ will be isomorphic (but not identical). We call the objects in $\mathcal{H}^{\text{std}}$ given by a single $s \in S$ *reflection indices*.

Morphisms in $\mathcal{H}^{\mathrm{std}}$ are $\Bbbk$-linear combinations of isotopy classes of *standard diagrams*, diagrams of dotted strings with top and bottom boundaries being objects in $\mathcal{H}^{\mathrm{std}}$, such that the only vertices are $2m_{st}$-valent vertices, together with polynomials in boxes. Here is an example of a standard diagram:




How should we interpret these diagrams algebraically, i.e. what is the image of $\mathcal{F}^{\mathrm{std}} : \mathcal{H}^{\mathrm{std}} \to \mathit{Std}\mathrm{Bim}$? A vertical dashed strand represents the identity on the corresponding $R_s$. A polynomial box represents multiplication by that polynomial. Then, we have an isomorphism $R_s \otimes_R R_s \xrightarrow{\sim} R$ sending the 1-tensor to 1. This isomorphism is drawn diagrammatically as a cap, and its inverse as a cup.



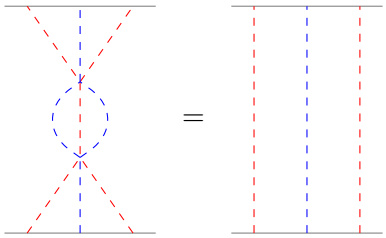

(8.3)

This gives rise to the following relations.


(8.4)

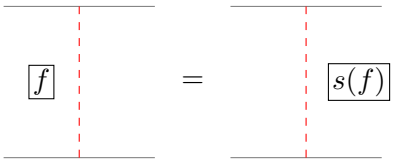

(8.5)

(The empty diagram on the right of (8.5) gives the identity on $R$). Similarly, if $m_{st} = 3$, we have an isomorphism $R_s \otimes R_t \otimes R_s \cong R_{sts} \cong R_t \otimes R_s \otimes R_t$ sending the 1-tensor to the 1-tensor. This isomorphism is represented diagrammatically as a dashed $2m_{st}$-valent vertex, with relation



(8.6)

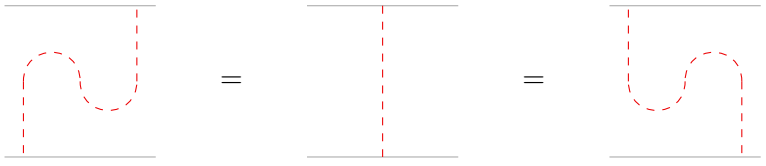
Polynomial boxes multiply as we would expect. We can also slide polynomials across dashed lines by applying the corresponding $s$.



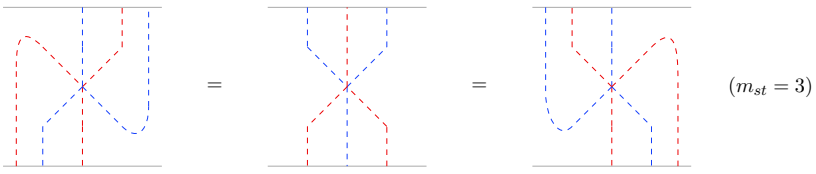

(8.7)

The final relations are the three-colour relations, which are exactly the same as the three-colour relations in $\mathcal{H}_{BS}$, only with dashed lines instead of solid lines[1]. See Section 4 of [EW16] to see these relations explicitly in each type. This concludes the presentation of $\mathcal{H}^{\text{std}}$.

To see that the functor $\mathcal{F}^{\text{std}}$ is well-defined up to isotopy, it suffices to check that the following equalities are maintained under $\mathcal{F}^{\text{std}}$:



(8.8)




(8.9)

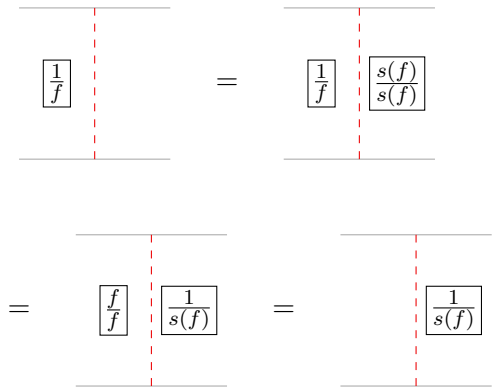
[1]The exception is in type $H_3$, where we no longer have the "lower terms".

This follows from the fact that all these morphisms are isomorphisms sending 1-tensors to 1-tensors.

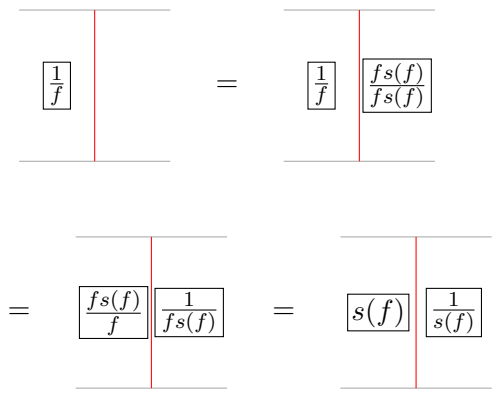
**Theorem 8.1.1.** *The functor $\mathcal{F}^{std} : \mathcal{H}^{std} \to StdBim$ is faithful and essentially surjective. When our realization is faithful, $\mathcal{F}^{std}$ is an equivalence.*

*Proof.* See Theorem 4.11 of [EW16].

We can also localize everything to get an equivalence $\mathcal{F}^{\mathrm{std}}_Q : \mathcal{H}^{\mathrm{std}}_Q \xrightarrow{\sim} Std\mathrm{Bim}_Q$. Here, $\mathcal{H}^{\mathrm{std}}_Q$ is the category which is the same as $\mathcal{H}^{\mathrm{std}}$, except that now we are allowed to have elements in $Q$ on the right of our diagrams. Note that this means we can have elements in $Q$ anywhere in our diagrams, since using (8.7) we have:




Similarly, we can localize $\mathcal{H}_{BS}$ to form $\mathcal{H}_{BS,Q}$, in which we allow elements in $Q$ on the right of our diagrams. Once again, this means we are allowed elements in $Q$ anywhere in our diagrams, since using polynomial forcing (4.11) we have:



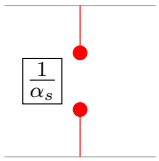

(8.10)

## 8.2 Mixed diagrammatics: the category $Kar(\mathcal{H}_{BS,Q})$

Recall from Section 3.9 that we localized $\mathbb{BS}$Bim to form $\mathbb{BS}\text{Bim}_Q$, and then included directs sums and summands to get $\mathbb{S}\text{Bim}_Q$. Using the terminology of Section 4.5, we have that $\mathbb{S}\text{Bim}_Q = Kar(\mathbb{BS}\text{Bim}_Q)$.[2] Then we had that $\mathbb{S}\text{Bim}_Q$ is equivalent to $Std\text{Bim}_Q$, the category of direct sums standard $Q$-bimodules. We will take an analogous path with our diagrammatic categories. We have localized $\mathcal{H}_{BS}$ to form $\mathcal{H}_{BS,Q}$. Now we want to construct a diagrammatic category $Kar(\mathcal{H}_{BS,Q})$, the Karoubian closure of $\mathcal{H}_{BS,Q}$.

Here we will describe how to take the Karoubian closure in a slightly different manner to Section 4.5, which will be more conducive to our current purposes. Taking the Karoubian closure means formally including the images of idempotents as new objects in our category. Take a general category $\mathcal{C}$, and an object $X \in \mathcal{C}$ with idempotent $e : X \to X$. Let $Y$ be the new object representing the image of $e$.[3] To append $Y$ as a new object in $\mathcal{C}$, we need inclusion and projection morphisms $\iota : Y \hookrightarrow X$, $\pi : X \twoheadrightarrow Y$, satisfying relations $\pi \circ \iota = id_Y$ and $\iota \circ \pi = e$.

Let us now apply this to the category $\mathcal{H}_{BS,Q}$. We have an idempotent $e : s \to s$.

$e =$ 

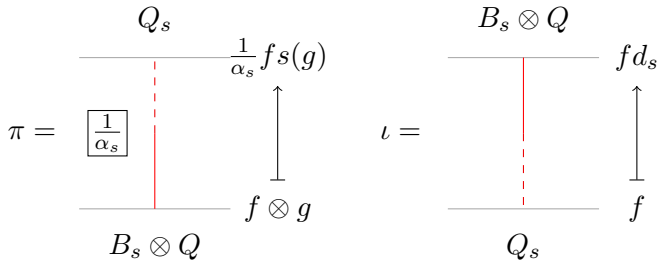

Recall from equation (3.6) that we have

$$BS(\underline{w}) \otimes_R Q \cong \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{e}}.$$

[2]When we defined the Karoubian closure, it required the addition of grading shifts of objects. However, there are no grading considerations here, since we do not consider $Q$ to be a graded ring. Therefore, when a category $\mathcal{C}$ is not a graded category, we slightly abuse notation by not requiring $Kar(\mathcal{C})$ to have grading shifts.

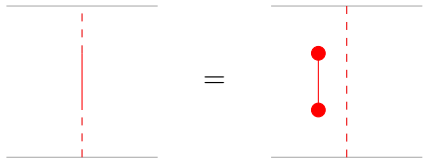
[3]This $Y$ corresponds to the pair $(X, e)$ in the terminology of Section 4.5.

In particular, we have $B_s \otimes_R Q \cong Q \oplus Q_s$. The above idempotent $e$ corresponds to projecting from $B_s \otimes Q$ to $Q$, and including back in. Therefore the other idempotent $1 - e$ gives projection onto $Q_s$, followed by inclusion. But there is no object in $\mathcal{H}_{BS,Q}$ corresponding to $Q_s$. Thus to construct $Kar(\mathcal{H}_{BS,Q})$ we add new objects $s^{\text{std}}$ for each $s \in S$ called reflection indices, representing the standard bimodules $Q_s$. More precisely, we will have a functor $\mathcal{G} : Kar(\mathcal{H}_{BS,Q}) \to Std\text{Bim}$, sending $s^{\text{std}}$ to $Q_s$. Taking inspiration from our category $\mathcal{H}^{\text{std}}$, we represent the identity on $s^{\text{std}}$ as a dashed line. Then, we construct morphisms $\pi : s \to s^{\text{std}}$ and $\iota : s^{\text{std}} \to s$. We include the algebraic images of these morphisms under $\mathcal{G}$ alongside the diagrams.

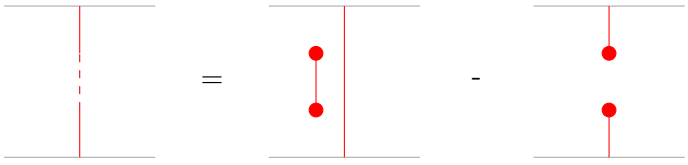



(8.11)

Here, $d_s := \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$. We have chosen to put $\frac{1}{\alpha_s}$ as part of the projection map, simply to make the diagrammatics work out more nicely. We call these maps *bivalent vertices* (ignoring the $\frac{1}{\alpha_s}$). The requirements $\pi \circ \iota = id_Y$ and $\iota \circ \pi = e$, after multiplying by $\alpha_s$, give diagrammatic relations




(8.12)



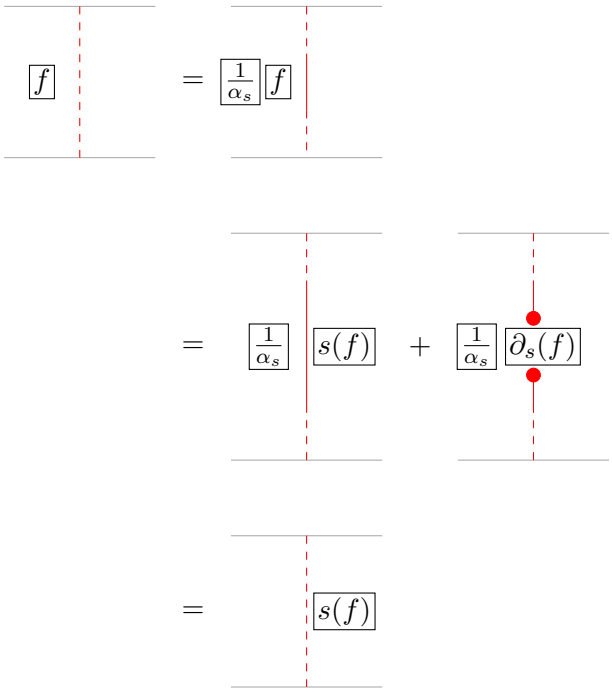

(8.13)

It is not too hard to check that $\mathcal{G}$ respects these relations. Note that we are not yet considering our diagrams up to isotopy. In fact, in the end isotopic diagrams in $Kar(\mathcal{H}_{BS,Q})$ will only be the same up to sign. Using the above relations we can obtain the following

crucial consequence:

(8.14)

This in turn tells us that we can move polynomials across dotted strands in the same way as in $\mathcal{H}^{\text{std}}$ (see equation (8.7)).



(8.15)

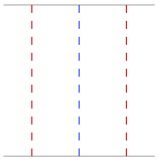
We defined the functor $\mathcal{G} : Kar(\mathcal{H}_{BS,Q}) \to Std\text{Bim}_Q$ on the reflection indices and the bivalent vertices. All other objects and generators of $Kar(\mathcal{H}_{BS,Q})$ lie in $\mathcal{H}_{BS,Q}$, and on here we define $\mathcal{G}$ via the functors

$$\mathcal{H}_{BS,Q} \xrightarrow{\mathcal{A}\otimes Q} BS\text{Bim}_Q \hookrightarrow Std\text{Bim}_Q$$

where $\mathcal{A}$ is the functor from $\mathcal{H}_{BS}$ to $BS$Bim defined in Chapter 4.

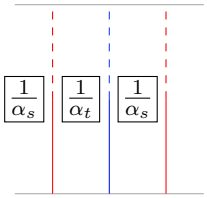
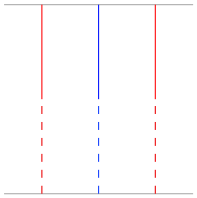
Now that we have appended objects representing $Q_s$ to $\mathcal{H}_{BS,Q}$, we also have objects representing all standard $Q$-bimodules $Q_w$, for $w \in W$. That is, if $\underline{w} = (s_1, ..., s_n)$ is an

expression for $w$, then we have an object $\underline{w}^{\text{std}} := (s_1^{\text{std}}, ..., s_n^{\text{std}})$. For example, the identity on $(s,t,s)^{\text{std}}$ would be drawn as



.

The corresponding projection and inclusion morphisms are just tensor products of those for $s^{\text{std}}$. For example, the projection and inclusion between $(s,t,s)$ and $(s,t,s)^{\text{std}}$ are

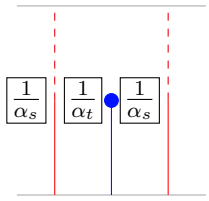
$\pi =$  $\iota =$ 

.

More generally, recall from equation (3.6) that we have

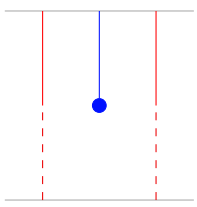
$$BS(\underline{w}) \otimes_R Q \cong \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{e}}.$$

The projection from $BS(\underline{w}) \otimes Q$ to $Q_{\underline{e}}$ will be drawn in $Kar(\mathcal{H}_{BS,Q})$ by tensoring the projections to $s$ and $s^{\text{std}}$ appropriately. For example, let $\underline{w} = (s,t,s)$ and $\underline{e} = (1,0,1)$. Then we have a projection $\pi : (s,t,s) \to (s,s)^{\text{std}}$ drawn as

$\pi =$ 

.

The inclusion is similar.

$\iota =$ 

The fact that these maps $\pi$ and $\iota$ satisfy the required relations for projections and inclusions is a result of the corresponding fact for projections and inclusions between $s$ and $s^{\mathrm{std}}$ for $s \in S$ — i.e. relations (8.12) and (8.13).

In essence we are done defining $Kar(\mathcal{H}_{BS,Q})$[4]. In fact it is not yet totally clear that $Kar(\mathcal{H}_{BS,Q})$ is the Karoubian closure of $\mathcal{H}_{BS,Q}$ since we do not yet know that it is idempotent complete (see Remark 4.5.4). However we will soon see (by Corollary 8.2.2) that $Kar(\mathcal{H}_{BS,Q})$ is equivalent to $Std\mathrm{Bim}_Q$, and therefore is idempotent complete, so that $Kar(\mathcal{H}_{BS,Q})$ is indeed the Karoubian closure.

It will be useful, however, to introduce some diagrammatic shorthands to the category $Kar(\mathcal{H}_{BS,Q})$. Let us briefly zoom out to provide some motivation for this. We have already noted that the dashed lines we are introducing in $Kar(\mathcal{H}_{BS,Q})$ have a strong resemblance to our standard diagrammatic category $\mathcal{H}^{\mathrm{std}}$. In fact, we will construct a functor $\mathcal{S}td : \mathcal{H}_Q^{\mathrm{std}} \to Kar(\mathcal{H}_{BS,Q})$, such that the following diagram commutes:

$$\begin{array}{ccc} & Kar(\mathcal{H}_{BS,Q}) & \\ \mathcal{S}td \nearrow & \circlearrowleft & \searrow \mathcal{G} \\ \mathcal{H}_Q^{\mathrm{std}} & \xrightarrow{\mathcal{F}_Q^{\mathrm{std}}} & Std\mathrm{Bim}_Q \end{array} \tag{8.16}$$

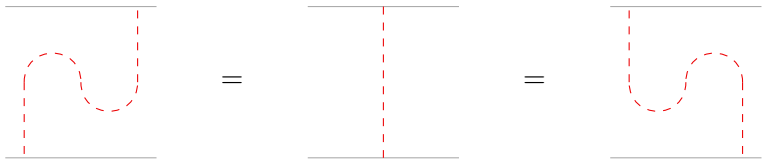
This functor $\mathcal{S}td$ will be an equivalence of categories. This is the diagrammatic version of the equivalence $Std\mathrm{Bim}_Q \cong \mathbb{S}\mathrm{Bim}_Q$. Note that since by Theorem 8.1.1 the functor $\mathcal{F}_Q^{\mathrm{std}}$ is an equivalence, we will have that $\mathcal{G}$ is also an equivalence.

It is currently unclear how to construct this functor $\mathcal{S}td$, since there are currently no dashed cups, caps, or $2m_{st}$-valent vertices in $Kar(\mathcal{H}_{BS,Q})$. The lack of cups and caps also means we do not yet have a notion of isotopy in $Kar(\mathcal{H}_{BS,Q})$. We will rectify this now.

We introduce dashed caps and cups into $Kar(\mathcal{H}_{BS,Q})$, such that their image under $\mathcal{G}$ is the same as the image of dashed caps and cups under $\mathcal{F}^{\mathrm{std}}$. We then include the following

[4]We are not literally done defining it, since we are about to add new generators and relations. However, these will not change anything: each new generator can be expressed using generators we already have, and each new relation is implied by relations we already have. Therefore, these new generators should be considered diagrammatic shorthand.

new relations in $Kar(\mathcal{H}_{BS,Q})$:

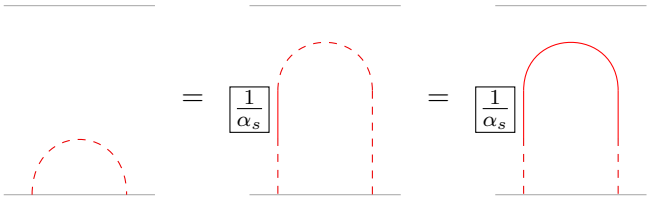


(8.17)

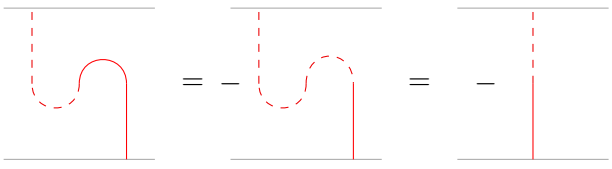
(8.18)

(8.19)

From these we can see that by adding dashed caps and cups, we are not actually adding anything new to our category. For example, we can re-express a dashed cap as

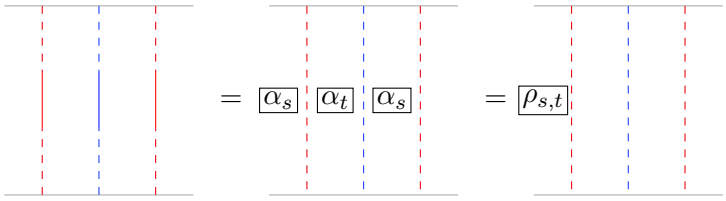


Furthermore, these relations tell us that isotopic diagrams in $Kar(\mathcal{H}_{BS,Q})$ are only the same up to sign. For example,



To define the image of the dashed $2m_{st}$-valent vertex under the functor $\mathcal{S}td$, we need to

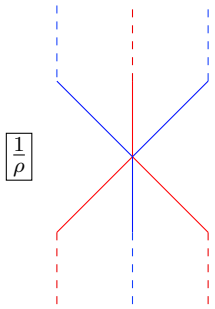
define an element $\rho_{s,t} \in R$. It is defined by the diagram



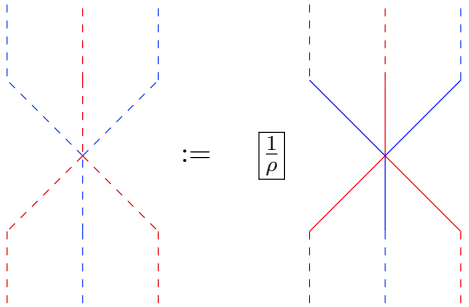

(8.20)

Figure (8.20) depicts the situation for $m_{st} = 3$. In general we have $m_{st}$ strands of alternating colours. We define $\rho_{s,t}$ by moving the $\alpha$'s to the left, using (8.15). This is the product of *positive roots* corresponding to the dihedral group $\langle s, t \rangle$ (see Section 3.4 of [Eli16b] for a definition of positive roots). This means in particular that $\rho_{s,t}$ is well-defined, i.e. it doesn't matter which colour is on the left in (8.20). Now, if we moved all the $\alpha$'s to the right rather than the left, we would end up with $(-1)^{m_{st}}\rho_{s,t}$. This tells us that moving $\rho_{s,t}$ from left to right through the dotted strands multiplies it by $(-1)^{m_{st}}$.

We then set the following morphism to be the image of the dashed $2m$-valent vertex under $\mathcal{S}td$.

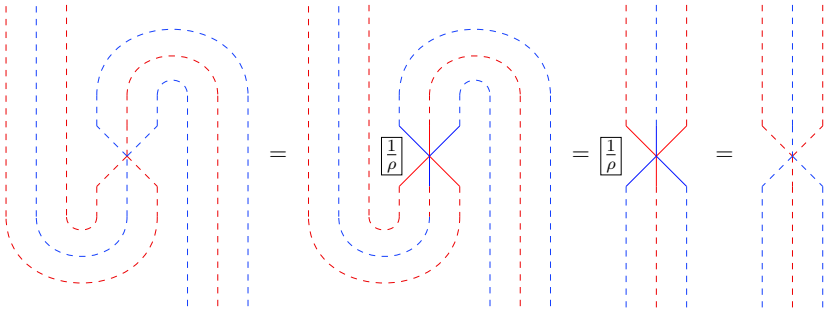



We use the dashed $2m$-valent vertex as a short-hand for this morphism in $Kar(\mathcal{H}_{BS,Q})$. That is, in $Kar(\mathcal{H}_{BS,Q})$ we set



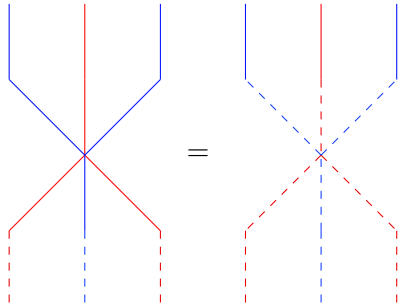

. (8.21)

The reason for the $1/\rho$ out the front is two-fold. First, this renders the morphism a degree 0 morphism, just as the $2m$-valent vertex is degree 0 in $\mathcal{H}^{\text{std}}$ and in $\mathcal{H}_{BS}$. Second, it means that in $Kar(\mathcal{H}_{BS,Q})$, the $2m$-valent vertex is cyclic.



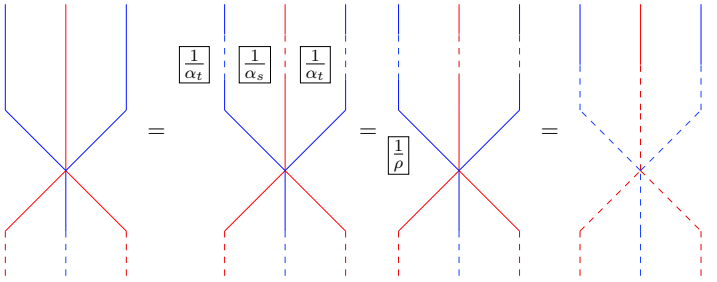

The second equality above comes from repeated applications of (8.18) and (8.19).

We can also pull bivalent vertices through $2m$-valent vertices. That is, in $Kar(\mathcal{H}_{BS,Q})$ we have the relation




. (8.22)

Let us justify this relation now. First, note that if we put an up-dot anywhere on the top of the left-hand diagram in (8.22), then by Proposition 4.2.1 we can pull this dot through to the bottom and get zero by (8.14). The identity on $(s,t,s)$ is a sum of projection and inclusion morphisms to/from its summands. The only non-zero projection is onto $(s,t,s)^{\text{std}}$, since the other projections contain up-dots. We therefore have




.

We can now define the functor $\mathcal{S}td : \mathcal{H}_Q^{\mathrm{std}} \to Kar(\mathcal{H}_{BS,Q})$. It sends $\underline{w}$ to $\underline{w}^{\mathrm{std}}$. It sends dashed cups and caps to dashed cups and caps. It sends the dashed $2m$-valent vertex to the morphism in (8.21).

**Proposition 8.2.1.** *The functor $\mathcal{S}td$ is well-defined, and is an equivalence of categories.*

*Proof.* See Proposition 5.23 of [EW16]. □

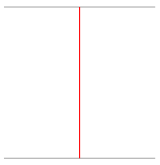
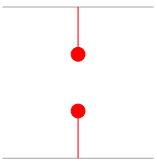
**Corollary 8.2.2.** *When our realization is faithful, the functor $\mathcal{G} : Kar(\mathcal{H}_{BS,Q}) \xrightarrow{\sim} StdBim_Q$ in (8.16) is an equivalence of categories.*

*Proof.* This is due to the equivalences $\mathcal{F}_Q^{\mathrm{std}} : \mathcal{H}_Q^{\mathrm{std}} \to Std\mathrm{Bim}_Q$ and $\mathcal{S}td : \mathcal{H}_Q^{\mathrm{std}} \to Kar(\mathcal{H}_{BS,Q})$ in (8.16). □

The fact that $\mathcal{G}$ is an equivalence implies that $Kar(\mathcal{H}_{BS,Q})$ is indeed the Karoubian closure of $\mathcal{H}_{BS,Q}$. This is because we know that all objects in $Kar(\mathcal{H}_{BS,Q})$ decompose as direct sums of sequences of reflection indices $\underline{w}^{\mathrm{std}}$, and this goes to the indecomposable $Q_w$ under $\mathcal{G}$, so that $Kar(\mathcal{H}_{BS,Q})$ is idempotent complete.

This equivalence $\mathcal{G}$ is the equivalence which will be most significant for us[5]. It allows us to think of morphisms in $\mathcal{H}_{BS,Q}$ as matrices of polynomials in $Q$. Let us illustrate this with an example.

**Example 8.2.3.** We consider two morphisms $(s) \to (s)$ in $\mathcal{H}_{BS,Q}$.

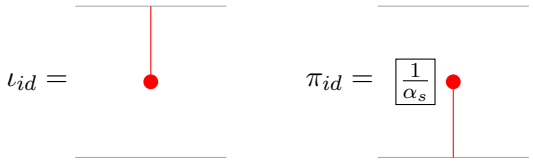
$id =$  $\varphi =$ 

We know that in $Kar(\mathcal{H}_{BS,Q})$, $(s)$ decomposes as $s^{\mathrm{std}} \oplus \mathbb{1}$, where $\mathbb{1}$ is the monoidal identity. When we pass to $Std\mathrm{Bim}_Q$ via $\mathcal{G}$, this becomes $B_s \otimes Q \cong Q \oplus Q_s$, and our two morphisms

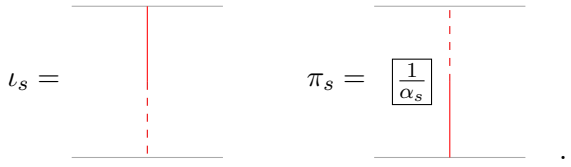
[5]More precisely, the analogous equivalence for our spherical diagrammatic categories. See Corollary 8.4.6.

become 2-by-2 matrices. By the formula (3.7), these matrices should be zero off the diagonal. We can use the diagrammatics of $Kar(\mathcal{H}_{BS,Q})$ to find out what the diagonal entries should be.

The entries of the matrix corresponding to each morphism are found by pre- and post-composing with the appropriate inclusion and projection morphisms. Inclusion and projection morphisms to/from $\mathbb{1}$ are

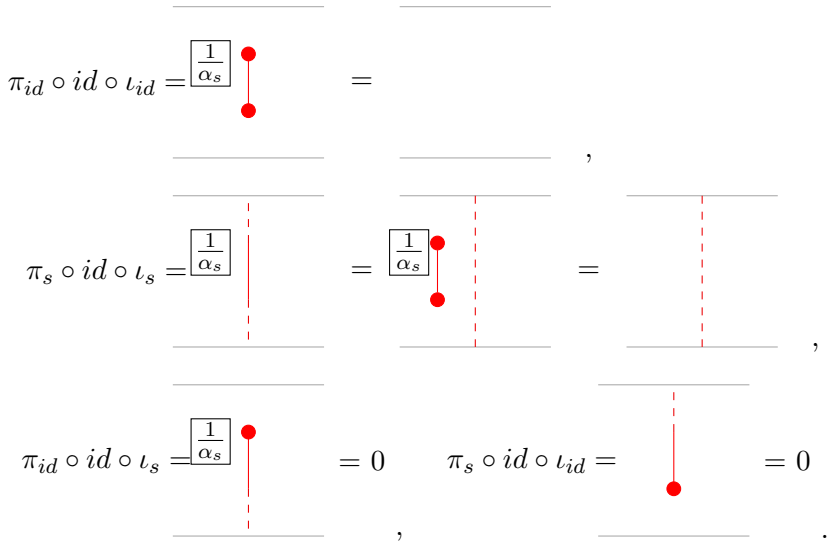



while those to/from $s^{\text{std}}$ are




.

From this we can find the entries of the matrix corresponding to $id$:



This tells us that the matrix corresponding to $id$ is indeed the identity matrix. Similar computations tell us that the matrix corresponding to $\varphi$ is

$$\begin{pmatrix} \alpha_s & 0 \\ 0 & 0 \end{pmatrix}.$$

## 8.3 Spherical standard diagrammatics: the categories

As mentioned, we wish to prove the linear independence of spherical double-leaves using localization techniques. These double-leaves live in $\mathcal{M}_{BS}$, so when we localize them, they will live in $\mathcal{M}_{BS,Q}$: the category which is identical to $\mathcal{M}_{BS}$ except that we allow polynomials in $Q$ on the right of our morphisms. By the same argument as in (8.10), this means we are allowed elements of $Q$ anywhere in our diagrams.

We will now repeat the procedures of the previous section in the case of our spherical categories. That is, we will take $Kar(\mathcal{M}_{BS,Q})$ and construct the following commutative diagram of equivalences, analogous to (8.16).

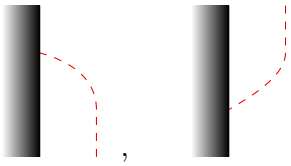
$$\begin{array}{ccc} & Kar(\mathcal{M}_{BS,Q}) & \\ \tilde{\mathcal{S}td} \nearrow & \circlearrowleft & \searrow \tilde{\mathcal{G}} \\ \mathcal{M}_Q^{\mathrm{std}} & \xrightarrow{\tilde{\mathcal{F}}_Q^{\mathrm{std}}} & JStd\mathrm{Bim}_Q \end{array} \tag{8.23}$$

In particular, the equivalence $\tilde{\mathcal{G}} : Kar(\mathcal{M}_{BS,Q}) \xrightarrow{\sim} JStd\mathrm{Bim}_Q$ will allow us to express morphisms in $\mathcal{M}_{BS,Q}$ (such as our localized double-leaves) as matrices with entries in $Q$. Of course, we have yet to define the categories involved in (8.23).

**Definition 8.3.1.** The category $Kar(\mathcal{M}_{BS,Q})$ is the same category as $Kar(\mathcal{H}_{BS,Q})$, except that we have a wall on the left which solid $J$-strands can plug into, satisfying all the same relations (7.2) - (7.4) of $\mathcal{M}_{BS,Q}$.

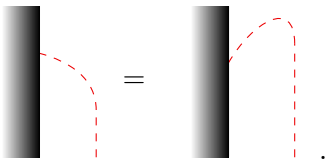
Once again, $Kar(\mathcal{M}_{BS,Q})$ is temporarily abusive notation since we have not yet shown that $Kar(\mathcal{M}_{BS,Q})$ is idempotent complete. However, this will be proven in Corollary 8.4.6.

**Definition 8.3.2.** The category $\mathcal{M}^{\text{std}}$ is the same as $\mathcal{H}^{\text{std}}$, except that morphisms have a wall on the left which dashed strands can plug into, if the strand corresponds to some $s \in J$. That is, we have new generators



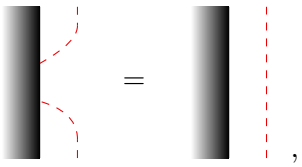
(8.24)

for each $s \in J$, satisfying the relation

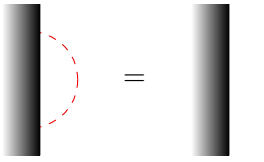


(8.25)

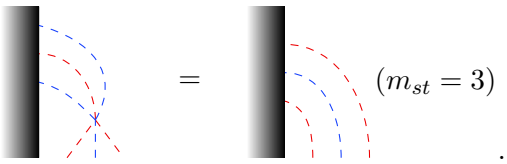
This means that, like $\mathcal{H}^{\text{std}}$, morphisms in $\mathcal{M}^{\text{std}}$ are well-defined up to isotopy. We also have the relations



(8.26)



(8.27)



(8.28)

(Of course, this last relation generalizes for all $m_{st}$).

The localization $\mathcal{M}_Q^{\text{std}}$ is the same as $\mathcal{M}^{\text{std}}$, but we allow elements of $Q$ on the right of our morphisms. By a now familiar argument, this means we allow elements of $Q$ in any region of our diagrams.

Let us turn to the final undefined category in (8.23).

**Definition 8.3.3.** The category $JStd\text{Bim}$ is the same as $Std\text{Bim}$, but we now consider objects as $(R^J, R)$-bimodules, rather than $(R, R)$-bimodules. That is, we can consider each standard bimodule $R_x$ (for $x \in W$) as an $(R^J, R)$-bimodule, and we let $JStd\text{Bim}$ be the full subcategory of $(R^J, R)$-Bim consisting of direct sums of grading shifts of standard bimodules.

When we localize these standard modules in $JStd\text{Bim}$, we once again have an isomorphism $R_x \otimes_R Q \cong Q_x$ of $(R^J, Q)$-bimodules. We can therefore consider $R_x \otimes_R Q$ to be a $(Q^J, Q)$-bimodule, so that we have a localization functor $JStd\text{Bim} \to (Q^J, Q)$-Bim.

**Definition 8.3.4.** We let $JStd\text{Bim}_Q$ be the essential image of this localization functor $JStd\text{Bim} \to (Q^J, Q)$-Bim.

The objects of $JStd\text{Bim}_Q$ are clearly the same as $Std\text{Bim}_Q$: they are direct sums of standard $Q$-bimodules $Q_x$ (considered as $(Q^J, Q)$-bimodules). However, we have more morphisms in $JStd\text{Bim}_Q$ than in $Std\text{Bim}_Q$. Recall from formula (3.7) that

$$\mathrm{Hom}_{Std\text{Bim}_Q}(Q_x, Q_y) \cong \begin{cases} Q & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

That is, all morphisms between different standard bimodules are 0. In $JStd\text{Bim}_Q$ we have a bit more freedom.

**Lemma 8.3.5.** *For $x, y \in W$, we have*

$$Hom_{JStdBim_Q}(Q_x, Q_y) \cong \begin{cases} Q & \text{if } W_J x = W_J y \\ 0 & \text{otherwise.} \end{cases} \tag{8.29}$$

*Proof.* For $q \in Q_x$ and $r \in Q$, we let $q \cdot_x r$ denote the right action of $r$ on $q$, i.e. $q \cdot_x r := qx(r)$. Then, let $\varphi : Q_x \to Q_y$ be a $(Q^J, Q)$-bimodule morphism. This is determined by $\varphi(1) \in Q$: for $q \in Q_x$, $\varphi(q) = \varphi(1 \cdot_x x^{-1}(q)) = \varphi(1) \cdot_y x^{-1}(q) = \varphi(1) y x^{-1}(q)$. Then, for $q \in Q^J$, we also have $\varphi(q) = \varphi(q1) = q\varphi(1) = yx^{-1}(q)\varphi(1)$, so that for $\varphi$ to be well-defined, we need $yx^{-1}(q) = q$ for all $q \in Q^J$. If $W_J x = W_J y$, then $yx^{-1} \in W_J$ and so $yx^{-1}(q) = q$ for all $q \in Q^J$ and $\varphi$ is well-defined.

We need to prove the converse; that is, if $yx^{-1} \notin W_J$, then there exists $q \in Q^J$ such that $yx^{-1}(q) \neq q$. Set $u := yx^{-1}$. Our realization $\mathfrak{h}$ is faithful. This means that the only element of $W$ which stabilizes all of $\mathfrak{h}$ is $id$. Therefore, since $\Bbbk$ is infinite (see the end of Section 3.4) and since $W_J$ is finite, there is some element $v \in \mathfrak{h}$ which is not stabilized by any element of $W_J$, nor any element of $uW_J$. This in turn implies that all points in the orbits $W_J(v)$ and $W_J u^{-1}(v)$ are different. Therefore there is some function $f \in R = \mathrm{Sym}(\mathfrak{h}^*)$ such that $f$ is 1 on all points of $W_J(v)$, and $f$ is zero on all points of $W_J u^{-1}(v)$. Then set

$$q := \frac{1}{|W_J|} \sum_{w \in W_J} w(f).$$

It is immediate that $q$ is invariant under $W_J$, so that $q \in Q^J$. We claim that $u(q) \neq q$. To see this, note that

$$\begin{aligned} q(v) &= \frac{1}{|W_J|} \sum_{w \in W_J} (w(f))(v) \\ &= \frac{1}{|W_J|} \sum_{w \in W_J} f(w^{-1}(v)) \\ &= \frac{1}{|W_J|} \sum_{w \in W_J} f(w(v)) \\ &= 1, \end{aligned}$$

whereas

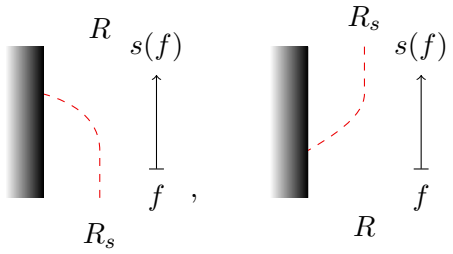
$$\begin{aligned} (u(q))(v) &= q(u^{-1}(v)) \\ &= \frac{1}{|W_J|} \sum_{w \in W_J} f(wu^{-1}(v)) \\ &= 0. \end{aligned}$$

□

## 8.4 Spherical standard diagrammatics: the functors

We have now defined all categories in (8.23). We will now define the functors involved and prove that they are equivalences.

First, let us define $\tilde{\mathcal{F}}^{\mathrm{std}} : \mathcal{M}^{\mathrm{std}} \to JStd\mathrm{Bim}$. The sequence $\underline{w} = (s_1, ..., s_n)$ of reflection indices in $\mathcal{M}^{\mathrm{std}}$ gets sent to the object $R_{s_1} \otimes_R R_{s_2} \otimes_R \cdots \otimes_R R_{s_n} \cong R_w$ in $JStd\mathrm{Bim}$ (where $\underline{w}$ is an expression for $w$). On any morphism not involving one of our new "wall generators" (8.24), $\tilde{\mathcal{F}}^{\mathrm{std}}$ is defined the same way as $\mathcal{F}^{\mathrm{std}} : \mathcal{H}^{\mathrm{std}} \to Std\mathrm{Bim}$. The image of these wall generators in $JStd\mathrm{Bim}$ is defined as follows.




(8.30)

(Oddly enough, the map given by each generator has the same expression, only with different domain and codomain). It is easy to check that these are both $(R^J, R)$-bimodule morphisms. It is also easy to check that this functor is well-defined with respect to the relations (8.25), (8.26) and (8.27). As for relation (8.28), both diagrams go to morphisms $R_{sts\cdots} \to R$ which send $1 \mapsto 1$, so they are the same map.

**Proposition 8.4.1.** *When our realization is faithful, the functor $\tilde{\mathcal{F}}^{std}$ is an equivalence of categories.*

It is clear that $\tilde{\mathcal{F}}^{\mathrm{std}}$ is essentially surjective. So we need to show that it is fully faithful. For this it is enough to show that for two expressions $\underline{x}, \underline{y}$ of $x, y \in W$, we have

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{std}}}(\underline{x}, \underline{y}) \xrightarrow{\sim} \mathrm{Hom}_{JStd\mathrm{Bim}}(R_x, R_y).$$

We will first show this on degree 0 morphisms. That is, define $\mathcal{M}_0^{\mathrm{std}}$ to be $\mathcal{M}^{\mathrm{std}}$, but with no polynomial boxes. The image of $\mathcal{M}_0^{\mathrm{std}}$ under $\tilde{\mathcal{F}}^{\mathrm{std}}$ lands in $JStd\mathrm{Bim}_0$, the subcategory of $JStd\mathrm{Bim}$ with only degree 0 morphisms. Then we have[6]

$$\mathrm{Hom}_{JStd\mathrm{Bim}_0}(R_x, R_y) \cong \begin{cases} \Bbbk & \text{if } W_J x = W_J y \\ 0 & \text{otherwise.} \end{cases} \tag{8.31}$$

[6]The formula in Lemma 8.3.5 for $JStd\mathrm{Bim}_Q$ has an obvious analogue for $JStd\mathrm{Bim}$, which is true by virtually the same proof. The formula (8.31) is a result of that.

So we want to show that $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y})$ contains a single non-zero morphism up to scalar if $W_J x = W_J y$, and is empty otherwise (and that non-zero morphisms in $\mathcal{M}_0^{\mathrm{std}}$ get sent to non-zero morphisms in $JStd\mathrm{Bim}_0$). In fact, we know the analogous fact is true in our non-spherical categories.

**Proposition 8.4.2.** *Let $\mathcal{F}_0^{std} : \mathcal{H}_0^{std} \to StdBim_0$ be the restriction of $\mathcal{F}^{std}$ to degree 0 morphisms. Then $\mathcal{F}_0^{std}$ is faithful. When our realization is faithful, $\mathcal{F}_0^{std}$ is an equivalence of categories.*

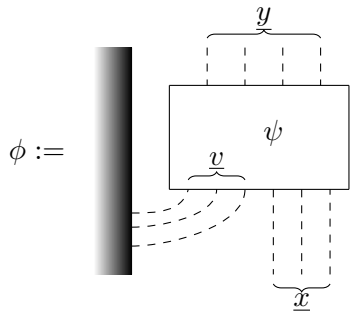
*Proof.* See Theorem 4.11 of [EW16]. □

This will be very helpful to us in proving Proposition 8.4.1. It says, for example, that two rex moves between elements $\underline{x}, \underline{y} \in \mathcal{H}_0^{\mathrm{std}}$ are the same, since the morphism space is one-dimensional, and both morphisms take the 1-tensor to the 1-tensor.

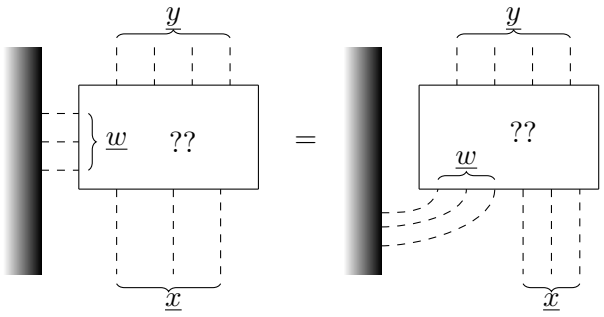
**Lemma 8.4.3.** *Let $\tilde{\mathcal{F}}_0^{std} : \mathcal{M}_0^{std} \to JStdBim_0$ be the restriction of $\tilde{\mathcal{F}}^{std}$ to degree 0 morphisms. When our realization is faithful, $\tilde{\mathcal{F}}_0^{std}$ is an equivalence of categories.*

*Proof.* Assume our realization is faithful. First, let us take two expressions $\underline{x}, \underline{y}$ of $x, y \in W$ such that $W_J x = W_J y$. We want to show that there is only one morphism $\underline{x} \to \underline{y}$ in $\mathcal{M}_0^{\mathrm{std}}$ up to scalar. We will do this by constructing a specific morphism, and then showing that any other morphism $\underline{x} \to \underline{y}$ is the same as the one we constructed.

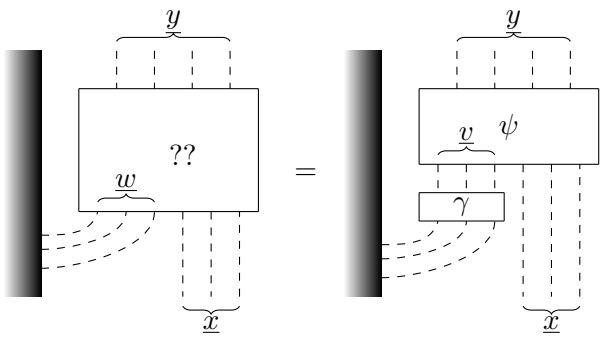
We construct a morphism $\phi : \underline{x} \to \underline{y}$ as follows. Let $x = uz$ and $y = u'z$, for $u, u' \in W_J$ and $z \in {}^J W$. We know these decompositions are unique by Lemma 6.1.2. Set $v := u'u^{-1}$, and fix a rex $\underline{v}$ of $v$. Then $vx = vuz = u'z = y$. This means that $(\underline{v}, \underline{x})$ and $\underline{y}$ are both expressions for $y$. Therefore by Proposition 8.4.2, there is a unique morphism (up to scalar) $\psi$ in $\mathcal{H}_0^{\mathrm{std}}$ from $(\underline{v}, \underline{x})$ to $\underline{y}$. We construct our morphism $\phi : \underline{x} \to \underline{y}$ as in the following diagram.




Now, consider an arbitrary non-zero morphism $\underline{x} \to \underline{y}$ in $\mathcal{M}_0^{\mathrm{std}}$. We will shift any wall-strands to the bottom via isotopy.




So we have some expression $\underline{w}$ of $w \in W$ coming in from the wall. This means that $wx = y$, since in the box labelled '??' we have a non-zero morphism $(\underline{w}, \underline{x}) \to \underline{y}$ in $\mathcal{H}_0^{\mathrm{std}}$. This means in particular that $wx = y = vx$, where $v$ is defined above, so that $w = v$. Now, we can in fact choose the box '??' to be any morphism in $\mathcal{H}_0^{\mathrm{std}}$ that we want, since they will all be the same up to scalar. Let us choose the '??' box to be a map $\gamma : \underline{w} \to \underline{v}$ consisting of rex moves and caps[7], followed by the morphism $\psi : (\underline{v}, \underline{x}) \to \underline{y}$ we chose above.




[7]This is possible since $\underline{v}$ is reduced.

Since $\gamma$ consists of $2m$-valent vertices and caps, it can be pulled into the wall using relations (8.27) and (8.28). We end up with the map $\phi$.

Thus we have seen that if $W_J x = W_J y$, then there is only one non-zero morphism $\underline{x} \to \underline{y}$ in $\mathcal{M}_0^{\mathrm{std}}$ up to scalar. This means that the morphism spaces $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y})$ and $\mathrm{Hom}_{JStd\mathrm{Bim}_0}(R_x, R_y)$ are both isomorphic to $\Bbbk$. Furthermore, the image of any non-zero morphism in $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y})$ under $\tilde{\mathcal{F}}^{\mathrm{std}}$ sends the 1-tensor to (a non-zero scalar multiple of) the 1-tensor, so this is a non-zero map. This also means that in the case that $W_J x \neq W_J y$, there are no non-zero diagrams from $\underline{x} \to \underline{y}$ in $\mathcal{M}_0^{\mathrm{std}}$, since if there were, they would go to something non-zero in $\mathrm{Hom}_{JStd\mathrm{Bim}}(R_x, R_y)$, which is empty. Therefore in each case, $\tilde{\mathcal{F}}^{\mathrm{std}}$ maps $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y})$ isomorphically to $\mathrm{Hom}_{JStd\mathrm{Bim}_0}(R_x, R_y)$. □

We now just need to extend from degree 0 morphisms to all degrees.

*Proof of Proposition 8.4.1.* In any morphism in $\mathrm{Hom}_{\mathcal{M}^{\mathrm{std}}}(\underline{x}, \underline{y})$, we can move all polynomial boxes to the right via relation (8.15). This means that we have a surjective morphism $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y}) \otimes_{\Bbbk} R \to \mathrm{Hom}_{\mathcal{M}^{\mathrm{std}}}(\underline{x}, \underline{y})$. To see that this is also injective, note that there is only one morphism $\phi$ (up to scalar) in $\mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y})$, and if $\phi \otimes r \in \mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y}) \otimes_{\Bbbk} R$ gets sent to 0 under the map, this must mean that $r = 0$. This gives us an isomorphism

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{std}}}(\underline{x}, \underline{y}) \cong \mathrm{Hom}_{\mathcal{M}_0^{\mathrm{std}}}(\underline{x}, \underline{y}) \otimes_{\Bbbk} R.$$

Similarly, any morphism in $\mathrm{Hom}_{JStd\mathrm{Bim}}(R_x, R_y)$ is a degree 0 morphism times a polynomial. Therefore the same reasoning as above tells us that

$$\mathrm{Hom}_{JStd\mathrm{Bim}}(R_x, R_y) \cong \mathrm{Hom}_{JStd\mathrm{Bim}_0}(R_x, R_y) \otimes_{\Bbbk} R.$$

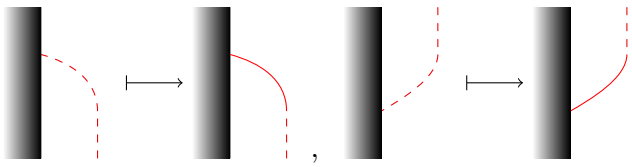
Since $\tilde{\mathcal{F}}^{\mathrm{std}}$ is $R$-linear, we have that $\tilde{\mathcal{F}}^{\mathrm{std}} = \tilde{\mathcal{F}}_0^{\mathrm{std}} \otimes_{\Bbbk} R$. Since $R$ is a free $\Bbbk$-module, it is flat, so that $\tilde{\mathcal{F}}^{\mathrm{std}} = \tilde{\mathcal{F}}_0^{\mathrm{std}} \otimes_{\Bbbk} R$ is an isomorphism. □

**Corollary 8.4.4.** *The functor $\tilde{\mathcal{F}}_Q^{std} = \tilde{\mathcal{F}}^{std} \otimes_R Q : \mathcal{M}_Q^{std} \to JStdBim_Q$ is an equivalence of categories.*

*Proof.* The field of fractions $Q$ is a flat $R$-module. □

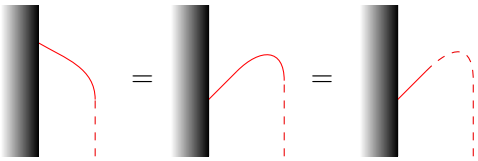
We have now constructed one side to the commutative triangle in (8.23) and shown that it is an equivalence. Our next task is to construct the other two functors $\tilde{\mathcal{S}td}$ and $\tilde{\mathcal{G}}$ and show that they are also equivalences.

The functor $\tilde{\mathcal{S}td} : \mathcal{M}_Q^{\text{std}} \to Kar(\mathcal{M}_{BS,Q})$ extends the functor $\mathcal{S}td : \mathcal{H}_Q^{\text{std}} \to Kar(\mathcal{H}_{BS,Q})$, so we only need to define it on the new wall-generators in (8.24). This is done as follows.

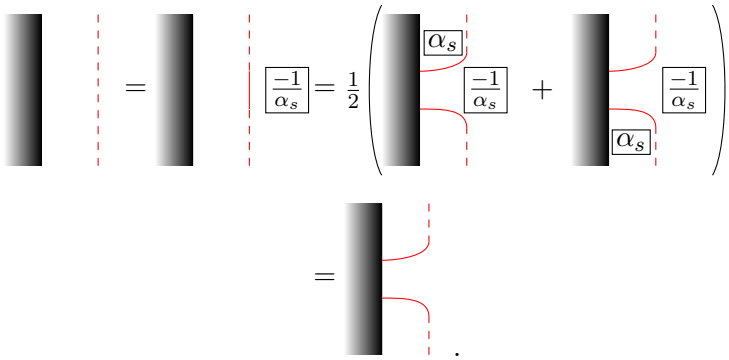


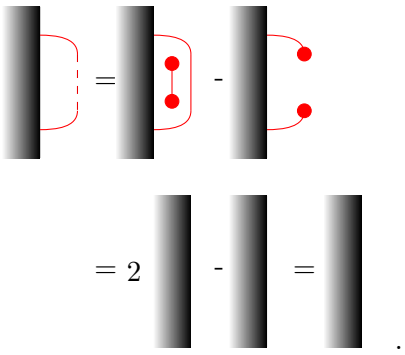
(8.32)

We must show that this functor is well-defined; i.e. it respects the relations (8.25) - (8.28). The fact that it respects (8.25) is a result of the following computation in $Kar(\mathcal{M}_{BS,Q})$.

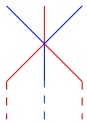


Relations (8.26) and (8.27) are the result of the following.

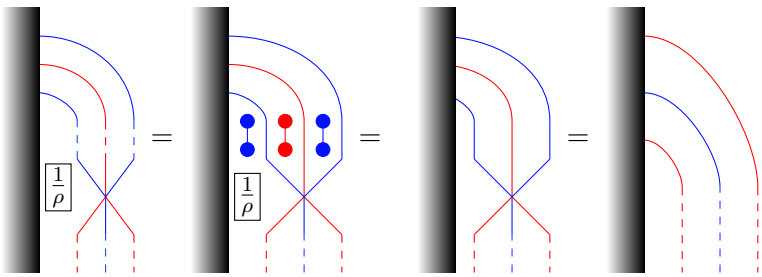




As for the third, recall that placing an up-dot anywhere on-top of the following diagram will yield 0.



This is because we can pull dots through $2m$-valent vertices, yielding zero-morphisms as in (8.14). This leads to the following computation.




In the first equality we have applied (8.12) to all three strands, along with the fact above that up-dots pull through $2m$-valent vertices to yield zero-maps. In the second equality, we have shifted all barbells to the left using polynomial forcing, yielding $\rho$, which cancels with $\frac{1}{\rho}$. The third equality is just (7.4). Since the first diagram in the above picture is the image of the $2m$-valent vertex, this shows that $\tilde{\mathcal{S}td}$ respects relation (8.28), and thus the functor is well-defined.

Next, we define $\tilde{\mathcal{G}} : Kar(\mathcal{M}_{BS,Q}) \to JStd\text{Bim}_Q$. To do this, we need to define $J\mathbb{BS}\text{Bim}_Q$. The category $J\mathbb{BS}\text{Bim}_Q$ is defined exactly analogously to $\mathbb{BS}\text{Bim}_Q$. That is, we have a

localization functor $J\mathbb{BS}\mathrm{Bim} \to (Q^J, Q)$-Bim, and we define $J\mathbb{BS}\mathrm{Bim}_Q$ to be the essential image. We also have an embedding $J\mathbb{BS}\mathrm{Bim}_Q \hookrightarrow JStd\mathrm{Bim}_Q$ coming directly from the embedding $\mathbb{BS}\mathrm{Bim}_Q \hookrightarrow Std\mathrm{Bim}_Q$.

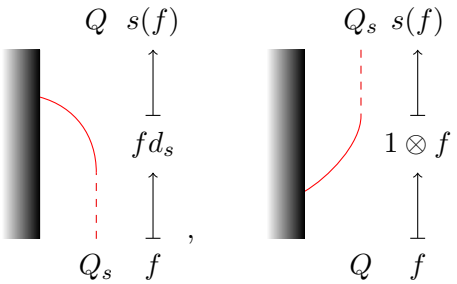
Then, the functor $\tilde{\mathcal{G}}$ extends $\mathcal{G} : Kar(\mathcal{H}_{BS,Q}) \to Std\mathrm{Bim}_Q$, so we only need to define it on the "wall-generators". Here is it defined via the functors

$$\mathcal{M}_{BS,Q} \xrightarrow{\tilde{\mathcal{A}} \otimes Q} J\mathbb{BS}\mathrm{Bim}_Q \hookrightarrow JStd\mathrm{Bim}_Q$$

where $\tilde{\mathcal{A}} : \mathcal{M}_{BS} \to J\mathbb{BS}\mathrm{Bim}$ was defined in Section (7.1). In other words, $\tilde{\mathcal{G}}$ is defined such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}_{BS,Q} & \longrightarrow & Kar(\mathcal{M}_{BS,Q}) \\ & \searrow^{\tilde{\mathcal{A}} \otimes Q} \quad \swarrow_{\tilde{\mathcal{G}}} & \\ & J\mathbb{BS}\mathrm{Bim}_Q \hookrightarrow JStd\mathrm{Bim}_Q & \end{array} \tag{8.33}$$

Now we want to show that the diagram in (8.23) does indeed commute. It is immediate that it is commutative on objects. For any morphism which does not interact with the wall, the commutativity is a result of that of (8.16). So we only need to show that it is commutative on wall-generators (8.24). The images of the wall-generators under $\tilde{\mathcal{S}td}$ were given in (8.32). We then want to find the image of these under $\tilde{\mathcal{G}}$. Using the algebraic images given in (8.11) and (7.8), we get that the image under $\tilde{\mathcal{G}}$ is




(8.34)

which matches the images of the wall-generators under $\tilde{\mathcal{F}}_Q^{\mathrm{std}}$ given in (8.30)[8]. Therefore the diagram (8.23) commutes.

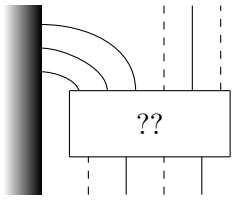
[8]The morphisms in (8.34) come from straight-forward algebraic computations using the Demazure operators, defined in (3.1).

Our final goal is to show that all functors in (8.23) are equivalences (when our realization is faithful). We already know from Corollary 8.4.4 that $\tilde{\mathcal{F}}_Q^{\mathrm{std}}$ is an equivalence. So it will be enough to show that $\tilde{\mathcal{S}td}$ is an equivalence.

**Proposition 8.4.5.** *When our realization is faithful, the functor $\tilde{\mathcal{S}td} : \mathcal{M}_Q^{std} \to Kar(\mathcal{M}_{BS,Q})$ is an equivalence of categories.*

*Proof.* Every object in $Kar(\mathcal{M}_{BS,Q})$ decomposes as a direct sum of reflection indices, just as in $Kar(\mathcal{H}_{BS,Q})$. These reflection indices are clearly in the image of $\tilde{\mathcal{S}td}$. Therefore the functor is essentially surjective.

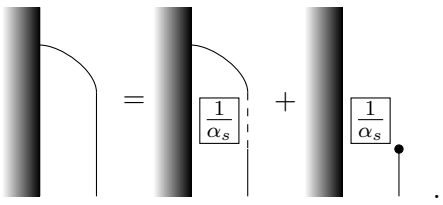
Now we want to show that $\tilde{\mathcal{S}td}$ is fully faithful. First we show that it is full. We can take any morphism in $Kar(\mathcal{M}_{BS,Q})$ and move the wall-strands to the top[9].




(8.35)

Then, the diagram in the box labelled '??' is in $Kar(\mathcal{H}_{BS,Q})$. Since $\mathcal{S}td : \mathcal{H}_Q^{\mathrm{std}} \to Kar(\mathcal{H}_{BS,Q})$ is an equivalence (see Proposition 8.2.1), there is some (sum of) diagrams in $\mathcal{H}_Q^{\mathrm{std}}$ which is mapped to '??' under $\mathcal{S}td$, and therefore also under $\tilde{\mathcal{S}td}$.

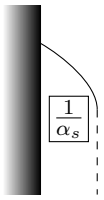
Now, the solid strands plugging into the wall in the above diagram can also be seen to be in the image of $\tilde{\mathcal{S}td}$, using the relation



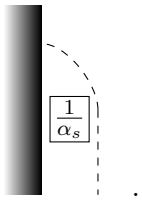

. (8.36)

[9]Recall that not all diagrams in $Kar(\mathcal{M}_{BS,Q})$ are well-defined up to isotopy. For example, see (8.2). However, the problem only comes from rotating bivalent vertices, which we need not do to move these wall-strands to the top.

We obtained the above relation by dividing (8.13) by $\alpha_s$, and then pulling the dot into the wall. The right-hand side of this relation can be seen to be in the image of $\tilde{\mathcal{S}td}$. This is because bivalent vertices are in the image of $\mathcal{S}td : \mathcal{H}_Q^{\mathrm{std}} \to Kar(\mathcal{H}_{BS,Q})$, and the map




is the image of




.

Therefore in (8.35) both the box labelled '??' and the wall-strands above it are in the image of $\tilde{\mathcal{S}td}$. Therefore the functor is full.

Finally, to show $\tilde{\mathcal{S}td}$ is faithful, take two morphisms $\phi, \psi$ in $\mathcal{M}_Q^{\mathrm{std}}$ such that $\tilde{\mathcal{S}td}(\phi) = \tilde{\mathcal{S}td}(\psi)$. This implies $\tilde{\mathcal{G}} \circ \tilde{\mathcal{S}td}(\phi) = \tilde{\mathcal{G}} \circ \tilde{\mathcal{S}td}(\psi)$. But since we know the diagram (8.23) is commutative, this means $\tilde{\mathcal{F}}_Q^{\mathrm{std}}(\phi) = \tilde{\mathcal{F}}_Q^{\mathrm{std}}(\psi)$. If our realization is faithful, then by Corollary 8.4.4 the functor $\tilde{\mathcal{F}}_Q^{\mathrm{std}}$ is an equivalence. Thus we get $\phi = \psi$, and the functor $\tilde{\mathcal{S}td}$ is faithful. □

**Corollary 8.4.6.** *When our realization is faithful, $\tilde{\mathcal{G}} : Kar(\mathcal{M}_{BS,Q}) \to JStdBim_Q$ is an equivalence.*

Once again, this proves that $Kar(\mathcal{M}_{BS,Q})$ is indeed the Karoubian closure of $\mathcal{M}_{BS,Q}$.

This equivalence $\tilde{\mathcal{G}} : Kar(\mathcal{M}_{BS,Q}) \to JStd\mathrm{Bim}_Q$ is the key result of this section. We will now use it to prove the linear independence of the double-leaves.

# Chapter 9

# Linear independence

Throughout the remainder of the paper, we fix a reflection-faithful representation $\mathfrak{h}$. We will prove that the localized double-leaves form a basis.

**Proposition 9.0.1.** *Given $\underline{x}, \underline{y} \in \mathcal{M}_{BS}$, let $\mathcal{SDL}_{\underline{x},\underline{y}}$ contain one double-leaf map $\mathbb{SDL}_{\underline{f},\underline{e}}$ for every pair of subexpressions $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $W_J \underline{x}^{\underline{e}} = W_J \underline{y}^{\underline{f}}$. Then, after localization, this set of double-leaf maps forms a basis of $Hom_{\mathcal{M}_{BS,Q}}(\underline{x}, \underline{y})$ as a right $R$-module.*

Our double-leaf maps lie in $\mathcal{M}_{BS}$. We take the images of them under the functors[1]

$$\mathcal{M}_{BS} \xrightarrow{(-)\otimes Q} \mathcal{M}_{BS,Q} \hookrightarrow Kar(\mathcal{M}_{BS,Q}) \cong JStd\mathrm{Bim}_Q. \tag{9.1}$$

We will show that these images form a basis of the Hom spaces in $Kar(\mathcal{M}_{BS,Q}) \cong JStd\mathrm{Bim}_Q$. Then, because $\mathrm{Hom}_{\mathcal{M}_{BS,Q}}(\underline{x}, \underline{y}) \cong \mathrm{Hom}_{Kar(\mathcal{M}_{BS,Q})}(\underline{x}, \underline{y})$ (see Section 4.5), this implies that the localized double-leaves form a basis of $\mathcal{M}_{BS,Q}$. This in turn means that the double-leaves are linearly independent in $\mathcal{M}_{BS}$. For if

$$\sum \mathbb{SDL}_{\underline{f},\underline{e}} \cdot c_{\underline{f},\underline{e}} = 0,\ c_{\underline{f},\underline{e}} \in R,$$

[1]Here, the equivalence $Kar(\mathcal{M}_{BS,Q}) \xrightarrow{\sim} JStd\mathrm{Bim}_Q$ is the functor $\tilde{\mathcal{G}}$ described in section 8.4.

then

$$\sum \mathbb{SDL}_{\underline{f},\underline{e}} \cdot c_{\underline{f},\underline{e}} \otimes id_Q = \sum \mathbb{SDL}_{\underline{f},\underline{e}} \otimes id_Q \cdot c_{\underline{f},\underline{e}} = 0,$$

so that $c_{\underline{f},\underline{e}} = 0$ for all $(\underline{f}, \underline{e})$.

## 9.1 A partial order on pairs of subexpressions

We will abuse notation by letting $\mathbb{SDL}_{\underline{f},\underline{e}}$ denote the image of the double-leaf in $Kar(\mathcal{M}_{BS,Q}) \cong JStd\text{Bim}_Q$ under the functors in (9.1). It will be clear from context exactly which category we are working in at a given time. In $JStd\text{Bim}_Q$, the double-leaf map $\mathbb{SDL}_{\underline{f},\underline{e}} : \underline{x} \to \underline{y}$ is a map $\bigoplus_{\underline{e'} \subset \underline{x}} Q_{\underline{e'}} \to \bigoplus_{\underline{f'} \subset \underline{y}} Q_{\underline{f'}}$. We can therefore think of it as a matrix, with entries $p^{e,f}_{e',f'} \in Q$. To be precise, we let $p^{e,f}_{e',f'}$ be the composition $Q_{\underline{e'}} \hookrightarrow \underline{x} \xrightarrow{\mathbb{SDL}_{\underline{f},\underline{e}}} \underline{y} \twoheadrightarrow Q_{\underline{f'}}$. (Note that by Lemma 8.3.5, $p^{e,f}_{e',f'} = 0$ unless $W_J \underline{x^{e'}} = W_J \underline{y^{f'}}$). We will prove Proposition 9.0.1 by showing that the maps $\mathbb{SDL}_{\underline{f},\underline{e}}$ in $JStd\text{Bim}_Q$ are upper-triangular with respect to a certain partial order on the set of pairs $(\underline{e}, \underline{f})$. To define this partial order, we first define a partial order each of the components; i.e. a partial order on subexpressions.

Take an expression $\underline{w} = (s_1, ..., s_n)$, and let $E$ be the set of subexpressions $\underline{e} \subset \underline{w}$. There is a partial order on $E$ called the *path-dominance order*: for sub-expressions $\underline{e} = (e_1, ..., e_n)$ and $\underline{f} = (f_1, ..., f_n)$, we can take their strolls $x_0, ..., x_n$ and $y_0, ..., y_n$ respectively (i.e. $x_i := s_1^{e_1} \cdots s_i^{e_i}$). Then we say that $\underline{f} \leq \underline{e}$ in the path dominance order if $y_i \leq x_i$ in the Bruhat order, for all $i$. For example, let $\underline{w} = (t, s, t)$ in $S_3$ and set $\underline{e} = (1, 0, 0)$, $\underline{f} = (0, 0, 1)$. Then the stroll of $\underline{e}$ is $\mathrm{id}, t, t, t$, whereas that of $\underline{f}$ is id, id, id, $t$. We can see that at each step, the stroll of $\underline{e}$ is higher (or equal) in the Bruhat order, so $\underline{e} > \underline{f}$.

We will put a partial order $\preceq$ on $E$ which is a slight variation of this. Rather than taking the strolls associated to $\underline{e}$ and $\underline{f}$, we will take coset strolls, introduced in Section 7.2: The coset stroll $cos(\underline{e})$ of $\underline{e}$ is $\tilde{x_0}, ..., \tilde{x_n}$, where $\tilde{x_i}$ is the minimal coset representative of $x_i = s_1^{e_1} \cdots s_k^{e_k}$. We write $cos(\underline{f}) \leq cos(\underline{e})$ if $\tilde{y_i} \leq \tilde{x_i}$ for all $i$, where $\tilde{y_0}, ..., \tilde{y_n}$ and $\tilde{x_0}, ..., \tilde{x_n}$ are the coset strolls of $\underline{f}$ and $\underline{e}$ respectively.

**Definition 9.1.1.** We define a partial order $\preceq$ on the set $E$ of subexpressions of $\underline{w}$ as follows: for $\underline{f}, \underline{e} \subset \underline{w}$, $\underline{f} \preceq \underline{e}$ if either $cos(\underline{f}) < cos(\underline{e})$, or $cos(\underline{f}) = cos(\underline{e})$ and there is no step $k$ at which $\underline{e}$ is $X0$ and $\underline{f}$ is $X1$.

**Example 9.1.2.** Let $(W, S)$ be type $A_2$, and let $J = \{s\}$. Take $\underline{w} = (s, t, s, t)$. Let $\underline{e}, \underline{f}, \underline{g} \subset \underline{w}$ be

$$\underline{e} = (0, 1, 1, 1),\ \underline{f} = (0, 1, 1, 0),\ \underline{g} = (1, 1, 1, 0).$$

The coset strolls of each of these expressions is $id, id, t, ts, ts$, so $cos(\underline{e}) = cos(\underline{f}) = cos(\underline{g})$. The decoration (see (7.9)) of $\underline{e}$ is $X0, U1, U1, X1$, while that of $\underline{f}$ is $X0, U1, U1, X0$. There is no step at which $\underline{e}$ is $X0$ and $\underline{f}$ is $X1$, so we have $\underline{f} \prec \underline{e}$. Similarly, we have $\underline{f} \prec \underline{g}$. However, $\underline{e}$ and $\underline{g}$ are not comparable.

Now we use this partial order on subexpressions to define a partial order (abusively denoted $\prec$) on *pairs* of subexpressions $(\underline{e}, \underline{f})$, where $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$.

**Definition 9.1.3.** For $\underline{e}, \underline{e}' \subset \underline{x}$, $\underline{f}, \underline{f}' \subset \underline{y}$, we write $(\underline{e}', \underline{f}') \preceq (\underline{e}, \underline{f})$ if $\underline{e}' \preceq \underline{e}$ and $\underline{f}' \preceq \underline{f}$.

## 9.2 Proof of linear independence

We will now show that the maps $\mathbb{SDL}_{\underline{f},\underline{e}}$ are upper triangular with respect to this partial order.

**Proposition 9.2.1.** *(a) If $p^{e,f}_{e',f'} \neq 0$, then $(\underline{e}', \underline{f}') \preceq (\underline{e}, \underline{f})$.*

*(b) Moreover, $p^{e,f}_{e,f} \neq 0$ for any $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$.*

This proposition will be proven over the next few lemmas. Recall that $p^{e,f}_{e',f'}$ is given by the composition

$$Q_{\underline{e}'} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z} \xrightarrow{\overline{SLL_{\underline{y},\underline{f}}}} \underline{y} \twoheadrightarrow Q_{\underline{f}'} \tag{9.2}$$

where $\underline{z}$ is a rex of the minimal coset representative $z$ of $W_J \underline{x}^{\underline{e}} = W_J \underline{y}^{\underline{f}}$. Consider the first half of this map (we will call this $\phi_{\underline{e},\underline{e}'}$):

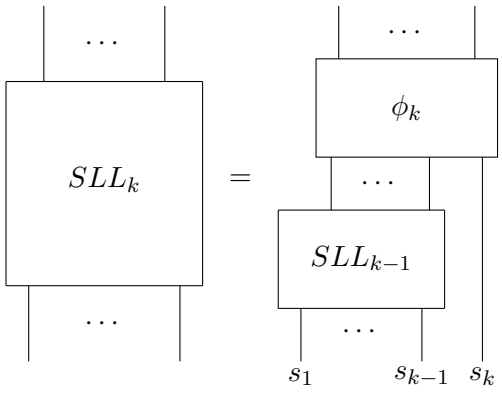
$$\phi_{\underline{e},\underline{e}'} := Q_{\underline{e}'} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z}. \tag{9.3}$$

We will first show the following statement:

**Lemma 9.2.2.** *If $\phi_{\underline{e},\underline{e}'} \neq 0$, then $\underline{e}' \preceq \underline{e}$.*

*Proof.* First, consider the case where $cos(\underline{e}) = cos(\underline{e}')$. Suppose $\underline{e}' \not\preceq \underline{e}$. Then there is some step $k$ at which $\underline{e}'$ is X1 and $\underline{e}$ is X0. At this step in the construction of $SLL_{\underline{x},\underline{e}}$, the map $SLL_k$ will have an up-dot on the right. When we then precompose with the inclusion map $Q_{\underline{f}} \hookrightarrow \underline{x}$, we will end up with a strand , rendering the whole map zero[2]. Therefore $\phi_{\underline{e},\underline{e}'} = 0$.

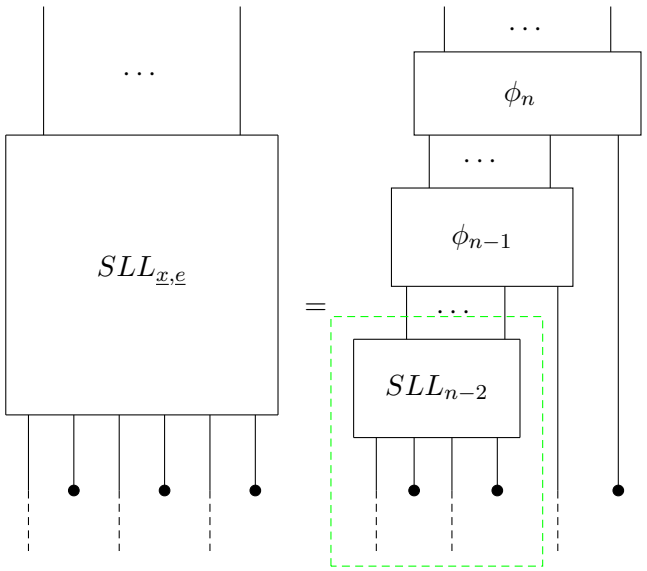
Now, take the case where $cos(\underline{e}) \neq cos(\underline{e}')$. We suppose that $\phi_{\underline{e},\underline{e}'} \neq 0$, and we will show that $cos(\underline{e}') < cos(\underline{e})$ (and that therefore $\underline{e}' \prec \underline{e}$). Recall the recursive construction of the spherical light-leaves:




(9.4)

At the end of this construction we precompose with the inclusion $\iota_{\underline{e}'} : Q_{\underline{e}'} \hookrightarrow \underline{x}$ to get $\phi_{\underline{e},\underline{e}'} = SLL_{\underline{x},\underline{e}} \circ \iota_{\underline{e}'}$. Now, at any step $k$, let $\underline{x}_{\leq k} = (s_1, s_2, ..., s_k)$, and let $\underline{e}'_{\leq k}$ be the subexpression $(e'_1, e'_2, e'_3, ..., e'_k) \subset \underline{x}_{\leq k}$. Let $y_k := \underline{x}_{\leq k}^{\underline{e}'_{\leq k}} = s_1^{e'_1} \cdots s_k^{e'_k}$ (i.e. $y_0, ..., y_n$ is the stroll of $\underline{e}'$). Then, let $\iota_{\leq k} : Q_{y_k} \hookrightarrow \underline{x}_{\leq k}$. We can see that the composition $SLL_k \circ \iota_{\leq k}$ will appear inside $SLL_{\underline{x},\underline{e}} \circ \iota_{\underline{e}'}$, depicted below inside the green box (for $k = n - 2$, $\underline{e}' = (1, 0, 1, 0, 1, 0)$):

[2]Here, we are of course considering the image of $SLL_{\underline{x},\underline{e}}$ in $Kar(\mathcal{M}_{BS,Q}) \cong JStd\text{Bim}_Q$.




(9.5)

From this, we can see that if any $SLL_k \circ \iota_{\leq k}$ is zero, then so is $\phi_{\underline{e},\underline{e}'} = SLL_{\underline{x},\underline{e}} \circ \iota_{\underline{e}'}$.

Now, the map $SLL_k \circ \iota_{\leq k}$ is a map $Q_{y_k} \hookrightarrow \underline{x}_{\leq k} \to \underline{z}_k$, where $\underline{z}_k$ is a rex for the minimal coset representative $z_k$ of $\underline{x}_{\leq k}^{\underline{e}_{\leq k}}$ (i.e. $z_0, ..., z_n$ is the coset stroll $cos(\underline{e})$). For this map to be non-zero, we need that $Q_{y_k}$ can map to some summand $Q_w$ in $\underline{z}_k$ (i.e. $w \leq z_k$). By Lemma 8.3.5, this means that $W_J y_k = W_J w$. If $\tilde{y}_k$ is the minimal coset representative of $y_k$, then $W_J \tilde{y}_k = W_J y_k = W_J w$, so that $\tilde{y}_k \leq w$ (since $\tilde{y_k}$ is minimal). We stated above that $w \leq z_k$, so that $\tilde{y}_k \leq w \leq z_k$. So $\tilde{y}_k \leq z_k$ for all $k$, i.e. $cos(\underline{e}') \leq cos(\underline{e})$. Since we assumed $cos(\underline{e}') \neq cos(\underline{e})$, we get $cos(\underline{e}') < cos(\underline{e})$, and therefore $\underline{e}' \prec \underline{e}$. □

We can use this to prove part $(a)$ of Proposition 9.2.1:

*Proof of Proposition 9.2.1* $(a)$. Suppose $p_{e',f'}^{e,f} \neq 0$. We want to show that $(\underline{e}', \underline{f}') \preceq (\underline{e}, \underline{f})$, i.e. that $\underline{e}' \preceq \underline{e}$ and $\underline{f}' \preceq \underline{f}$. Recall (again) that $p_{e',f'}^{e,f}$ is given by the composition

$$Q_{\underline{e}'} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z} \xrightarrow{\overline{SLL_{\underline{y},\underline{f}}}} \underline{y} \twoheadrightarrow Q_{\underline{f}'}. \tag{9.6}$$

The first half of this we called $\phi_{\underline{e},\underline{e}'}$, and so if $p_{e',f'}^{e,f} \neq 0$, then $\phi_{\underline{e},\underline{e}'} \neq 0$. By Lemma 9.2.2, this means $\underline{e}' \preceq \underline{e}$.

Now consider the second half of the map $p^{e,f}_{e',f'}$:

$$\underline{z} \xrightarrow{\overline{SLL_{\underline{y},\underline{f}}}} \underline{y} \twoheadrightarrow Q_{\underline{f'}}.$$

Note that (up to a constant) this is just $\overline{\phi_{\underline{f},\underline{f'}}}$, i.e. $\phi_{\underline{f},\underline{f'}}$ flipped upside down. This is because projection map $\underline{y} \twoheadrightarrow Q_{\underline{f'}}$ is (up to a constant) the inclusion map $Q_{\underline{f'}} \hookrightarrow \underline{y}$ flipped upside-down (see Section 8). So $p^{e,f}_{e',f'} \neq 0$ implies $\overline{\phi_{\underline{f},\underline{f'}}} \neq 0$, which implies $\phi_{\underline{f},\underline{f'}} \neq 0$. Again by Lemma 9.2.2, this means $\underline{f'} \preceq \underline{f}$. Thus we have $\underline{e'} \preceq \underline{e}$ and $\underline{f'} \preceq \underline{f}$, i.e. $(\underline{e'}, \underline{f'}) \preceq (\underline{e}, \underline{f})$. $\square$

To prove part $(b)$ of Proposition 9.2.1, we need another lemma. Take any expression $\underline{w}$ and a subexpression $\underline{e} \subset \underline{w}$. This gives us a light-leaf map $SLL_{\underline{w},\underline{e}} : \underline{w} \to \underline{z}$, where $\underline{z}$ is a reduced expression for the minimal coset rep $z$ of $\underline{w}^{\underline{e}}$. Then in $JStd\mathrm{Bim}_Q$, $\underline{z} = {}_{R^J}BS(\underline{z}) \otimes_R Q \cong \bigoplus_{\underline{e'} \subset \underline{z}} Q_{\underline{e'}}$, so we have a projection map $p : \underline{z} \to Q_z$ onto the unique summand $Q_z$.

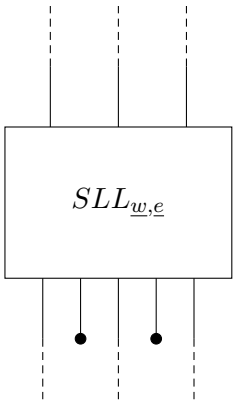
Consider the composition

$$Q_{\underline{e}} \hookrightarrow \underline{w} \xrightarrow{SLL_{\underline{w},\underline{e}}} \underline{z} \twoheadrightarrow Q_z$$

Since $z$ is the minimal coset rep of $\underline{w}^{\underline{e}}$, we have that $W_J \underline{w}^{\underline{e}} = W_J z$, so that by Lemma 8.3.5 the composition above is given by some $q_{\underline{e}} \in Q$.

**Lemma 9.2.3.** *For any $\underline{e} \subset \underline{w}$, $q_{\underline{e}} \neq 0$.*

*Proof.* This proof follows almost exactly the proof of Proposition 6.6 in [EW16]. We have that $q_{\underline{e}}$ is given by the map $Q_{\underline{e}} \hookrightarrow \underline{w} \xrightarrow{SLL_{\underline{w},\underline{e}}} \underline{z} \twoheadrightarrow Q_z$. Diagrammatically in $Kar(\mathcal{M}_{BS,Q})$ this will look something like the following picture (for $\underline{e} = (1,0,1,0,1)$ and $\underline{w}$ unspecified).




Note in particular that we attach dashed lines to every strand at the top. We want to show that this map is non-zero. First, suppose $\underline{e}$ contains a '0' somewhere, say $e_k = 0$. If $e_k$ is $U0$ or $X0$, then at this step in the light-leaf construction, we will have a up-dot $\overset{\bullet}{|}$. We see from the above picture that this will connect to a down-dot $|\!\!\downarrow$ to form a barbell, which is $\alpha_s$ for some $s \in S$. This can move to the right through the dashed lines, as in (8.15), and so this will contribute some non-zero constant. If $e_k$ is $D0$, then we get a trivalent vertex at this step in the construction of the light leaf, so that the downward dot in the inclusion map (as depicted in the above picture) can be pulled into a strand: $\curlyvee_{\bullet} = |$. What we are left with is a case where $\underline{e}$ consists of only 1's, so it remains to prove it in this case.

If $\underline{e}$ consists entirely of '1's then, by the construction of the light-leaves, $SLL_{\underline{w},\underline{e}}$ will consist entirely of caps, rex moves, and strands plugging into the wall. To compute $q_{\underline{e}}$, we are taking this $SLL_{\underline{w},\underline{e}}$ and adding dashed lines to all the top and bottom strands. Take one of these dashed strands at the bottom, and follow it upwards. From relation (8.22), we can move the dashed strand through $2m_{st}$-valent vertices, and therefore through rex moves. If we continue upwards in this way, we will eventually hit either: a strand at the top, a cap, or the wall. If we reach the top, then by the relation (8.12), we end up with a dotted line and a barbell, which slides to the right to become some (non-zero) constant. If we reach a cap, then relations (8.18) and (8.19) say something similar: we are left with a dashed cap, and a barbell which slides to the right. Finally, if we reach the wall, then we get the dashed strand going into the wall, as in (8.34). This map sends $1 \mapsto 1$, and the dashed caps and dashed $2m_{st}$-valent vertices both send $1 \otimes \cdots \otimes 1 \mapsto 1 \otimes \cdots \otimes 1$. Therefore as

a map, $q_{\underline{e}}$ sends the 1-tensor to some non-zero constant times the 1-tensor, so is non-zero. $\square$

We can now prove Proposition 9.2.1 $(b)$:

*Proof of Proposition 9.2.1 $(b)$:* We want to show that $p^{e,f}_{e,f} \neq 0$. We will show that in fact, $p^{e,f}_{e,f} = c \cdot q_{\underline{e}} \cdot q_{\underline{f}}$, where $c \in Q$ is non-zero. Since by Lemma 9.2.3 we know that $q_{\underline{e}}$ and $q_{\underline{f}}$ are non-zero, this will imply $p^{e,f}_{e,f} \neq 0$.

We know that $p^{e,f}_{e,f}$ is given by the map

$$Q_{\underline{e}} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z} \xrightarrow{\overline{SLL}_{\underline{y},\underline{f}}} \underline{y} \twoheadrightarrow Q_{\underline{f}}. \tag{9.7}$$

Let us add a projection and inclusion onto/from $Q_z$ in the middle:

$$Q_{\underline{e}} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z} \twoheadrightarrow Q_z \hookrightarrow \underline{z} \xrightarrow{\overline{SLL}_{\underline{y},\underline{f}}} \underline{y} \twoheadrightarrow Q_{\underline{f}}. \tag{9.8}$$

We claim that this does not change anything, i.e. that the maps (9.7) and (9.8) are the same. To see this, look at the first two maps $Q_{\underline{e}} \hookrightarrow \underline{x} \to \underline{z}$ in (9.8): $Q_{\underline{e}}$ can only land in direct summands $Q_w$ of $\underline{z}$ such that $W_J w = W_J \underline{x}^{\underline{e}} = W_J z$. Since $z$ is a minimal coset rep, this implies $z \leq w$. But since $\underline{z}$ is reduced, we also have $w \leq z$, so that $w = z$. So the first two maps $Q_{\underline{e}} \hookrightarrow \underline{x} \to \underline{z}$ land in $Q_z$, so that projecting onto $Q_z$ and including back into $\underline{z}$ does nothing. Thus, the maps (9.7) and (9.8) are equal.

Now, consider the first half of (9.8):

$$Q_{\underline{e}} \hookrightarrow \underline{x} \xrightarrow{SLL_{\underline{x},\underline{e}}} \underline{z} \twoheadrightarrow Q_z$$

This is exactly $q_{\underline{e}}$. Similarly, the second half is (again, up to a non-zero constant) the map $\overline{q_{\underline{f}}}$, i.e. $q_{\underline{f}}$ flipped upside down. But since $q_{\underline{f}} \in Q$, we have $\overline{q_{\underline{f}}} = q_{\underline{f}}$. Thus, the map in (9.8) is $c \cdot q_{\underline{e}} \cdot q_{\underline{f}}$ for some non-zero $c \in Q$, and so $p^{e,f}_{e,f} = c \cdot q_{\underline{e}} \cdot q_{\underline{f}}$. So $p^{e,f}_{e,f} \neq 0$ for any $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$. $\square$

Thus, the localized maps $\mathbb{SDL}_{\underline{f},\underline{e}} \otimes id_Q$ are upper-triangular with respect to the partial order '$\preceq$', and thus form a basis, proving Proposition 9.0.1. $\square$

**Corollary 9.2.4.** *Given $\underline{x}, \underline{y} \in \mathcal{M}_{BS}$, let $\mathcal{SDL}_{\underline{x},\underline{y}}$ contain one double-leaf map $\mathbb{SDL}_{\underline{f},\underline{e}}$ for every pair of subexpressions $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $W_J \underline{x}^{\underline{e}} = W_J \underline{y}^{\underline{f}}$. Then the maps in $\mathcal{SDL}_{\underline{x},\underline{y}}$ are linearly independent.*

# Chapter 10

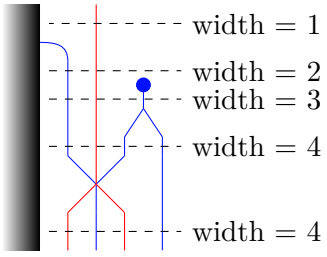
# Spanning

We will now prove that the double-leaves span. More precisely:

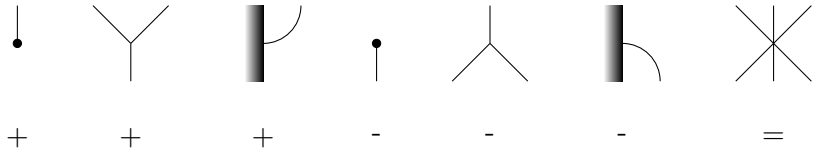
**Proposition 10.0.1.** *Given two expressions $\underline{x}, \underline{y}$, let $\mathcal{SDL}_{\underline{x},\underline{y}}$ be a collection of maps containing one double-leaf $\mathbb{SDL}_{\underline{e},\underline{f}}$ for every pair of subexpressions $\underline{e} \subset \underline{x}$, $\underline{f} \subset \underline{y}$ such that $W_J\underline{x}^{\underline{e}} = W_J\underline{y}^{\underline{f}}$. Then $\mathcal{SDL}_{\underline{x},\underline{y}}$ spans $Hom_{\mathcal{M}_{BS}}(\underline{x}, \underline{y})$.*

Our proof draws heavily on the proof that double-leaves span in $\mathcal{H}_{BS}$ presented in section 7 of [EW16].

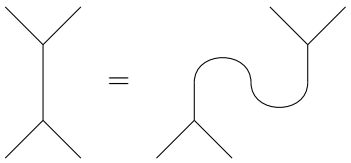
First, some terminology. Let us take a diagrammatic morphism in $\mathcal{M}_{BS}$. We can arrange this diagram so that it has no horizontal lines. Then, at any particular height, this diagram has a certain *width*; that is, we pass a horizontal line through our morphism and count the number of strands it passes through.



Then, we classify each of our generators as *positive* (+), *negative* (-) or *neutral* (=) depending on whether they increase, decrease or preserve the width.




We don't consider polynomials to affect the width. We say that a diagram is *negative-positive* if it consists of negative and neutral (non-positive) generators, followed by positive and neutral (non-negative) generators. A diagram is *strictly* negative-positive if the width actually shrinks and then increases again non-trivially. Note that this notion is not well-defined up to isotopy. For example, consider the following map.




The left-hand diagram is negative-positive, whereas the right-hand diagram is not. We say that a map is (strictly) negative-positive if it can be moved via isotopy into a diagram which is (strictly) negative-positive. For example, the above map is strictly negative-positive.

Take a reduced expression $\underline{z}$ for a minimal coset representative $z$. Let $I_{\underline{z}}$ be the right ideal (top ideal, thinking diagrammatically) of morphisms which are positive on top. Again, this notion is not well-defined up to isotopy; We say that a map is in $I_{\underline{z}}$ if it can be moved via isotopy into a map which is positive on top.

So, a map in $I_{\underline{z}}$ (up to isotopy) finishes with one of our positive generators, followed by rex moves. But note that since $\underline{z}$ is a reduced expression for a minimal coset representative, this positive generator cannot be the trivalent vertex, nor a strand coming from the wall. Therefore a morphism in $I_{\underline{z}}$ must end in an up-dot  followed by rex moves. By proposition 4.2.1, we know that this dot will pull through our rex moves to the top. Therefore, $I_{\underline{z}}$ is the right (top) ideal of morphisms with an up-dot at the top.

## 10.1 Preliminaries

Before getting to the heart of the proof, we need to introduce a few lemmas.

**Lemma 10.1.1.** *Let $\underline{w}$ and $\underline{w}'$ be two reduced expressions for the same element, and let $\beta$ and $\beta'$ be two different rex moves from $\underline{w}$ to $\underline{w}'$. Then $\beta - \beta'$ has a strictly negative-positive decomposition.*

*Proof.* Lemma 7.4 of [EW16] gives this in the case of $\mathcal{H}_{BS}$, so it is also true in $\mathcal{M}_{BS}$. □

**Lemma 10.1.2.** *Let $\underline{x}$ be an arbitrary expression. Then the map $id_{\underline{x}}$ is a sum of negative-positive maps which each factor through some reduced expression $\underline{w}$ of some $w \leq \underline{x}$.*

*Proof.* Again, lemma 7.5 of [EW16] gives this in the case of $\mathcal{H}_{BS}$, so it is true in $\mathcal{M}_{BS}$ as well. □

In fact, we can say something slightly stronger than the above lemma: the elements $w$ can be minimal coset representatives.

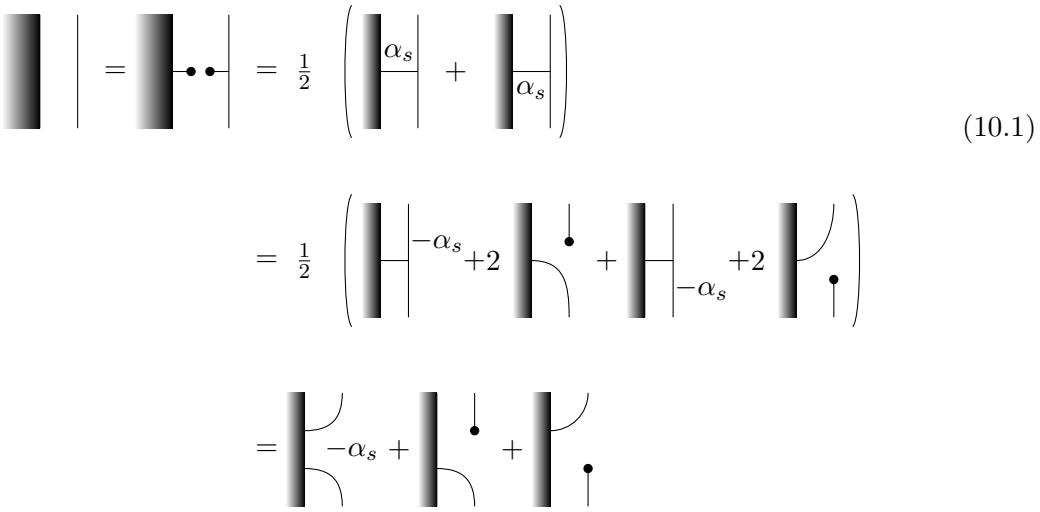
**Lemma 10.1.3.** *Let $\underline{x}$ be an arbitrary expression. Then the map $id_{\underline{x}}$ is a sum of negative-positive maps which each factor through some reduced expression $\underline{z}$ for a minimal coset representative $z \leq \underline{x}$.*

*Proof.* We proceed by induction on the length of $\underline{x}$. The base case $\underline{x} = \varnothing$ is trivial. If $\underline{x}$ is not a reduced expression, then we can apply Lemma 10.1.2 to write $id_{\underline{x}}$ as a sum of negative-positive maps which factor through reduced expressions $\underline{w}$ for $w < \underline{x}$. In particular, $l(\underline{w}) < l(\underline{x})$, so we can apply induction.

Then, suppose $\underline{x}$ is a reduced expression for some $x$. If $x$ is an m.c.r., then the statement is immediate. If $x$ is not an m.c.r., then there is some other reduced expression $\underline{x}'$ of $x$ which has some $s \in J$ on the left. Let $\alpha : \underline{x} \to \underline{x}'$ and $\alpha' : \underline{x}' \to \underline{x}$ be rex moves. We can compose these to get $\alpha' \circ \alpha : \underline{x} \to \underline{x}' \to \underline{x}$. So we now have two rex moves from $\underline{x}$ to itself: $id_{\underline{x}}$ and $\alpha' \circ \alpha$. By Lemma 10.1.1, the difference between these two is a sum

of strictly negative-positive maps, so that they each factor through expressions $\underline{y}$, with $l(\underline{y}) < l(\underline{x})$. We can apply the inductive hypothesis to these $\underline{y}$'s, so now we only need to show that $\alpha' \circ \alpha : \underline{x} \to \underline{x}' \to \underline{x}$ is a sum of negative-positive maps factoring through reduced expressions $\underline{z}$ for m.c.r.'s $z$. It will be sufficient to show this for the map $id_{\underline{x}'}$, since then we can pre- and post-compose with $\alpha$ and $\alpha'$.

To the left strand of $id_{\underline{x}'}$ we apply the following relation:



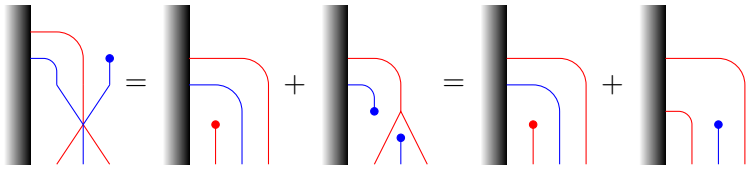

(10.1)

This means $id_{\underline{x}'}$ is a sum of maps which factor through an expression of smaller length; i.e. $\underline{x}'$ but with the $s \in J$ on the left removed. We can then apply the inductive hypothesis. □

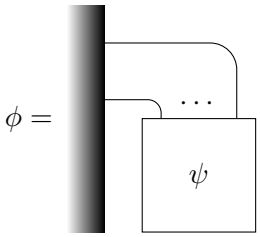
Let us first prove spanning in the simplest possible situation. Take an expression $\underline{x}$ in $W_J$. The light-leaves from $\underline{x}$ will all go to $\underline{\varnothing}$. Given a subexpression $\underline{e} \subset \underline{x}$, the decoration $d'_1, ..., d'_n$ of $\underline{e}$ will consist of only $X0$ and $X1$. The corresponding light-leaf map $SLL_{\underline{x},\underline{e}}$ will thus consist only of up-dots and identity strands plugged into the wall. We show that these light-leaves span $\mathrm{Hom}_{\mathcal{M}_{BS}}(\underline{x}, \underline{\varnothing})$.

**Lemma 10.1.4.** *Take an expression $\underline{x}$ in $W_J$ and a map $\phi : \underline{x} \to \underline{\varnothing}$ in $\mathcal{M}_{BS}$. Then $\phi$ lies in the right $R$-span of light-leaves $SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{\varnothing}$.*

Here is an example to illustrate the idea (using two-colour dot contraction (4.16)):



*Proof.* By shifting the wall plug-ins to the top using isotopy, we can consider our map $\phi$ as a map $\psi$ in $\mathcal{H}_{BS}$ (i.e. with no wall plug-ins), followed by plugging all strands into the wall.



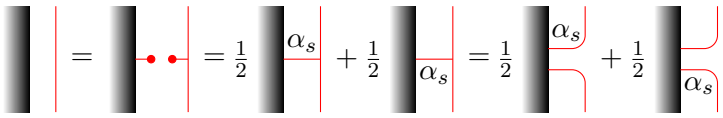

First, we can shift any polynomials in $\psi$ to the right using polynomial forcing (4.11). Now, take any up-dots that show up in $\psi$ and pull them as low as possible. They pull through rex moves via proposition 4.2.1. Eventually each up-dot will reach either the bottom of the diagram, a down-dot (forming a barbell), a splitting trivalent vertex (thus disappearing), or a merging trivalent vertex (forming a cap). In the case where it reaches a down-dot, forming a barbell, we can again move this to the right using polynomial forcing. Any up-dots formed in this process we again move as low as possible[1]. After this process, we end up with all our up-dots at the bottom of the diagram; the others are all either part of caps, or have turned into barbells and been shifted to the right.

Then, we rearrange the diagram so that no two generators are at the same height (other than all the up-dots at the bottom of the diagram). For the purposes of this proof we will consider a cap as a single generator, rather than the combination of a merging trivalent and an up-dot. We can assume the first generator from the top in $\psi$ is a cap: if it is a down-dot, a merging trivalent or a $2m$-valent vertex, it gets pulled into the wall by

[1]There's a danger here of an infinite loop: each barbell moves to the right and creates more barbells. In Appendix A, we show that this doesn't happen.

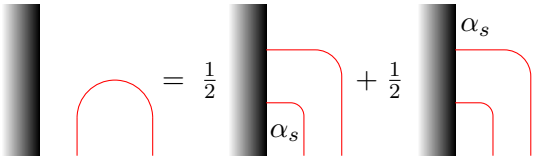
relations (7.2) - (7.4). If it is a splitting trivalent, then the whole map is zero by (7.6). If it is one of the up-dots at the bottom of the diagram, then the diagram is one of our light-leaves.

Now we induct on the number of $2m$-valent vertices present in the diagram. We first work with the base-case, where there are no $2m$-valent vertices. A key relation (throughout the rest of the chapter) will be the following.




(10.2)

When we apply this to the cap at the top of the diagram, we can pull the cap into the wall.



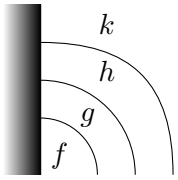

We have thus reduced the number of caps by 1. We then pull these $\alpha_s$'s to the right using polynomial forcing, creating more up-dots in the process, which we then pull downwards through the diagram as before. Then we again take the top most generator in each resulting diagram, which we can again assume is a cap. We repeat the process, each time eliminating the top most cap. Note that more caps can be created during the procedure of moving up-dots downward: one of these up-dots may meet a merging trivalent vertex, forming a cap. But there are a finite number of merging trivalent vertices, and since by assumption there are no $2m$-valent vertices in the diagram, there is no way for new merging trivalents to be formed. Therefore eventually both the merging trivalents and caps will eventually run out. We will be left with a sum of diagrams which are just up-dots, and identity strands plugged into the wall (with polynomials on the right). This is a linear combination of light-leaves.

Now assume that when $\psi$ has less than $M$ $2m$-valent vertices, it is in the span of light-leaves. Take a map $\psi$ with $M$ $2m$-valent vertices. We perform exactly the same procedure:

move any polynomials to the right using polynomial forcing, move any up-dots as low as possible in the diagram. By the same procedure as before, we eliminate caps on the top of the diagram one by one, and then pull the resulting up-dots down through the diagrams. If at any point one of these up-dots gets pulled through a $2m$-valent vertex, that vertex disappears using (4.16), so that by induction we are done. Otherwise, as before, the caps will eventually run out, so that we are left with only light-leaves. □

**Corollary 10.1.5.** *Take an expression $\underline{x}$ in $W_J$ and a map $\phi : \underline{x} \to \underline{\varnothing}$ in $\mathcal{M}_{BS}$. Then $\phi$ is a sum of maps which are made only of strands from $\underline{x}$ into the wall, with polynomials in the gaps.*

Here is an example of such a map.




*Proof.* By lemma 10.1.4, $\phi$ is in the span of light-leaves $\underline{x} \to \underline{\varnothing}$. Each of these is made of strands into the wall, and up-dots. Each of these up-dots can be pulled into the wall using (7.7), leaving polynomials on either side. This will yield a sum of maps of the desired form. □

The bulk of the work in this chapter will go towards proving the following:

**Proposition 10.1.6.** *Let $\underline{z}$ be a reduced expression for a minimal coset representative $z$, and let $\underline{x}$ be an arbitrary expression. Choose a light-leaf map $SLL_{\underline{x},\underline{e}}$ for each $\underline{e} \subset \underline{x}$ such that $W_J \underline{x}^{\underline{e}} = W_J z$. These maps span $Hom(\underline{x}, \underline{z})/I_{\underline{z}}$.*

We note that the analogous statement has been proven in the context of $\mathcal{H}_{BS}$.

**Proposition 10.1.7.** *Let $\underline{w}$ be a reduced expression for $w \in W$, and let $\underline{x}$ be an arbitrary expression. Choose a light-leaf map $LL_{\underline{x},\underline{e}}$ for each $\underline{e} \subset \underline{x}$ such that $\underline{x}^{\underline{e}} = w$. These maps*

*span* $Hom_{\mathcal{H}_{BS}}(\underline{x}, \underline{w})/I_{\underline{w}}{}^2$.

*Proof.* See Proposition 7.6 of [EW16]. □

The fact that this has already been proven in the case of $\mathcal{H}_{BS}$ will be extremely helpful to us in doing so for $\mathcal{M}_{BS}$ (i.e. proving Proposition 10.1.6). Once we have proved Proposition 10.1.6, it is not too hard to prove that double-leaves span.

*Proof of Proposition 10.0.1 given 10.1.6:* Take any expressions $\underline{x}, \underline{y}$, and an arbitrary morphism $\phi : \underline{x} \to \underline{y}$. We want to show that $\phi$ lies in the (right) $R$-span of double-leaves. We know from Lemma 10.1.3 that $id_{\underline{x}}$ is a sum of maps which factor through some reduced expression $\underline{z}$ for an m.c.r. $z$. So we may assume $\phi$ factors through such a $\underline{z}$, i.e. we have $\phi : \underline{x} \to \underline{z} \to \underline{y}$. We will induct on $z$ in the Bruhat order.

To prove the base-case, take a map $\underline{x} \xrightarrow{g} \varnothing \xrightarrow{f} \underline{y}$. Notice that $I_{\varnothing}$ is empty. Thus, by proposition 10.1.6 we have that the map $\underline{x} \xrightarrow{g} \varnothing$ is spanned by light-leaves. Similarly, $\varnothing \xrightarrow{f} \underline{y}$ is spanned by upside-down light-leaves. Therefore $\underline{x} \to \varnothing \to \underline{y}$ is spanned by double-leaves.

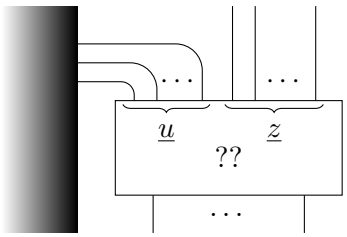
Now, assume that for any m.c.r. $v$, with $v < z$, any map $\underline{x} \to \underline{v} \to \underline{y}$ is in the span of double-leaves (for $\underline{v}$ a rex of $v$). Take our map $\phi : \underline{x} \xrightarrow{g} \underline{z} \xrightarrow{f} \underline{y}$. By proposition 10.1.6, we can write $g = g_z + g_l$, where $g_z$ is in the span of light-leaves from $\underline{x}$ to $\underline{z}$, and $g_l \in I_{\underline{z}}$. Similarly, we can write $f = f_z + f_l$, where $f_z$ is in the span of inverted light-leaves, and $f_l \in \overline{I}_z$. So $\phi = fg = f_z g_z + f_z g_l + f_l g_z + f_l g_l$. Now, $f_z g_z$ is clearly spanned by double-leaves. As for the other maps, we will prove that for any map $\psi : \underline{z} \to \underline{y}$, the composition $\psi g_l$ is spanned by double leaves. This will take care of $f_z g_l$ and $f_l g_l$. Then $f_l g_z$ can be dealt with using the same argument upside-down.

[2]We technically only defined $I_{\underline{w}}$ as an ideal of $\mathcal{M}_{BS}$ rather than $\mathcal{H}_{BS}$, and only when $\underline{w}$ is a rex of some $w \in {}^J W$. The ideal $I_{\underline{w}}$ in $\mathcal{H}_{BS}$, for general $\underline{w}$, is defined exactly analogously; i.e. it is maps to $\underline{w}$ which are positive on top. When $\underline{w}$ is reduced, this is exactly maps with an up-dot at the top.

The map $g_l : \underline{x} \to \underline{z}$ is in $I_{\underline{z}}$ and thus passes through an expression $\underline{w}$ shorter than $\underline{z}$. Then by Lemma 10.1.3, $id_{\underline{w}}$ factors through reduced expressions $\underline{v}$ of m.c.r.'s $v$, so that $\psi g_l$ is a sum of maps $\underline{x} \to \underline{v} \to \underline{y}$. It is not too hard to see that $v < z$, so we can apply our inductive hypothesis. □

So now it remains to prove Proposition 10.1.6. This will be the task of the remainder of this chapter.

Let $\underline{z}$ be a reduced expression of an m.c.r. $z$, and let $\underline{x}$ be arbitrary. We take a map $\phi : \underline{x} \to \underline{z}$, and we want to show that it is a linear combination of spherical light-leaves, plus maps in $I_{\underline{z}}$. We first note that all the strands which plug into the wall can be moved to the top, so that we have

$\phi =$ 

(10.3)

where $\underline{u}$ is some sequence of $s_i \in J$. We can in fact assume that $\underline{u}$ is reduced, using relations (7.3) and (7.4)[3]. Since $\underline{z}$ is a reduced expression of an m.c.r., we get that the concatenation $(\underline{u}, \underline{z})$ is in fact a reduced expression[4]. Therefore by Proposition 10.1.7, we know that the box labelled '??' above is in the span of (non-spherical) light leaves, plus maps in $I_{(\underline{u},\underline{z})}$.[5] (In fact, we get to choose which light-leaf maps to use here. This will be important, and we will describe this choice in detail in the section below.) Maps in $I_{(\underline{u},\underline{z})}$ are those with down-dots at the top. If this dot is part of $\underline{z}$, then our map is in $I_{\underline{z}}$. If this dot is part of $\underline{u}$, then it gets pulled into the wall using (7.2). So $\underline{u}$ gets replaced by a shorter expression $\underline{u}'$. We can then play the same game: we can turn $\underline{u}'$ into a reduced

[3]Any non-reduced expression either has a doubled simple reflection (in which case we apply (7.3)) or can be moved to an expression with a doubled reflection via braid relations (in which case we apply (7.4)).

[4]This is a consequence of Lemma 6.1.2.

[5]Here, $I_{(\underline{u},\underline{z})}$ is the ideal in $\mathcal{H}_{BS}$ of maps which are positive at the top.

expression $\underline{u}''$ (shorter than $\underline{u}$) and replace the '??' box with light-leaves plus maps in $I_{(\underline{u}'',\underline{z})}$. Since we are shortening our expression each time we go through this procedure, this must eventually terminate. What we end up with is a sum of non-spherical light-leaf maps, followed by plugging strands into the wall, and maps in $I_z$.

## 10.2 Construction of non-spherical light-leaves

We noted above that we get to choose the particular construction of these non-spherical light-leaves. That is, at each step in the light-leaves construction, we have a choice about our intermediate expressions $\underline{w}_k$, and which rex moves to apply (see Section 5.1). We will now describe in detail precisely how we choose these intermediate expressions and rex moves. This choice will be crucial in proving Proposition 10.1.6. The description is somewhat involved, as this choice will do most of the heavy-lifting in our proof.

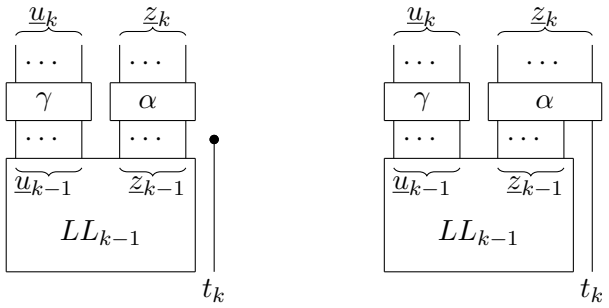
Roughly speaking, the idea is to make these non-spherical light-leaves look as much like spherical light-leaves as possible. That is, our spherical light-leaves have various strands which plug into the wall on the left. Non-spherical light-leaves don't have any wall to plug into, so we will instead collect all these "wall-strands" on the left of the diagram.

We choose intermediate expressions $\underline{w}_k$ so as to move strands $s \in J$ to the left after each step. More precisely, suppose we are constructing a light-leaf $LL_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$ for $\underline{x} = (t_1, ..., t_M)$. At step $k$ we construct a map $LL_k$ to a rex $\underline{w}_k$ of $w_k := \underline{x}^{\underline{e}_{\leq k}}_{\leq k} = t_1^{e_1} \cdots t_k^{e_k}$. How do we choose this rex $\underline{w}_k$? We know from Lemma 6.1.2 that $w_k$ can be uniquely decomposed as $u_k z_k$, where $u_k \in W_J$ and $z_k \in {}^J W$. We also know from the same lemma that $l(w_k) = l(u_k) + l(z_k)$, so $w_k$ has a rex $(\underline{u}_k, \underline{z}_k)$ for rex's $\underline{u}_k,\ \underline{z}_k$ of $u_k,\ z_k$ respectively. We set $\underline{w}_k := (\underline{u}_k, \underline{z}_k)$. This ensures that the image of each $LL_k$ has the maximal number of $J$-strands on the left.

We also choose our rex moves so as to leave these left $J$-strands alone as much as possible. The remainder of this section will describe in detail how we choose these rex moves. The choice of rex move at a given step will depend on the labelling of the subexpression. Recall

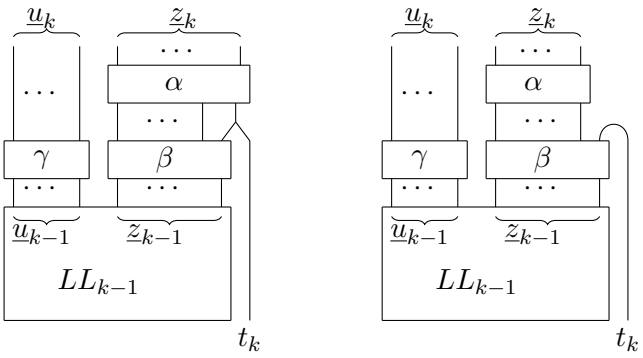
that the subexpression $\underline{e} \subset \underline{w}$ has two different labellings: the one used in the non-spherical light-leaves construction, and the one used in the spherical light-leaves construction. We will refer to these as $d_1, ..., d_M$ and $d'_1, ..., d'_M$ respectively. For example, consider $W = S_3 = \langle s, t \rangle$, $J = \{s\}$, $\underline{w} = (s, t, s)$, and $\underline{e} = (1, 0, 0)$. Then the non-spherical labelling of $\underline{e}$ would be $d_1, d_2, d_3 = U1, U0, D0$, whereas the spherical labelling would be $d'_1, d'_2, d'_3 = X1, U0, X0$.

Consider the map $\phi_k : (\underline{u}_{k-1}, \underline{z}_{k-1}, t_k) \to (\underline{u}_k, \underline{z}_k)$ in the $k$'th step of our light-leaf construction. To decide which rex moves to use in $\phi_k$, we have two cases: first, when $d_k = d'_k$, and second, when $d_k \neq d'_k$. If $d_k = d'_k$, then in particular $d'_k$ must be either $U$ or $D$ (not $X$). If $d_k = d'_k = U0$ or $U1$, then we can apply separate rex moves $\gamma$ and $\alpha$ to the $\underline{u}$'s and the $\underline{z}$'s respectively:



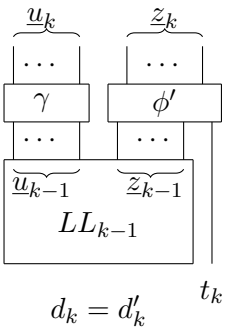

$d_k = d'_k = U0$ $\qquad$ $d_k = d'_k = U1$ (10.4)

(We include $LL_{k-1}$ in our diagrams, so that the diagrams depict the map $LL_k$). If $d_k = d'_k = D0$ or $D1$, then we need to apply some rex move $\beta$ to move $t_k$ to the right. Since $d'_k = D$, this $\beta$ can be applied solely to $\underline{z}_{k-1}$:




(10.5)

Here, $\alpha$ is some other rex move to $\underline{z}_k$. The point is that the $\underline{u}$ and $\underline{z}$ components can be dealt with separately, so that we have something of the following form.

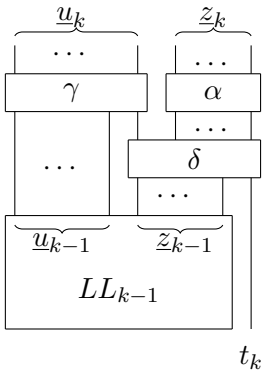



(10.6)

We can notice that in each of the above four cases, the map $\phi'$ is exactly the $\phi$ associated to $d'_k$ in the $k^{\text{th}}$ step of the spherical light-leaf construction. Thus we are making our non-spherical light-leaves look like spherical light-leaves.

The second case, where $d_k \neq d'_k$, is more involved. This can only happen if $d'_k = X$. We will describe how to choose $\phi_k$ on a case by case basis. Unlike in the above $d_k = d'_k$ case, we can't keep the $\underline{u}$ and $\underline{z}$ strands entirely separate, but we keep them as separate as possible.

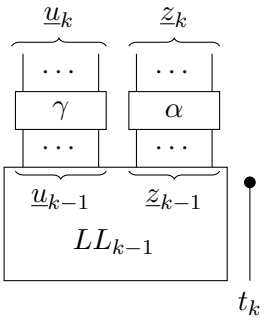
- $d_k = U1$, $d'_k = X1$:




$$d_k = U1,\ d'_k = X1 \tag{10.7}$$

If $d'_k = X1$, then by Lemma 6.1.3 (b) there is some $s \in J$ such that $z_{k-1}t_k = sz_{k-1}$. By Lemma 6.1.3 (a), $(\underline{z}_{k-1}, t_k)$ is reduced so that there is some rex move $\delta : (\underline{z}_{k-1}, t_k) \to (s, \underline{z}_{k-1})$. Then, since $d_k = U1$, $(\underline{u}_{k-1}, s)$ must also be reduced, so there is some rex move $\gamma : (\underline{u}_{k-1}, s) \to \underline{u}_k$. Again, $\alpha$ is one of our 'optional' rex moves.

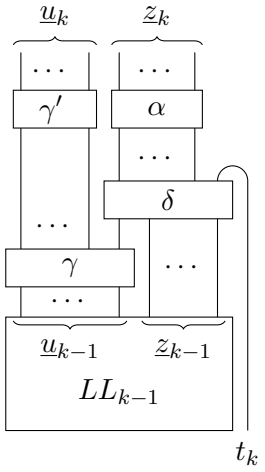
- $d_k = U0,\ d'_k = X0$:




$$d_k = U0,\ d'_k = X0 \tag{10.8}$$

In this case we apply separate rex moves $\gamma : \underline{u}_{k-1} \to \underline{u}_k$ and $\alpha : \underline{z}_{k-1} \to \underline{z}_k$.

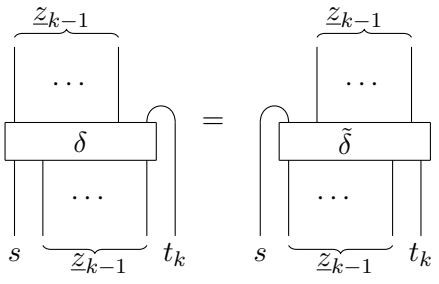
- $d_k = D1, d'_k = X1$:




(10.9)

Since $d_k = D1$, we need to apply some rex move $\beta$ to $(\underline{u}_{k-1}, \underline{z}_{k-1})$ to get $t_k$ to the right of the expression. This is done as follows: Let $\underline{u}_{k-1} = (s_1, ..., s_l)$, $\underline{z}_{k-1} = (r_1, ..., r_m)$. Since $d_k = D1$, the exchange condition (Theorem 2.3.2) tells us that $u_{k-1}z_{k-1}t_k = s_1 \cdots s_l r_1 \cdots r_m t_k = s_1 \cdots \hat{s_i} \cdots s_l r_1 \cdots r_m$ for some $i$, where the hat denotes deletion (the deleted generator cannot be among the $r_i$'s by Lemma 6.1.3 (a)). Since $d'_k = X1$, there is some $s \in J$ such that $z_{k-1}t_k = sz_{k-1}$. This means $u_{k-1}z_{k-1}t_k = s_1 \cdots s_l s z_{k-1} = s_1 \cdots \hat{s_i} \cdots s_l z_{k-1}$, meaning $s$ is in the right descent set of $u_{k-1}$. Therefore there is some rex of $u_{k-1}$ with $s$ on the right (in fact it is $(s_1, ..., \hat{s_i}, ..., s_l, s)$). We apply a rex move $\gamma$ to $\underline{u}_{k-1}$ to move this $s$ to the right, and then apply a rex move $\delta : (s, \underline{z}_{k-1}) \to (\underline{z}_{k-1}, t_k)$. We have thus moved $t_k$ to the right as required. After applying a cap to this $t_k$, we apply rex moves $\gamma'$ and $\delta'$ to make sure the targets are correct.

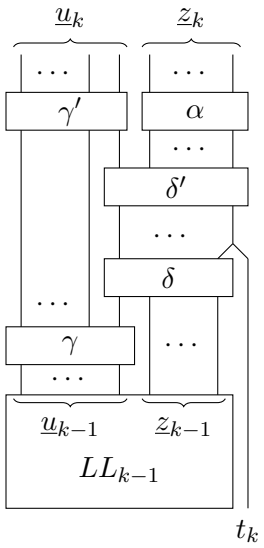
The key in this case will be that we can rotate the $s$ and $t_k$ strands in $\delta$:




(10.10)

Here $\tilde{\delta}$ is some other rex move. This will assist us in turning non-spherical light-leaves into spherical light-leaves.

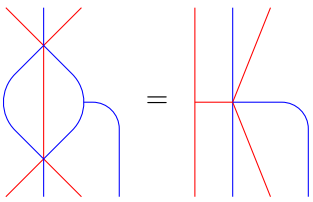
- $d_k = D0,\ d'_k = X0$:



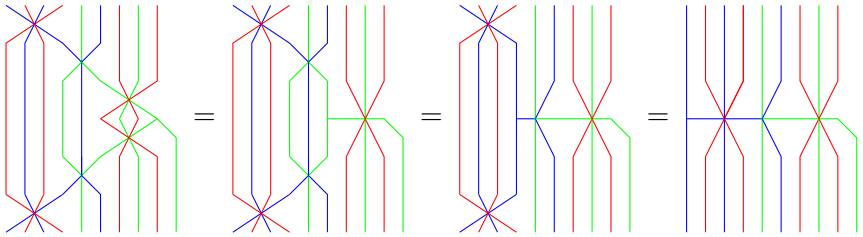

(10.11)

This final case is the most daunting. It is in fact similar to the previous case of $d_k = D1$, $d'_k = X1$: we first apply $\gamma$ as $\delta$ in almost the same way. The difference will be that $\delta$ will be a carefully chosen rex move $(s, \underline{z}_{k-1}) \to (\tilde{\underline{z}}_{k-1}, t_k)$, where $\underline{z}_{k-1}$ and $\tilde{\underline{z}}_{k-1}$ are not necessarily the same. We then apply a third rex move $\delta' : (\tilde{\underline{z}}_{k-1}, t_k) \to (s, \tilde{\underline{z}}_{k-1})$. Finally we apply rex moves $\gamma'$ and $\alpha$ to ensure our map has the correct target $(\underline{u}_k, \underline{z}_k)$.

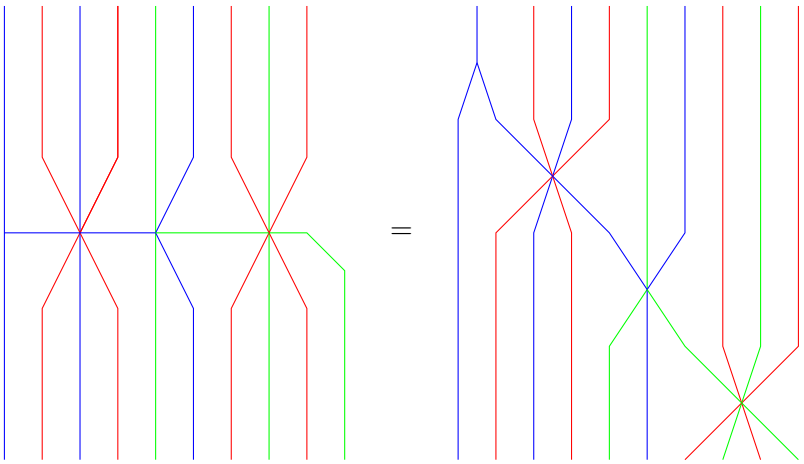
We want to choose $\delta$ and $\delta'$ so that we can repeatedly apply two-colour associativity (4.17) in order to pull the trivalent vertex from the right to the left (This will be necessary when we turn our light-leaves into spherical light-leaves). This is best illustrated through examples. If $\delta$ is a single $2m$-valent vertex, then we can set $\delta' = \overline{\delta}$ (i.e. $\delta$ upside-down), and we can pull the trivalent vertex to the left using a single application of two-colour associativity:



This also works if $\delta$ is a chain of $2m$-valent vertices right next to one another, moving left to right:



This final result we can arrange as



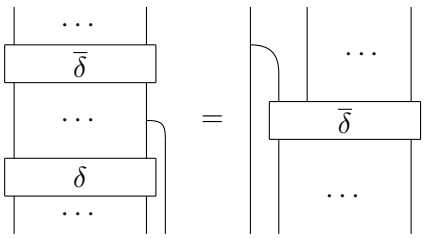
This is just $\delta' = \overline{\delta}$ with a merging trivalent at the top left.

Of course, not all rex moves consist of a chain of $2m$-valent vertices going left to right in this way. But it turns out (and we will show) that in our situation we can always choose a $\delta$ that *ends* in such a chain of $2m$-valent vertices. This will be enough for us to pull the trivalent vertex from right to left.

To make this precise, we define a *sweep* to be a sequence of applications of the braid relation to an expression, moving left to right: the first one acting on letters $1, ..., m_1$, the second to letters $m_1, ..., m_2$, and so on, right to the end of the expression. For example, let $m_{st} = 3$, $m_{su} = 2$ and $m_{tu} = 3$. Here is a sweep from $(t, s, t, u, t, s)$ to $(s, t, u, t, s, t)$:

$$(t, s, t, u, t, s) \mapsto (s, t, s, u, t, s) \mapsto (s, t, u, s, t, s) \mapsto (s, t, u, t, s, t)$$

We will abuse terminology by using 'sweep' to refer to the corresponding rex move in $\mathcal{H}_{BS}$ or $\mathcal{M}_{BS}$, i.e. a map consisting of $2m$-valent vertices moving left to right. We saw above that if $\delta$ is a sweep, then we can apply two-colour associativity to get the following.




(10.12)

**Lemma 10.2.1.** *Take some $z \in W$ and $s, t \in S$ such that $sz = zt > z$. Then there is some rex $\underline{\tilde{z}}$ of $z$ such that $(s, \underline{\tilde{z}})$ can be transformed into $(\underline{\tilde{z}}, t)$ via a sweep.*

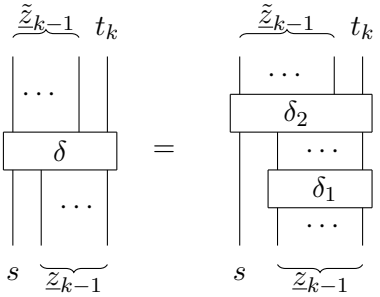
*Proof.* We proceed by induction on $l(z)$. For the base case, if $l(z) = 0$, then $s = t$ and the statement is trivial. Suppose the statement is true when $l(z) < M$, and suppose $l(z) = M$. Take any rex $\underline{z}$ of $z$. Note that since $zt > z$, we know by Corollary 2.3.3 that $\underline{z}$ does not end in $t$, so neither does $(s, \underline{z})$. But $(\underline{z}, t)$ does end in $t$, and there is some rex move from $(s, \underline{z})$ to $(\underline{z}, t)$. Therefore at some step in this rex move, the last letter is changed to a $t$ from (say) a $u$. This can only happen via an application of the braid relation. That is, at this step in the rex move we have $(s_1, ..., s_n, u, t, u) \mapsto (s_1, ..., s_n, t, u, t)$ for some $s_i \in S$ (here we let $m_{tu} = 3$ for clarity of notation, but everything works the same for different $m_{tu}$). Then, we have $zt = s_1 \cdots s_n tut$, so that $z = s_1 \cdots s_n tu$. Thus $sz = ss_1 \cdots s_n tu$. But $sz = zt = s_1 \cdots s_n utu$, so that $ss_1 \cdots s_n = s_1 \cdots s_n u$. We can thus apply our inductive hypothesis to $z' := s_1 \cdots s_n$, with $sz' = z'u > z$.[6] That is, there is some rex $\underline{z}'$ of $z'$ such that $(s, \underline{z}')$ can be turned into $(\underline{z}', u)$ by a sweep. Then set $\underline{\tilde{z}} := (\underline{z}', t, u)$. We can turn $(s, \underline{\tilde{z}})$ into $(\underline{\tilde{z}}, t)$ via a sweep:

$$(s, \underline{\tilde{z}}) = (s, \underline{z}', t, u) \xmapsto{sweep} (\underline{z}', u, t, u) \mapsto (\underline{z}', t, u, t) = (\underline{\tilde{z}}, t).$$

□

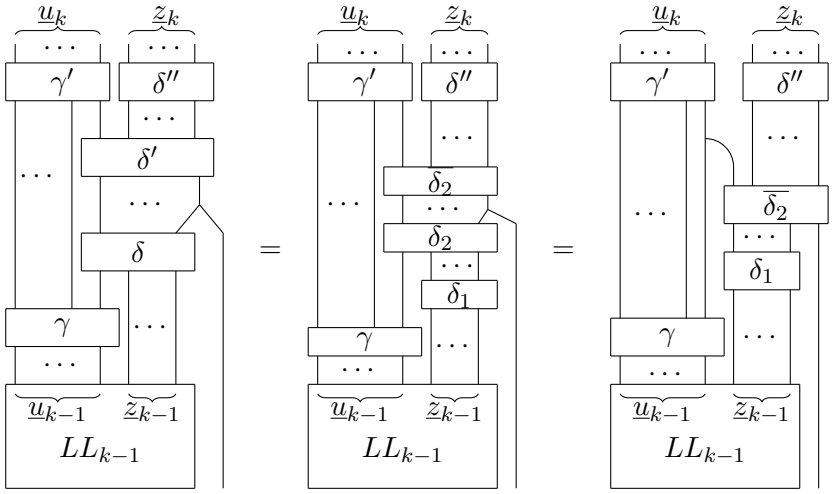
[6]We get $z'u > z$ by the fact that $(s_1, ..., s_n, u, t, u)$ is reduced, so that $(s_1, ..., s_n, u)$ is a rex of $z'u$.

This tells us how we should construct our $\delta$. Recall (see (10.11)) that we are applying $\delta$ to the expression $(s, \underline{z}_{k-1})$, where $sz_{k-1} = z_{k-1}t_k$. By the above lemma, there is some expression $\underline{\tilde{z}}_{k-1}$ of $z_{k-1}$ such that $(s, \underline{\tilde{z}}_{k-1})$ can be transformed into $(\underline{\tilde{z}}_{k-1}, t_k)$ via a sweep. First, apply a rex move $\delta_1$ from $\underline{z}_{k-1}$ to $\underline{\tilde{z}}_{k-1}$. Then apply a sweep $\delta_2$ from $(s, \underline{\tilde{z}}_{k-1})$ to $(\underline{\tilde{z}}_{k-1}, t_k)$:




(10.13)

We then choose $\delta' := \overline{\delta_2}$. Applying this choice to figure 10.11 we get

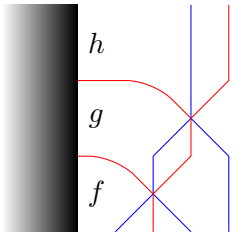



(10.14)

where we have applied (10.12) to $\delta_2$ and $\overline{\delta_2}$. The final diagram on the right of (10.14) will hopefully give some sense of why we have gone to so much trouble here. Once again, we are trying to make our non-spherical light-leaves look like spherical light-leaves. If we imagine the strands joining $\gamma$ and $\gamma'$ as a 'wall', then this diagram looks a lot like the step of the spherical light-leaves algorithm corresponding to $d'_k = X1$ (see diagram (7.10)).

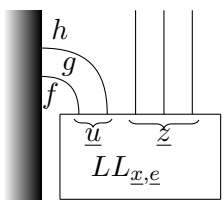
## 10.3 Proof of spanning

Again, let $\underline{z}$ be a reduced expression for $z$ an m.c.r., and let $\underline{x}$ be an arbitrary expression. We saw that any map $\phi : \underline{x} \to \underline{z}$ in $\mathcal{M}_{BS}$ can be considered a map $\psi : \underline{x} \to (\underline{u}, \underline{z})$ in $\mathcal{H}_{BS}$ (for some rex $\underline{u}$ in $W_J$), followed by plugging the $\underline{u}$-strands into the wall (see diagram (10.3)). We then saw that this is a sum of our carefully chosen (non-spherical) light-leaf maps, followed by plugging the $\underline{u}$-strands into the wall, as well as maps in $I_{\underline{z}}$. What we would like to show is the following: A light-leaf map to $(\underline{u}, \underline{z})$, followed by plugging the $\underline{u}$-strands into the wall, is in the span of our choice of spherical light-leaves and $I_{\underline{z}}$. This would prove Proposition 10.1.6, that any choice of spherical light-leaves spans $\mathrm{Hom}_{\mathcal{M}_{BS}}(\underline{x}, \underline{z})/I_{\underline{z}}$. This is eventually what we will show, but first we will show something weaker. Let $X_{\underline{x},\underline{z}}$ be the collection of *any* constructions of spherical light-leaves (SLL's) from $\underline{x}$ to $\underline{z}$, with polynomials $f \in R$ anywhere along the wall. Here is an example:

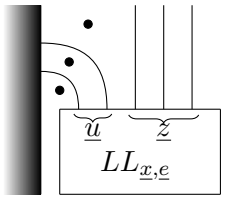



**Lemma 10.3.1.** *Let $\underline{z}$ be a rex of an m.c.r. $z$, and let $\underline{u}$ be a reduced expression in $W_J$. Let $\underline{x}$ be an arbitrary expression. Take a (non-spherical) light-leaf $LL_{\underline{x},\underline{e}} : \underline{x} \to (\underline{u}, \underline{z})$, constructed in the way described in Section 10.2. Then, plug the top $\underline{u}$-strands into the wall, and place some polynomials $f \in R$ anywhere along the wall. Then this new map is a sum of maps in $X_{\underline{x},\underline{z}}$.*

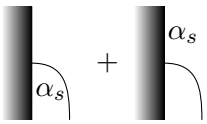
Here is an example of the kind of map being described:



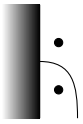
To avoid cluttering the diagrams, we will represent polynomials as black circles:



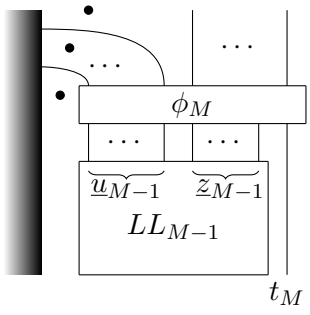

We will also use a single diagram to represent sums of a certain diagram with different polynomials. For example, the sum



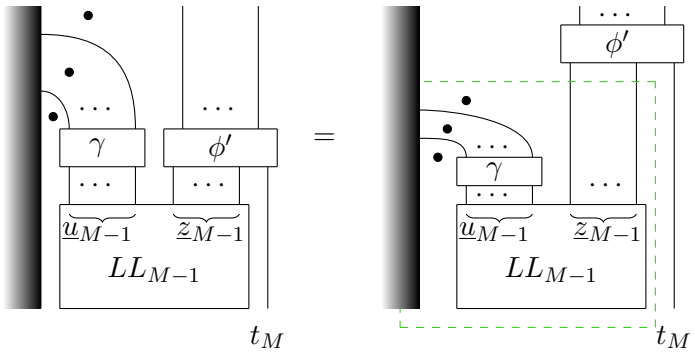

would be represented as



*Proof.* We proceed by induction on the length of $\underline{x}$. If $x = \varnothing$, then the statement is trivial. Then, suppose the statement is true for $l(\underline{x}) < M$, and suppose $\underline{x} = (t_1, ..., t_M)$ is of length $M$. Take some light-leaf $LL_{\underline{x},\underline{e}} : \underline{x} \to (\underline{u}, \underline{z})$, and plug the $\underline{u}$-strands into the wall and place polynomials along the wall. Consider $LL_{\underline{x},\underline{e}}$ as the map $LL_{M-1}$ followed by $\phi_M$. Let $LL_{M-1}$ have target $(\underline{u}_{M-1}, \underline{z}_{M-1})$, where $\underline{u}_{M-1}$ is a rex of $u_{M-1} \in W_J$, and $\underline{z}_{M-1}$ is a rex of $z_{M-1} \in {}^J W$ (see Section 10.2).



Once again, the subexpression $\underline{e} \subset \underline{x}$ has two different labellings $d_1, ..., d_M$ and $d'_1, ..., d'_M$ used in the non-spherical and spherical light-leaves construction respectively. The case where $d_M = d'_M$ is easiest. Here, we can take $\phi_M$ to act separately on $\underline{u}_{M-1}$ and $\underline{z}_{M-1}$ as in figure 10.6, yielding

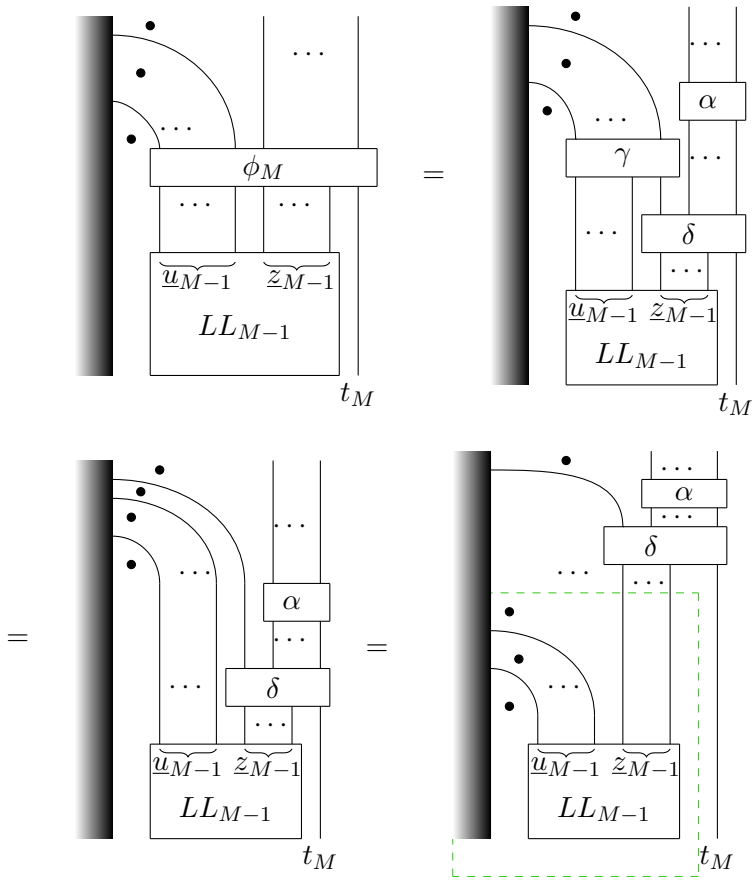



We can include $\gamma$ as part of $\phi_{M-1}$ in $LL_{M-1}$, so that by induction, the diagram in the green box above is in the span of SLL's with polynomials along the wall. We noted below figure (10.6) that the $\phi'$ here is exactly the $\phi$ associated to $d'_k$ in the $k^{\text{th}}$ step of $SLL_{\underline{x},\underline{e}}$. Thus the whole diagram is in the span of SLL's with polynomials along the wall.

The case where $d_k \neq d'_k$ is trickier, and this is where all the work we did in Section 10.2 will pay off. As in that section, we break it down into the four cases:

- $d_k = U1,\ d'_k = X1$:

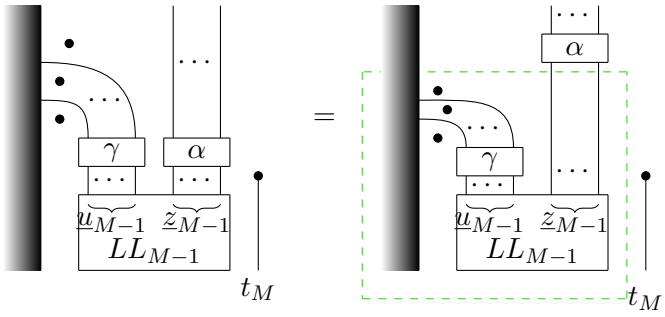
We apply the choice of $\phi_M$ that we chose in figure (10.7):




In the second equality we have applied Corollary 10.1.5. Once again by induction, the map in the green box is spanned by SLL's with polynomials. The part of the diagram outside the green box is exactly the $\phi$ associated with $d'_k = X1$ (see Section 7.3).

- $d_k = U0,\ d'_k = X0$:

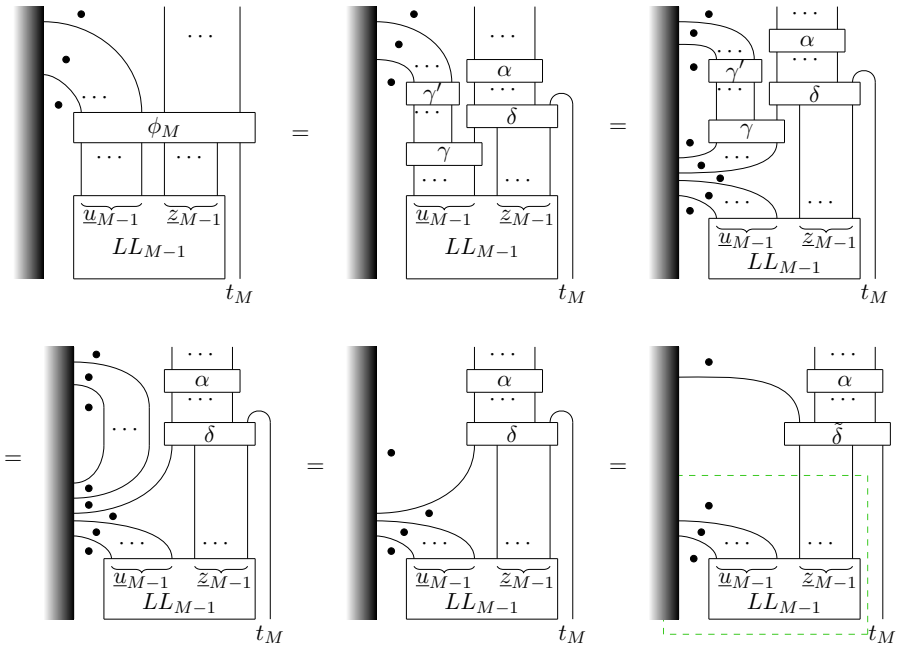
We apply our choice of $\phi_M$ as in figure (10.8):



We can include $\gamma$ as part of $LL_{M-1}$, and thus by induction the green box is spanned by SLL's with polynomials. The part of the map outside the green box is the $\phi$ associated with $d'_k = X0$.

- $d_k = D1,\ d'_k = X1$:

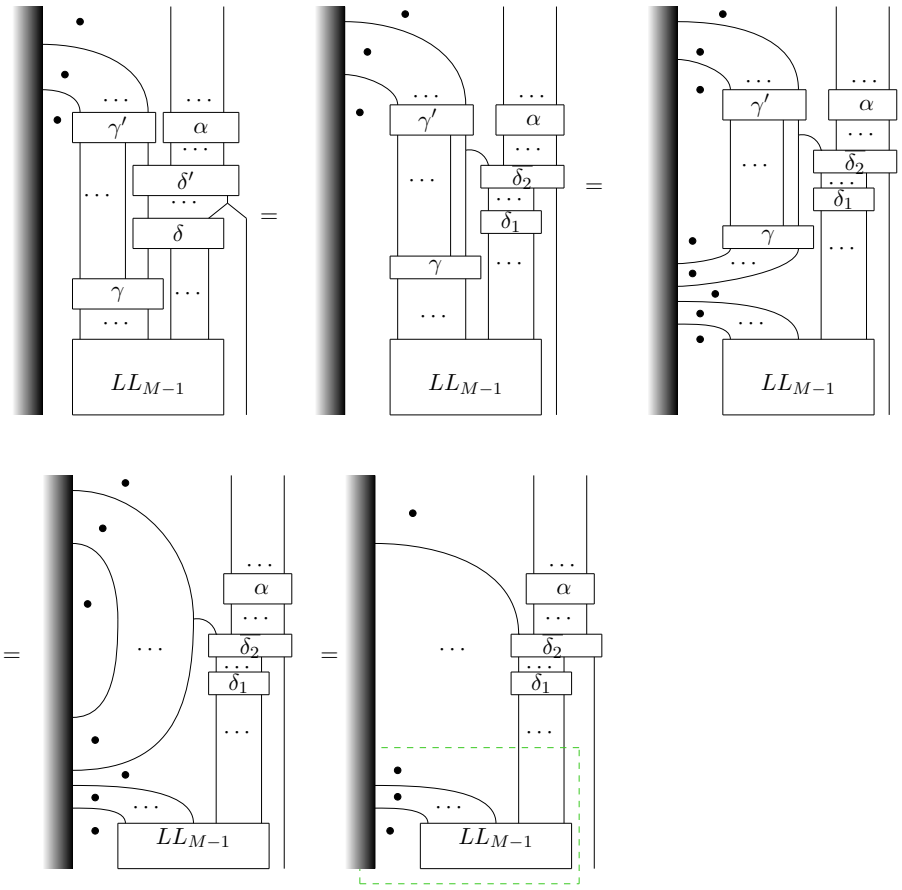
Applying our choice of $\phi_M$ in figure (10.9), we get




In the second equality, we have repeatedly applied (10.2). In the third, we have used Lemma 10.1.5 on $\gamma$ and $\gamma'$. In the fourth, we repeatedly apply (7.5). Finally in the fifth equality, we rotate $\delta$ into $\tilde{\delta}$ as described in figure (10.10). Then, as in the other cases, by induction the map in the green box is in the span of SLLs with polynomials. The part of the map outside the green box is the $\phi$ associated to $d'_k = X1$.

- $d_k = D0,\ d'_k = X0$:

We choose $\phi_M$ as in figure 10.11, and apply equation (10.14):



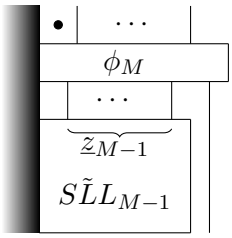

The first equality is from equation (10.14). The others follow very much like the above case of $d_k = D1,\ d_k = X1$. By induction the map in the green box is in the span of SLL's with polynomials. The part of the map outside the green box is the map $\phi$ associated with $X1$. (Note that it's okay that this is $X1$ and not $X0$: we just need the result to be *some* SLL, not necessarily the ones associated to $\underline{e}$.) □

We have now shown that, for a rex $\underline{z}$ of $z$ an m.c.r., and arbitrary $\underline{x}$, any map $\phi : \underline{x} \to \underline{z}$ in $\mathcal{M}_{BS}$ is in the span of *some* SLL's with polynomials along the wall, and maps in $I_{\underline{z}}$. If we want to prove Proposition 10.1.6, we are left with two issues to address. First, we want to show that any map is in the span of any *particular* choice of SLL constructions. Second, we need to somehow move these polynomials to the right. Our strategy will in fact be to move the polynomials up along the wall to the top-left corner of each diagram, and from there along the top of the diagram to the right.

The following lemma replaces the SLL's with any particular choice of SLL's, and moves polynomials up along the wall to the top-left corner. All that will be left after this is to move these polynomials along the top to the right.

**Lemma 10.3.2.** *Given an arbitrary expression $\underline{x}$, and $\underline{z}$ a rex of some $z \in {}^J W$, let $\mathcal{S}_{\underline{x},\underline{z}}$ be a collection of maps $\underline{x} \to \underline{z}$, containing one spherical light-leaf map $SLL_{\underline{x},\underline{e}}$ for each subexpression $\underline{e} \subset \underline{x}$ such that $W_J \underline{x}^{\underline{e}} = W_J z$. Then any map $\phi : \underline{x} \to \underline{z}$ is a sum of maps in $\mathcal{S}_{\underline{x},\underline{z}}$ with polynomials in the top-left corner, plus maps in $I_{\underline{z}}$.*

*Proof.* We prove the statement by induction on $l(\underline{x})$. Suppose the statement is true for any $\underline{x}$ with $l(\underline{x}) < M$. Take some $\underline{x}$ with $l(\underline{x}) = M$, and a map $\phi : \underline{x} \to \underline{z}$ (for $\underline{z}$ a rex of some $z \in {}^J W$). We know by Lemma 10.3.1 (and the discussion at the end of Section 10.1) that $\phi$ is a sum of *some* SLL maps (not necessarily the ones in $\mathcal{S}_{\underline{x},\underline{z}}$) with polynomials along the wall, plus maps in $I_{\underline{z}}$. Take one of these SLL maps with polynomials along the wall. It corresponds to some $\underline{e} = (e_1, ..., e_M) \subset \underline{x}$; in particular, $e_M$ will determine the map $\phi_M$. This SLL map can be pictured as

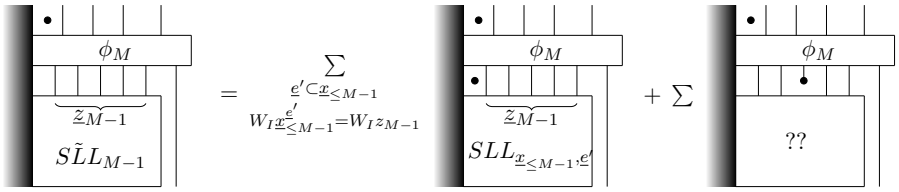



where $S\tilde{L}L_{M-1}$ is an SLL map with polynomials along the wall. Say this $S\tilde{L}L_{M-1}$ has target $\underline{z}_{M-1}$, a rex of $z_{M-1} \in {}^J W$. Now we apply the inductive hypothesis to this $S\tilde{L}L_{M-1}$; that is, we can replace it with SLL's *of our choice*, with polynomials in the top left corner, plus maps in $I_{\underline{z}_{M-1}}$. More precisely, for each subexpression $\underline{e}' \subset \underline{x}_{\leq M-1}$[7] such that $W_J \underline{x}^{\underline{e}'}_{\leq M-1} = W_J z_{M-1}$, we choose a construction of $SLL_{\underline{x}_{\leq M-1},\underline{e}'} : \underline{x}_{\leq M-1} \to \underline{z}_{M-1}$. We choose this construction as follows: Given such an $\underline{e}' \subset \underline{x}_{\leq M-1}$, we append $e_M$ to get a subexpression $\underline{f} := (\underline{e}', e_M) \subset \underline{x}$, with $W_J \underline{x}^{\underline{f}} = W_J z$. Thus there is some map

[7] A reminder about notation: Following Section 7.3, if $\underline{x} = (t_1, ..., t_M)$, then $\underline{x}_{\leq M-1} := (t_1, ..., t_{M-1})$.

$SLL_{\underline{x},\underline{f}} \in \mathcal{S}_{\underline{x},\underline{z}}$. At the $(M-1)$'th step in the construction of this $SLL_{\underline{x},\underline{f}}$, we have a map $SLL_{M-1} : \underline{x}_{\leq M-1} \to \underline{z}'_{M-1}$, where $\underline{z}'_{M-1}$ is a rex of $z_{M-1}$. However, $\underline{z}'_{M-1}$ may not be the same as $\underline{z}_{M-1}$. Thus, to construct $SLL_{\underline{x}_{\leq M-1},\underline{e}'} : \underline{x}_{\leq M-1} \to \underline{z}_{M-1}$, we take $SLL_{M-1} : \underline{x}_{\leq M-1} \to \underline{z}'_{M-1}$ and then compose with some rex move $\beta : \underline{z}'_{M-1} \to \underline{z}_{M-1}$.

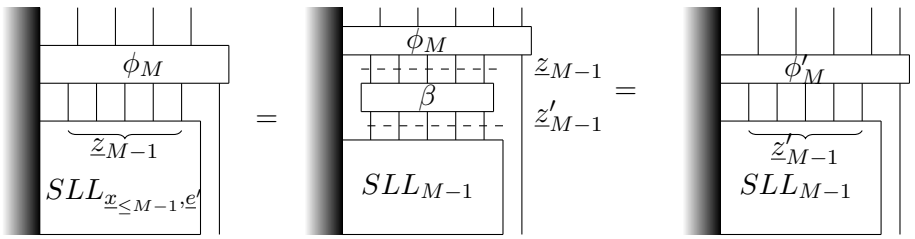
So now, we take the map $S\tilde{L}L_{M-1}$ in the above picture and replace it with these maps $SLL_{\underline{x}_{\leq M-1},\underline{e}}$ which we have constructed, plus maps in $I_{\underline{z}_{M-1}}$.




(10.15)

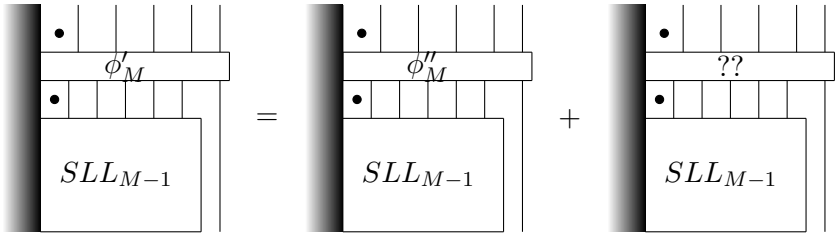
The map on the right of (10.15) represents some map in $I_{\underline{z}_{M-1}}$, followed by $\phi_M$. In particular, the up-dot may be at any position above the box labelled '??'.

Now, we want to show that this sum is a sum of maps in $\mathcal{S}_{\underline{x},\underline{z}}$ with polynomials in the top-left, plus maps in $I_{\underline{z}}$. Let us first look at one of the maps in the first sum in (10.15). By the way we have constructed $SLL_{\underline{x}_{\leq M-1},\underline{e}'}$, this will be of the form




where we have composed $\beta$ and $\phi_M$ to make $\phi'_M$, which is another valid construction of this step in the light-leaf construction. Again, $SLL_{M-1}$ is the $(M-1)$'th step in the construction of $SLL_{\underline{x},\underline{f}} \in \mathcal{S}_{\underline{x},\underline{z}}$. So the map on the right is a valid construction of $SLL_{\underline{x},\underline{f}}$, but may not be precisely the map $SLL_{\underline{x},\underline{f}} \in \mathcal{S}_{\underline{x},\underline{z}}$. In particular, the final step $\phi'_M$ may contain different rex moves. Call this final map $\phi''_M$ in $SLL_{\underline{x},\underline{f}}$. Using Lemma 10.1.1, we

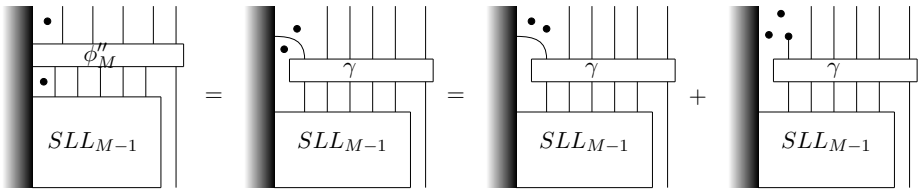
have




(10.16)

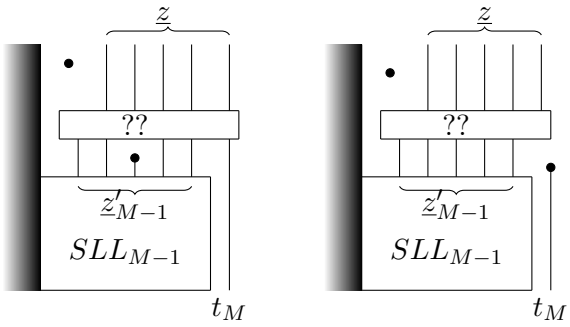
where the map labelled '??' has a strictly negative-positive decomposition. In particular, '??' has a down-dot at the top. If $d'_M \neq X1$, then both the maps on the right of (10.16) are of the form we want: the map $\phi''_M$ does not plug into the wall, so we can move the polynomial up past $\phi''_M$ to the top-left corner, leaving us with exactly $SLL_{\underline{x},\underline{f}}$ with polynomials in the top-left. And if $d'_M \neq X1$, then the down-dot on top of '??' means that map is in $I_{\underline{z}}$.

So the only difficult case is where $d'_M = X1$. In this case we have, using polynomial forcing




(10.17)

where $\gamma$ is some rex move. The first map on the right of (10.17) is again exactly $SLL_{\underline{x},\underline{f}}$ with polynomials in the top left. As for the map on the far right of (10.17), the up-dot pulls through $\gamma$ using Proposition 4.2.1, leaving us with maps of one of the two following forms:



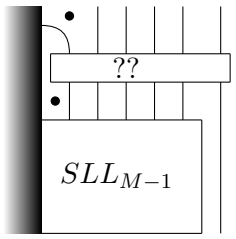

(10.18)

Now we use an argument that we will use again later, so we refer to it as the argument $(*)$. Let $\underline{z}'_{M-1} = (s_1, ..., s_k)$. In the first case in (10.18), the map '??' is a map

$(s_1, ..., \hat{s_i}, ..., s_k, t_M) \to \underline{z}$. By Proposition 10.1.7, this map '??' is spanned by light-leaves and maps in $I_{\underline{z}}$. Light-leaves are negative or neutral maps, and since the input and output of '??' have the same length, the light-leaves must be neutral. So if there is a light-leaf $LL : (s_1, ..., \hat{s_i}, ..., s_k, t_M) \to \underline{z}$, then it is a rex move and thus $(s_1, ..., \hat{s_i}, ..., s_k, t_M)$ is reduced, and $s_1 \cdots \hat{s_i} \cdots s_k t_M = z$. But $d'_M = X1$, so that by Lemma 6.1.3 (a), $zt_M > z$, whereas $s_1 \cdots \hat{s_i} \cdots s_k t_M t_M = s_1 \cdots \hat{s_i} \cdots s_k < s_1 \cdots \hat{s_i} \cdots s_k t_M$. So we have a contradiction, and thus there can be no light-leaf map $LL : (s_1, ..., \hat{s_i}, ..., s_k, t_M) \to \underline{z}$, so that the first map in (10.18) lies entirely in $I_{\underline{z}}$. (There ends argument $(*)$).

Then we consider the second map in (10.18). Once again, the box labelled '??' is a sum of light-leaves and maps in $I_{\underline{z}}$. Since $\underline{z}'_{M-1}$ and $\underline{z}$ are of the same length, a light-leaf map $LL : \underline{z}'_{M-1} \to \underline{z}$ must be a neutral map, i.e. a rex move. But placing a rex move in the box labelled '??' would leave us with an SLL map with $X0$ at the end; more precisely, it would leave us with a possible construction of $SLL_{\underline{x},\underline{f}'}$, where $\underline{f}' := (\underline{e}', 0) \subset \underline{x}$. Unfortunately, it may not leave us with *the* map $SLL_{\underline{x},\underline{f}'} \in \mathcal{S}_{\underline{x},\underline{z}}$. This is because $SLL_{M-1}$ was chosen to be part of $SLL_{\underline{x},\underline{f}} \in \mathcal{S}_{\underline{x},\underline{z}}$, not $SLL_{\underline{x},\underline{f}'}$. Thus, we need to reapply our inductive hypothesis to this $SLL_{M-1}$; that is, we replace $SLL_{M-1}$ with a sum of light-leaves and maps in $I_{\underline{z}'_{M-1}}$ just as in equation (10.15), and then proceed through the steps we have just carried out (or are about to carry out). Note that there is no danger here of an infinite loop: we only reached the situation in (10.18) because we had $d'_k = X1$. If we repeat the procedure with $d'_k = X0$, we will not run into the same problem.

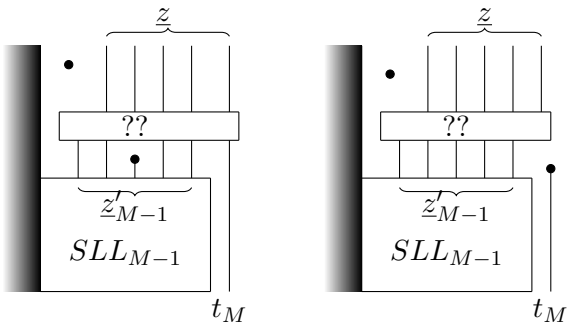
We have yet to address the right-hand map in (10.16), in the case that $d'_M = X1$:




(10.19)

Here, again, the box labelled '??' has a strictly negative-positive decomposition, so that it has a dot at the top and at the bottom. If the dot at the top is not connected to the wall strand, then this map is in $I_{\underline{z}}$. Otherwise, the dot gets pulled into the wall and we

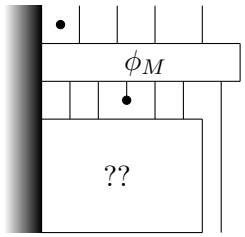
have a map that looks like one of the following two:




(10.20)

This is exactly the situation in (10.18), which we have already dealt with.

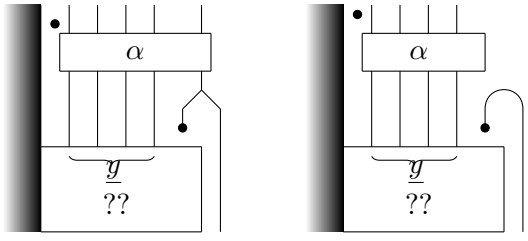
We have now dealt with maps in the first sum in (10.15). We now move on to maps in the second sum, of the form




(10.21)

where again, the up-dot could be positioned anywhere above '??'. First, if $d'_M$ is either $U0, U1$ or $X0$, then the up-dot will pull through any rex moves to the top, landing in $I_{\underline{z}}$.

If $d'_M = D0$ or $D1$, we can assume there are no 'necessary' rex moves (labelled $\beta$ in Section 5.1) as part of $\phi_M$, since if there were, we could pull the up-dots through $\beta$ and then incorporate what remained into '??'. Therefore, if the up-dot is not on the very right of '??', then it will again pull upward through any rex moves to land the map in $I_{\underline{z}}$. If the up-dot is on the very right of '??', then we are in one of the following situations:



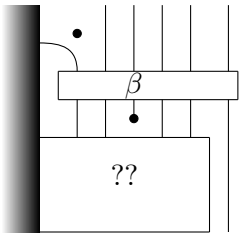

$d'_M = D0$ $\qquad$ $d'_M = D1$

(10.22)

where $\alpha$ is some rex move, and $\underline{y}$ is a rex of some $y \in {}^JW$ (because $d_k = D0$ or $D1$). Therefore we can reapply our inductive hypothesis to the box labelled '??', replacing it

with SLL's of our choice (with polynomials in the top-left), and maps in $I_{\underline{y}}$. These SLL maps will become (up to adjusting the final rex move $\alpha$, as in (10.16)) the SLL maps in $\mathcal{S}_{\underline{x},\underline{z}}$ corresponding to $d'_M = U1$ and $U0$ respectively. The dots at the top of the maps in $I_{\underline{y}}$ will pull through $\alpha$ to land in $I_{\underline{z}}$.

So we are left with the case $d'_M = X1$.




(10.23)

The up-dot will pull through $\beta$, either landing in $I_{\underline{z}}$ or pulling into the wall. If it pulls into the wall, then by the argument $(*)$, what remains will be in $I_{\underline{z}}$. □

Thus, we have shown that any map $\phi : \underline{x} \to \underline{z}$ is a sum of any particular choice of SLL's, with polynomials in the top-left. The only thing left to do is to move these polynomials to the right, using polynomial forcing. This leaves a right $R$-linear combination of our choice of SLL's, plus maps in $I_{\underline{z}}$. This proves Proposition 10.1.6, and thus also proposition 10.0.1. Therefore, we have proved that the double-leaves span, and also that they are linearly independent (Corollary 9.2.4). So we have proved Theorem 7.4.1, that the double-leaves form a basis. □

# Chapter 11

# Equivalence of Diagrammatic and Algebraic Categories

Now that we have constructed the double-leaves basis of $\mathcal{M}_{BS}$, we will use this to prove the following.

**Theorem 11.0.1.** *The functor $\tilde{\mathcal{A}} : \mathcal{M}_{BS} \to J\mathbb{BS}Bim$ defined in Section 7.1 is an equivalence of categories.*

The key will be that we have formulae for the graded ranks of the Hom spaces in each category, and that these are the same. Recall the Soergel Hom formula on Bott-Samelson bimodules:

$$\overline{\mathrm{rkHom}_{\mathbb{BS}\mathrm{Bim}}(BS(\underline{w}), BS(\underline{y}))} = \langle ch(BS(\underline{w})), ch(BS(\underline{y}))\rangle = \langle b_{\underline{w}}, b_{\underline{y}}\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the $\mathbb{Z}[v, v^{-1}]$-bilinear form on $H$ defined in (2.3). The two components of this formula — the character map $ch$ and the bilinear form $\langle \cdot, \cdot \rangle$ on $H$ — both generalize to all singular Soergel bimodules (see Section 6.2 for a description of singular Soergel bimodules). For all finitary subsets $I, J \subset S$, we have a character map $ch : {}^I\mathcal{B}^J \to {}^I\mathcal{S}^J$. In particular, we get a character map $ch : {}^J\mathcal{B} \to M(J)$, which we can restrict to $ch : J\mathbb{BS}\mathrm{Bim} \to M(J)$. From Theorem 6.2.2 (c), we know that the following diagram commutes.

$$\begin{array}{ccc} {}^J\mathcal{B} \times \mathcal{B} & \xrightarrow{- \otimes -} & {}^J\mathcal{B} \\ \downarrow{\scriptstyle ch \times ch} & & \downarrow{\scriptstyle ch} \\ M(J) \times H & \xrightarrow{- \cdot -} & M(J) \end{array}$$

By following where ${}_{R^J}R \otimes BS(\underline{w}) \in {}^J\mathcal{B} \times \mathcal{B}$ is taken in this diagram, we can determine that

$$ch({}_{R^J}BS(\underline{w})) = m_{id} b_{\underline{w}} = 1 \otimes b_{\underline{w}} \in M(J).$$

We also have a $\mathbb{Z}[v, v^{-1}]$-bilinear form $\langle \cdot, \cdot \rangle_{I,J}$ on ${}^I\mathcal{S}^J$ for all finitary subsets $I, J \subset S$. We will not define this form on general ${}^I\mathcal{S}^J$, since this would require defining the standard basis of each ${}^I\mathcal{S}^J$. We will simply define it on $M(J) = {}^J\mathcal{S}$. We define a $\mathbb{Z}[v, v^{-1}]$-bilinear form $\langle \cdot, \cdot \rangle_M$ on $M(J)$ by identifying $M(J) \cong b_{w_J} H$ as in Proposition 6.1.4 and setting[1]

$$\begin{aligned} \langle \cdot, \cdot \rangle_M &: b_{w_J} H \times b_{w_J} H \to \mathbb{Z}[v, v^{-1}], \\ \langle h_1, h_2 \rangle_M &:= v^{-d_J} \epsilon(i(h_1) *_J h_2) = \frac{v^{-d_J}}{\pi(J)} \langle h_1, h_2 \rangle. \end{aligned} \tag{11.1}$$

Finally, we have a formula for the graded rank of

$$\mathrm{Hom}_{J\mathbb{BS}\mathrm{Bim}}({}_{R^J}BS(\underline{w}), {}_{R^J}BS(\underline{y}))$$

as a right $R$-module.

**Proposition 11.0.2.** *We have*

$$\begin{aligned} \overline{rkHom_{J\mathbb{BS}Bim}({}_{R^J}BS(\underline{w}),\ {}_{R^J}BS(\underline{y}))} &= \langle ch(BS(\underline{w})), ch(BS(\underline{y})) \rangle_M \\ &= \langle 1 \otimes b_{\underline{w}}, 1 \otimes b_{\underline{y}} \rangle_M. \end{aligned}$$

*Proof.* This is a specific case of Theorem 7.9 of [Wil11][2]. □

[1] We have added in a factor of $v^{-d_J}$ to the formula given in Section 2.3 of [Wil11]. This is in order to make the standard basis of $M$ orthonormal with respect to this form. We have altered the Hom formula appropriately to adjust for this.

[2] Again, we have removed the grading shift that shows up in [Wil11] to match the extra factor of $v^{-d_J}$ in (11.1)

We now want to show that the graded ranks of the Hom spaces in $J\mathbb{BS}\mathrm{Bim}$ match those of $\mathcal{M}_{BS}$.

**Proposition 11.0.3.** *Given any two expressions $\underline{x}, \underline{y}$, we have*

$$rkHom_{\mathcal{M}_{BS}}(\underline{x}, \underline{y}) = rkHom_{J\mathbb{BS}Bim}({}_{R^J}BS(\underline{x}), {}_{R^J}BS(\underline{y})).$$

We saw in Remark 5.2.2 that this was the case for $\mathbb{BS}\mathrm{Bim}$ and $\mathcal{H}_{BS}$, and the proof here will be highly analogous to that. In particular, we had a version of the Soergel Hom formula (5.2) which was expressed in terms of the defects of subexpressions. This defect was defined in such a way that the degree of a light-leaf $LL_{\underline{w},\underline{e}}$ was exactly the defect $def(\underline{e})$. We will want a notion of defect which extends to our spherical light-leaves. We will now define a new notion of defect, the *spherical defect* $Def(\underline{e})$ of a subexpression $\underline{e} \subset \underline{x}$, in such a way that $\deg(SLL_{\underline{x},\underline{e}}) = Def(\underline{e})$.

Recall from Section 7.2 that given an expression $\underline{x} = (s_1, ..., s_n)$, a subexpression $\underline{e} \subset \underline{x}$ has a coset stroll $z_0, ..., z_n$, where $z_k$ is the minimal coset representative of $x_k := s_1^{e_1} \cdots s_k^{e_k}$. This coset stroll has an associated decoration $d'_1, ..., d'_n$, with $d'_k \in \{U1, U0, D1, D0, X1, X0\}$. These $d'_k$'s then determined the form of the spherical light-leaf $SLL_{\underline{x},\underline{e}}$, by specifying the maps $\phi_k$. If we want to know how to define $Def(\underline{e})$, we should look at the degrees of the $\phi_k$'s corresponding to each of the possible labels $U1, U0, D1, D0, X1, X0$. The degree of $\phi_k$ associated to $U1$ and $D1$ is zero, while $U0$ is +1 and $D0$ is -1. Then, $X0$ gives a degree +1 map, while $X1$ gives a degree -1 map. Therefore we should define the spherical defect as

$$Def(\underline{e}) := \#U0 + \#X0 - \#D0 - \#X1. \tag{11.2}$$

By the way we have defined it, we have $\deg(SLL_{\underline{x},\underline{e}}) = Def(\underline{e})$. By the fact that double-leaves form a basis, we have the following Hom formula for $\mathcal{M}_{BS}$.

$$\overline{\mathrm{rkHom}_{\mathcal{M}_{\mathcal{BS}}}(\underline{x}, \underline{y})} = \sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{f} \subset \underline{y} \\ W_J \underline{x}^{\underline{e}} = W_J \underline{y}^{\underline{f}}}} v^{Def(\underline{e}) + Def(\underline{f})} \tag{11.3}$$

We now wish to show that we can rewrite the Hom formula for $J\mathbb{BS}\mathrm{Bim}$ from Proposition 11.0.2 in terms of the spherical defect, so that it matches the above formula (11.3).

Recall that we have a standard basis $\{m_x \mid x \in {}^JW\}$ of the spherical module $M$, where $m_x = 1 \otimes \delta_x$.

**Lemma 11.0.4.** *The standard basis $\{m_x \mid x \in {}^JW\}$ of $M(J)$ is orthonormal with respect to the form $\langle \cdot, \cdot \rangle_M$.*

*Proof.* Under the identification $M(J) \cong b_{w_J}H$, the standard basis element $m_x \in M(J)$ goes to $b_{w_J}\delta_x \in b_{w_J}H$. Therefore by (11.1) we have

$$\langle m_x, m_y \rangle_M = \frac{v^{-d_J}}{\pi(J)} \langle b_{w_J}\delta_x, b_{w_J}\delta_y \rangle.$$

It is a straight-forward consequence of the definition (2.3) of $\langle \cdot, \cdot \rangle$ that

$$\begin{aligned} \langle b_{w_J}\delta_x, b_{w_J}\delta_y \rangle &= \langle i(b_{w_J}) b_{w_J}\delta_x, \delta_y \rangle \\ &= \langle b_{w_J}^2 \delta_x, \delta_y \rangle \\ &= \pi(J) \langle b_{w_J}\delta_x, \delta_y \rangle, \end{aligned}$$

where $i(b_{w_J}) = b_{w_J}$ by Remark 2.6.3. This gives us

$$\langle m_x, m_y \rangle_M = v^{-d_J} \langle b_{w_J}\delta_x, \delta_y \rangle.$$

We have from (2.5) that

$$b_{w_J} = \sum_{w \in W_J} v^{l(w_J)-l(w)} \delta_w,$$

and so

$$b_{w_J}\delta_x = \sum_{w \in W_J} v^{l(w_J)-l(w)} \delta_{wx},$$

since $x$ is a minimal coset representative. Therefore if $W_J x \neq W_J y$, then $wx \neq y$ for any $w \in W_J$, so that

$$\langle m_x, m_y \rangle_M = v^{-d_J} \langle b_{w_J}\delta_x, \delta_y \rangle = 0,$$

by the fact that the standard basis of $H$ is orthonormal with respect to $\langle \cdot, \cdot \rangle$.

Conversely if $W_J x = W_J y$, then $x = y$ since they are both minimal coset representatives. Therefore

$$\langle b_{w_J}\delta_x, \delta_y \rangle = \langle \sum_{w \in W_J} v^{l(w_J)-l(w)} \delta_{wy}, \delta_y \rangle = v^{l(w_J)} = v^{d_J}.$$

Therefore

$$\langle m_x, m_y \rangle_M = v^{-d_J} \cdot v^{d_J} = 1,$$

and so the $m_x$'s are orthonormal. □

Next we prove a lemma analogous to Deodhar's lemma (Lemma 3.8.3). We will use the notation $m_{\underline{x}^{\underline{e}}} := m_z$, where $z$ is the minimal coset representative of $W_J \underline{x}^{\underline{e}}$.

**Lemma 11.0.5.** *For any expression $\underline{x}$ we have*

$$1 \otimes b_{\underline{x}} = \sum_{\underline{e} \subset \underline{x}} v^{Def(\underline{e})} m_{\underline{x}^{\underline{e}}}. \tag{11.4}$$

In other words, the coefficient of $m_z$ in $1 \otimes b_{\underline{x}}$ is

$$\sum_{\substack{\underline{e} \subset \underline{x} \\ W_J \underline{x}^{\underline{e}} = W_J z}} v^{Def(\underline{e})}.$$

*Proof.* We proceed by induction on the length of $\underline{x}$. Suppose the statement is true when $l(\underline{x}) = n - 1$, and take $\underline{x} = (s_1, ..., s_n)$. Set $\underline{y} := (s_1, ..., s_{n-1})$. Then we know by the inductive hypothesis that

$$1 \otimes b_{\underline{y}} = \sum_{\underline{f} \subset \underline{y}} v^{Def(\underline{f})} m_{\underline{y}^{\underline{f}}},$$

so that

$$1 \otimes b_{\underline{x}} = 1 \otimes b_{\underline{y}} b_{s_n} = \sum_{\underline{f} \subset \underline{y}} v^{Def(\underline{f})} m_{\underline{y}^{\underline{f}}} b_{s_n}. \tag{11.5}$$

We can find the product $m_{\underline{y}^{\underline{f}}} b_{s_n}$ using the formula (6.1), which we reproduce below.

$$m_w b_s = \begin{cases} m_{ws} + v^{-1} m_w & \text{if } ws < w,\ ws \in {}^J W \\ m_{ws} + v m_w & \text{if } ws > w,\ ws \in {}^J W \\ (v + v^{-1}) m_w & \text{if } ws \notin {}^J W. \end{cases} \tag{11.6}$$

Thus each term $m_{\underline{y}^{\underline{f}}} b_{s_n}$ in (11.5) will yield two terms. These two terms will be $m_{\underline{x}^{\underline{e}}}$ for $\underline{e} = (\underline{f}, 0)$ and $(\underline{f}, 1)$, with some powers of $v$ in front. We just need to show that these powers of $v$ are exactly $Def(\underline{e})$.

Note that the three cases in (11.6) correspond exactly to the cases where $s$ is labelled by $D, U$ or $X$ in the labelling of the coset stroll (see Section 7.2). More precisely, if we replace $w$ with $\underline{y^f}$ and $s$ with $s_n$ in the above formula, then the three cases correspond to when the labellings of the coset strolls associated to $(\underline{f}, 0)$ and $(\underline{f}, 1)$ finish with $D$, $U$ or $X$ respectively. When they finish with $D$, then using the definition of $Def$ in (11.2) we will have

$$Def(\underline{f}, 0) = Def(\underline{f}) - 1,\ Def(\underline{f}, 1) = Def(\underline{f}),$$

which exactly corresponds to the powers of $v$ in the first case of (11.6). If the expressions finish with $U$, then

$$Def(\underline{f}, 0) = Def(\underline{f}) + 1,\ Def(\underline{f}, 1) = Def(\underline{f}),$$

which corresponds to the powers of $v$ in the second case of (11.6). Finally, if the labelling of the coset strolls of $(\underline{f}, 0),\ (\underline{f}, 1) \subset \underline{x}$ finish with $X$, then we will have

$$Def(\underline{f}, 0) = Def(\underline{f}) + 1,\ Def(\underline{f}, 1) = Def(\underline{f}) - 1,$$

which corresponds to the final case in (11.6).

Therefore the coefficient of $m_{\underline{x^e}}$ in each case will be $v^{Def(\underline{e})}$. □

*Proof of Proposition 11.0.3.* Proposition 11.0.2 tells us that

$$\overline{\mathrm{rkHom}_{J\mathbb{BS}\mathrm{Bim}}({}_{R^J}BS(\underline{w}), {}_{R^J}BS(\underline{y}))} = \langle 1 \otimes b_{\underline{w}}, 1 \otimes b_{\underline{y}} \rangle_M.$$

Lemmas 11.0.4 and 11.0.5 tell us that

$$\begin{aligned} \langle 1 \otimes b_{\underline{x}}, 1 \otimes b_{\underline{y}} \rangle_M &= \langle \sum_{\underline{e} \subset \underline{x}} v^{Def(\underline{e})} m_{\underline{x^e}}, \sum_{\underline{f} \subset \underline{y}} v^{Def(\underline{f})} m_{\underline{y^f}} \rangle_M \\ &= \sum_{\substack{\underline{e} \subset \underline{x} \\ \underline{f} \subset \underline{y} \\ W_J \underline{x^e} = W_J \underline{y^f}}} v^{Def(\underline{e}) + Def(\underline{f})}, \end{aligned}$$

which is exactly $\overline{\mathrm{rkHom}_{\mathcal{MBS}}(\underline{x}, \underline{y})}$ given in (11.3). □

We now know that $\mathcal{M}_{BS}$ and $J\mathbb{BS}\text{Bim}$ have matching Hom space ranks. We will use this fact to show that $\tilde{\mathcal{A}} : \mathcal{M}_{BS} \to J\mathbb{BS}\text{Bim}$ is an equivalence of categories.

*Proof of Theorem 11.0.1.* It is clear that $\tilde{\mathcal{A}}$ is essentially surjective, so we only need to show it is fully faithful.

We will first show that it is faithful. We take the diagram (8.33) which we used to define the functor $\tilde{\mathcal{G}}$ in Section 8.

$$\begin{array}{ccccc} \mathcal{M}_{BS,Q} & \longrightarrow & Kar(\mathcal{M}_{BS,Q}) & & \\ & \searrow^{\tilde{\mathcal{A}} \otimes Q} & & \searrow^{\tilde{\mathcal{G}}} & \\ & & J\mathbb{BS}\text{Bim}_Q & \hookrightarrow & JStd\text{Bim}_Q \end{array}$$

We can extend this diagram on the left.

$$\begin{array}{ccccccc} \mathcal{M}_{BS} & \xrightarrow{(-)\otimes Q} & \mathcal{M}_{BS,Q} & \longrightarrow & Kar(\mathcal{M}_{BS,Q}) & & \\ & \searrow^{\tilde{\mathcal{A}}} & & \searrow^{\tilde{\mathcal{A}} \otimes Q} & & \searrow^{\tilde{\mathcal{G}}} & \\ & & J\mathbb{BS}\text{Bim} & \xrightarrow[(-)\otimes Q]{} & J\mathbb{BS}\text{Bim}_Q & \hookrightarrow & JStd\text{Bim}_Q \end{array} \tag{11.7}$$

When we proved linear independence of the spherical double-leaves in Section 9, we in fact proved that they are linearly independent after mapping from $\mathcal{M}_{BS}$ to $Kar(\mathcal{M}_{BS,Q})$ across the top of (11.7). This means that the map $\mathcal{M}_{BS} \to Kar(\mathcal{M}_{BS,Q})$ across the top of (11.7) is faithful. Furthermore, since by Corollary 8.4.6 the functor $\tilde{\mathcal{G}}$ is an equivalence, we find that the map $\mathcal{M}_{BS} \to JStd\text{Bim}_Q$ in (11.7) is faithful. Therefore the map $\tilde{\mathcal{A}}$ must also be faithful.

Finally, since by Proposition 11.0.3 the graded ranks of the Hom spaces of $\mathcal{M}_{BS}$ and $J\mathbb{BS}\text{Bim}$ are the same, the fact that $\tilde{\mathcal{A}}$ is faithful implies that it is full. □

**Corollary 11.0.6.** *The diagrammatic spherical category $\mathcal{M} := Kar(\mathcal{M}_{BS})$ is equivalent to the algebraic spherical category $Kar(J\mathbb{BS}Bim)$.*

So far we have only shown that the diagrammatic and algebraic spherical categories, $\mathcal{M}$ and $Kar(J\mathbb{BS}\text{Bim})$, are equivalent as categories, with no additional structure. However, as mentioned in Remark 7.1.3, they are also equivalent as module categories.

**Corollary 11.0.7.** *The diagrammatic and algebraic spherical categories, $\mathcal{M}$ and $Kar(J\mathbb{BS}Bim)$, are equivalent as module categories over $\mathcal{H}$ and $\mathbb{S}Bim$ respectively. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{H} & \xrightarrow{-\otimes-} & \mathcal{M} \\ \downarrow & \circlearrowleft & \downarrow \\ Kar(J\mathbb{BS}Bim) \times \mathbb{S}Bim & \xrightarrow{-\otimes-} & Kar(J\mathbb{BS}Bim) \end{array}$$

*Proof.* This is a direct consequence of the fact that the corresponding Bott-Samelson categories are equivalent as module categories. That is, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}_{BS} \times \mathcal{H}_{BS} & \xrightarrow{-\otimes-} & \mathcal{M}_{BS} \\ \downarrow & \circlearrowleft & \downarrow \\ J\mathbb{BS}\text{Bim} \times \mathbb{BS}\text{Bim} & \xrightarrow{-\otimes-} & J\mathbb{BS}\text{Bim} \end{array}$$

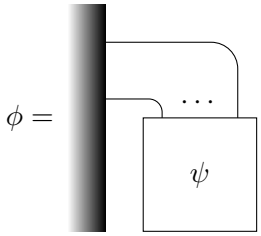
This in turn is a consequence of how the functor $\tilde{\mathcal{A}}$ was defined in Section 7.1. In particular, $\tilde{\mathcal{A}}$ clearly respects the right action of $\mathcal{H}_{BS}$ on $\mathcal{M}_{BS}$, both on the level of objects and morphisms. Taking Karoubi envelopes of this commuting diagram proves the result. □

# Appendix A

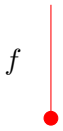
# The 'infinite loop' problem

Here we deal with a finicky problem that arose in proving Lemma 10.1.4, involving the possibility of an infinite loop. Let us restate that lemma here: Take an expression $\underline{x}$ in $W_J$ and a map $\phi : \underline{x} \to \underline{\varnothing}$ in $\mathcal{M}_{BS}$. Then $\phi$ lies in the right $R$-span of light-leaves $SLL_{\underline{x},\underline{e}} : \underline{x} \to \underline{\varnothing}$.

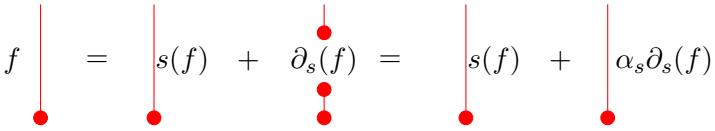
Let us now recall was the 'infinite loop' issue was. In the proof of this lemma, we start with an arbitrary map $\phi : \underline{x} \to \underline{\varnothing}$ and move all the wall plug-ins to the top.



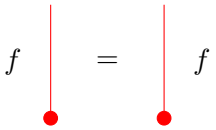

The first step was to move all polynomials to the right using polynomial forcing. The problem is that the polynomial forcing relation can create more barbells. These barbells will then need to be moved to the right using polynomial forcing, which could again create more barbells, and so on. For example, consider the following diagram (here we just draw the '$\psi$', and ignore the wall).



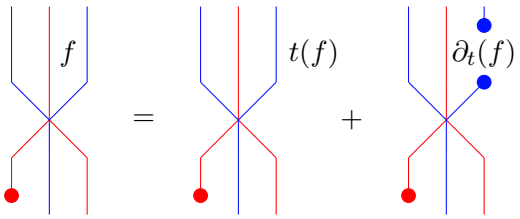

If we move $f$ to the right using polynomial forcing, we end up with the following.




In this instance it is clear that we do not have a problem. All our new polynomials ($s(f)$ and $\alpha_s\partial_s(f)$) are closer to the right of the diagram. Alternatively, we can just move $f$ underneath the down-dot[1].

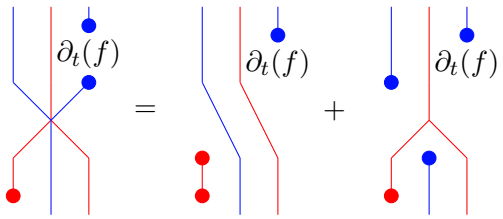



This is not always possible, however. In the following example, it is not possible to move $f$ under the blue strand; we must go through it using polynomial forcing.



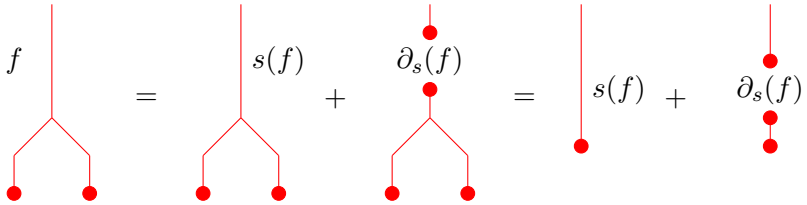

Let us expand out the second diagram using two-colour dot contraction (4.16).

[1]One can check, using the formula for the Demazure operator, that this gives the same result.




We have created a red barbell which is in fact further from the right of the diagram than the $f$ was. However, we have reduced the number of $2m$-valent vertices by 1. Therefore, since the number of $2m$-valent vertices in a given diagram is finite, this can only happen finitely many times.

In summary, moving polynomials to the right creates new barbells in two situations. One is the situation from the first example: we break a strand using polynomial forcing, and the bottom component of this broken strand is isolated:




As we saw, in this case there is no problem: all polynomials are closer to the right of the diagram (or alternatively, we just move $f$ underneath and to the right). The second case is via two-colour dot contraction. In this case we reduce the number of $2m$-valent vertices by 1, and this can only happen finitely many times.

Therefore, this process of moving polynomials to the right must eventually end.

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