BEILINSON-BERNSTEIN LOCALIZATION ON THE BASE AFFINE SPACE AND JANTZEN FILTRATIONS FOR FOR $SL_2(\mathbb{C})$

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Abstract

The study of D-modules - modules over a ring of differential operators - finds interest in various places. The development of algebraic D-modules in particular have seen much use in the representation theory of Lie algebras, after Beilinson and Bernstein showed in 1981 that there is an equivalence of categories between certain representations of a semisimple Lie algebra and twisted D-modules on the flag variety. This gave rise to the study of geometric representation theory. If we instead consider certain types of D-modules on the base affine space, which is a principal bundle over the flag variety, this Beilinson-Bernstein equivalence is obtained in a more general way. In this thesis, we explore this generalization, with a particular focus on examples, especially the example of $\mathfrak{sl}_2(\mathbb{C})$, where we shall show how various representations of $\mathfrak{sl}_2(\mathbb{C})$ can be obtained from certain D-modules which are relatively easy to construct.

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Chapter 1

Introduction

Analogously to how the study of group representations gives information about the structure of a group, the study of Lie algebra representations can tell us more about the structure of a given Lie algebra. Moreover, when that Lie algebra $\mathfrak{g}$ corresponds to a particular algebraic (or Lie) group G, the representation theory of G and that of $\mathfrak{g}$ are closely related, and so we can learn more about the structure of G by studying the representation theory of $\mathfrak{g}$. While the study of algebraic group representations is simplified via the study of the corresponding Lie algebra representations, attempts to fully classify all the representations of a Lie algebra are hopeless, due to the potential difficulty involved with infinite dimensional representations. Instead, we might turn our attention to certain nice representations of $\mathfrak{g}$, which, while still possibly infinite dimensional, have certain finiteness properties which simplify the study. In this thesis, we look to describing some such representations through the geometric approach of D-modules, using Beilinson and Bernstein's equivalence of categories. To discuss such representations, we fix some notation: let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$, let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. We say that a representation of $\mathfrak{g}$ has infinitesimal character $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ if any $z \in \mathcal{Z}(\mathfrak{g})$ acts by $\chi(z)$ - in other words, $z - \chi(z)$ acts by trivially, and we look to study such representations.

On the geometric side of the story are algebraic D-modules (modules over an algebra of differential operators), which have the immediate realization as tools to study solutions to linear partial differential equations. The classic example of an algebra of differential operators is the Weyl algebra, which is the subalgebra of the endomorphisms of a polynomial ring $k[x_1, \ldots, x_n]$ consisting of the differential endomorphisms; we can view this algebra as the noncommutative ring $k[x_1 \ldots, x_n, \frac{d}{dx_1}, \ldots, \frac{d}{dx_n}]$. D-modules on a smooth affine variety are then modules over the Weyl algebra. This notion can be globalized to general algebraic varieties, by considering sheaves of algebras of differential operators. Then a D-module is a quasi-coherent module over a sheaf of algebras of differential operators.

In 1981, it was shown by Beilinson and Bernstein in [BB81] that the category of Lie algebra representations with infinitesimal character χ is equivalent to a certain category of twisted D-modules on the flag variety $X := G/B$, where B is a Borel subgroup of G. In particular, the functor defining this equivalence from this category of twisted D-modules to the category of Lie algebra representations is the global sections functor. In this way, questions about the representations of a Lie algebra can be reframed in the language of algebraic geometry; by taking global sections of certain twisted D-modules, we obtain representations of a given Lie algebra. Hidden in the background of this picture is the base affine space $\widetilde{X} := G/N$, where N is a maximal unipotent subgroup of G. The base affine space has two particularly nice properties which reveal their use in this geometric approach to the study of Lie algebra representations. First, if H denotes a maximal torus of G, then the base-affine space is an H-torsor over the flag variety. Secondly it is a quasi-affine variety, so Theorem 6.1.1 tells us that the quasi-coherent sheaves on this H-torsor are generated by their global sections.

We can pull back twisted D-modules on X along the map $\pi : \widetilde{X} \to X$ to obtain D-modules on $\widetilde{X}$. In this way, the study of twisted D-modules on the flag variety is contained in the study of certain types of D-modules on the base affine space, called monodromic D-modules. The view on the base affine space actually provides more though, as the global sections of D-modules on the base affine space see Lie algebra representations of a *generalized* infinitesimal character χ, i.e. where for any $z \in \mathcal{Z}(\mathfrak{g})$ the action of $z - \chi(z)$ is nilpotent (as opposed to specifically zero as in the case of a (strict) infinitesimal character).

Those D-modules on $\widetilde{X}$ which are invariant under a certain action of the torus H will be the ones of interest here, and we shall see more precisely how they correspond with Lie algebra representations of generalized infinitesimal character. The utility of D-modules can be seen for instance in the direct image and inverse image functors, which allow us to push and pull D-modules along morphisms of varieties. In this way, using the Beilinson-Bernstein equivalence of categories, we can more easily obtain examples of Lie algebra representations. For instance, stratifying the base affine space by B-orbits, and pushing forward the structure sheaf will give the Verma and dual Verma modules, while doing the same for the inverse image of K-orbits on G/B (where K is a maximally compact subgroup of G) will give all the principle series and discrete series representations of $\mathfrak{g}$.

Given that Verma modules are building blocks of many other nice representations, various questions arise about their structure. In particular, one is interested in which simple modules appear withing the composition series of a particular Verma module and with what multiplicity. To study this question, it is sensible to look at filtrations of Verma modules, and there is one particular filtration, the Jantzen filtration which has a close connection with this problem. Typically, the Jantzen filtration for Verma modules is defined using deformed Verma modules, which have generalized infinitesimal character, so with the geometric interpretation of Lie algebra representations of generalized infinitesimal character at our disposal, we can then give a geometric description of the Jantzen filtration. This demonstrates why the base affine space is the desirable setting for defining the geometric version of the Jantzen filtration.

One of the primary aims of this thesis is to provide the reader with a more concrete understanding of this Beilinson-Bernstein localization on the base affine space. There will be a heavy focus on examples, in particular, we try to detail precisely what occurs in the example of $\mathfrak{sl}_2(\mathbb{C})$.

In Chapter 2, we give a very brief introduction to the basics of algebraic groups and Lie algebras, first running through some facts of algebraic groups, then of Lie algebras, in particular we work up to the definition of a semisimple Lie algebra, and some of its important subalgebras. The study of semisimple Lie algebras leads us inevitably to root systems, which provide a classification of semisimple Lie algebras. Of particular interest is the root space decomposition of a semisimple Lie algebra, which is very useful in the study of Lie algebra representations. Next, we briefly mention the correspondence between Lie algebras and algebraic groups, discussing how a Lie algebra can be obtained from the left invariant derivations on an algebraic group in a functorial way, and also we mention the adjoint of this functor, where one obtains an algebraic group from a given Lie algebra. Next, we take a closer look at the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, which is simple enough that we can give a nice concrete description of its structure. In particular, we explicitly detail how it can be obtained from the left invariant derivations of the algebraic group $\mathrm{SL}_2(\mathbb{C})$ via purely algebraic means. We finish the chapter with a definition and description of the flag variety and the base affine space, with a particular focus on the case of $G = \mathrm{SL}_n(\mathbb{C})$. We also prove that the flag variety is projective and the base affine space is quasi-affine in this case. Much of the material in this chapter is standard (except for the last section) so ideally, the reader would be familiar with the topics to some extent. In the interest of brevity, we do not always give full proofs, but refer the reader to a source with the full proof, and suggest here that the reader may wish to refer to [Hum72] and [Hum75] for a more detailed account of this background.

We then discuss representations of Lie algebras in more detail in Chapter3. In particular, we look at highest weight modules and the category $\mathcal{O}$, which give us "nice" infinite dimensional representations which we can study. In particular, we introduce Verma modules, which are the building blocks of category $\mathcal{O}$.

In Chapter 4, we give an introduction to the theory of algebraic D-modules, first looking at modules over the Weyl algebra, before globalizing the concept where we consider sheaves of rings (or algebras) of (twisted) differential operators and quasi-coherent sheaves of modules over those sheaves of differential operators. This chapter will be more detailed than both the preceding and succeeding chapters, as it will likely be a different point of view for many readers, while still accessible enough that we can go into sufficient detail. The subtle nature of sheaves at times makes this section rather involved; however, since we will always be working over smooth varieties some of the difficulties alleviated by using the fact that we can always localize. The main example the reader should keep in mind during this chapter is that of the projective line $\mathbb{P}^1$, and we do discuss briefly the twisted D-modules on $\mathbb{P}^1$ concretely. We move to a discussion of the D-module inverse and direct images, which require some care to define. Finally we end the chapter with a brief discussion on monodromic D-modules.

Chapter 5 gives some of the historical context for this thesis, starting with the Borel-Weil theorem, which, by considering the global sections of the line bundle, demonstrates the connection between certain G-equivariant line bundles on the flag variety with certain G-modules of dominant highest weight. The Borel-Weil-Bott theorem then generalizes this by looking at the higher cohomology of the line bundle. Again, the example to keep in mind is $G = \mathrm{SL}_2(\mathbb{C})$,with flag variety $G/B \simeq \mathbb{P}^1$. Next we look to another generalization of the Borel-Weil theorem, and that is the Beilinson-Bernstein localization theorem. We shall not undertake the proof here since it is quite involved, but we will at least try to give some insight into the theorem. There are two main aspects to understanding it. On the one hand, we would like the D-modules of interest to be generated by their global sections; another way to say this is that the flag variety is "D-affine", which loosely says that there are enough global differential operators on the flag variety. The other main aspect of the theorem is that the global sections of the twisted sheaf of differential operators on G/B coincides with the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. We shall discuss both of these points briefly.

In Chapter 6, we use the Beilinson-Bernstein equivalence of categories for H-monodromic D-modules in the base affine space to construct some interesting representations of $\mathfrak{g}$. This is the first main contribution of this thesis. In particular, we shall focus on the examples of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$, which give insight into the more general case of $\mathfrak{sl}_n(\mathbb{C})$. We first give an overview of the structure of the ring of differential operators on G/N before looking at some of the D-modules which correspond to Lie algebra representations. We look at the finite dimensional representations, which can all be obtained from the structure sheaf of the base affine space. Another natural D-module to consider is the pushforward of the structure sheaf of certain group orbits of G/N. We look first to the open B-orbit, which corresponds to the Verma and the dual Verma modules depending upon the type of pushforward taken. We also consider the structure sheaf of the inverse image (along $\pi : G/N \to G/B$) of the open K-orbit on G/B; pushing this forward to the base affine space will give the principal series representations.

In Chapter 7.4, we move on to the Jantzen filtration. We first discuss how the Jantzen filtration was initially defined for Verma modules, and a similar filtration can be defined in the case of dual Verma modules and for principal series representations. Before defining the geometric version of this Jantzen filtration, we take a look at the monodromy filtration for a general abelian category equipped with a nilpotent endomorphism, which gives us the tools to then define this geometric notion of the Jantzen filtration as done in [BB93] as the restriction of the monodromy filtration to either the kernel or cokernel of the nilpotent endomorphism. We then go through the example of computing the Jantzen filtration of Verma modules in the case of $\mathfrak{sl}_2(\mathbb{C})$ which will hopefully enlighten the reader to the reason this geometric view of the Jantzen filtration coincides with the original algebraic definition for Verma modules. This is the second main contribution of this thesis.

The assumed knowledge for this thesis involves all the past coursework the author has done. In particular, courses in the basics of category theory, representation theory and algebraic geometry, including some familiarity with sheaf theory and homological algebra. For background on this assumed knowledge, we suggest [Jac89] for any algebra, basics of representation theory and homological algebra, [Har77] for background on algebraic geometry including sheafs, and [Rie17] for any category theory.

Chapter 2

Algebraic groups and Lie algebras

Though this thesis is concerned primarily with Lie algebras and (their connection with) D-modules, we begin the story with algebraic groups, from where much geometric intuition and a good amount of motivation for the study of Lie algebras arises. We shall not require many tools from the study of Algebraic groups, so we just recall some basic definitions and provide examples. Of slightly more interest is the connection between an algebraic group and its corresponding Lie algebra; although we will not look at this in detail for the general case, it will be nice to view this connection for linear algebraic groups over $\mathbb{C}$, and in particular the case of $\mathrm{SL}_n(\mathbb{C})$. Throughout this section, unless otherwise stated, we shall take k to be an algebraically closed field of characteristic 0. The main resources we follow in this chapter will be [Hum72] and [TY05].

2.1 Algebraic group basics

Definition 2.1.1. An *algebraic group* $(G, \cdot)$ over an (arbitrary) field k is an algebraic variety with a compatible group structure on the points of the variety, in the sense that the group operations of multiplication and inversion are regular.

One may notice the similarity between this definition with the definition of a Lie group:

Definition 2.1.2. A *real (resp. complex) Lie group* $(G, \cdot)$ is a smooth (resp. complex) manifold with a compatible group operation between points of the manifold, in the sense that the group operations of multiplication and inversion are smooth (resp. holomorphic).

In particular, if the algebraic group G has the structure of a smooth (complex) manifold, then G will be a real (complex) Lie group. Throughout this paper (and in much of the surrounding literature), the algebraic groups we discuss in any sort of detail will have the structure of a complex Lie group. Hence, we shall mostly write only of algebraic groups, but the reader should keep in mind that analogous statements can be made of Lie groups where relevant, and in this section we will occasionally refer to the analogous nature of Lie groups.

Definition 2.1.3. A morphism of algebraic groups over k is a regular map which is also a group homomorphism.

Algebraic groups over k then form a category, which we denote $\mathcal{G}rp/k$. Later in this section, we shall see that associated to each algebraic group (or Lie group), there is a Lie algebra (Definition 2.2.1). In [Mil13] (Theorem 2.1 p. 158) it is shown that there exists a canonical functor from real (resp. complex) algebraic groups to real (resp. complex) Lie groups which preserves the corresponding Lie algebra.

Definition 2.1.4. Let G be an algebraic group. An algebraic subgroup of G is a subvariety H with an algebraic group structure such that the inclusion map $\iota : H \to G$ is a morphism of algebraic groups.

The task of showing that something is an algebraic group directly may sometimes be straightforward - such as in the case of $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$, where it is simply a matter of seeing that the multiplication and inversion are defined by polynomials (possibly along with division by a nonzero determinant) - however in other cases, it may be rather tedious or difficult. We give, without proof, a theorem which allows one to more easily identify more algebraic groups.

Theorem 2.1.5. *A Zariski closed subgroup of an algebraic group is an algebraic group.*

Some common examples of algebraic groups are, for instance, $(k^n, +)$, $(k^*, \cdot)$, where k is any field, $(S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}, \cdot)$. Of particular interest are linear algebraic groups, i.e. closed subgroups of $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$, the group of automorphisms of a finite n-dimensional k-vector space V; or equivalently upon choosing a basis, the group of $n \times n$ invertible matrices with entries in k. For instance the so-called classical groups, characterized below, are all linear algebraic groups.

Example 2.1.6. The classical groups, up to a choice of basis, are:

i) The general linear group itself $\mathrm{GL}_n(k)$;
ii) The special linear group $\mathrm{SL}_n(k)$, consisting of $n \times n$ matrices of determinant 1 with entries in k;

iii) The orthogonal group $\mathrm{O}_n(k)$ consisting of $n \times n$ orthogonal matrices with entries in k, i.e. invertible matrices whose inverse is equal to its transpose;

iv) The special orthogonal group $\mathrm{SO}_n(k)$, consisting of $n \times n$ orthogonal matrices with determinant 1 with entries in k;

v) The unitary group $\mathrm{U}_n(k)$, consisting of $n \times n$ unitary matrices with entries in k, i.e. invertible matrices whose inverse is equal to its conjugate transpose;

vi) The special unitary group $\mathrm{SU}_n(k)$ consisting of $n \times n$ unitary matrices of determinant 1 with entries in k;

vii) The symplectic group $\mathrm{Sp}_{2n}(k)$; let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity. The symplectic group consists of $2n \times 2n$ matrices A such that $A^T J A = J$.

If the underlying field of the vector space is the real or the complex numbers, then all of these are examples of Lie groups. One can check furthermore that $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, and $\mathrm{Sp}_{2n}(\mathbb{C})$ are complex Lie groups. It is somewhat interesting to note that, for instance, the special unitary group $\mathrm{SU}_n(\mathbb{C})$ is a real Lie group, but not a complex Lie group. The quickest way to see this is to note that $\mathrm{SU}_2(\mathbb{C})$ has real dimension 3.

Since an algebraic group G is a group, we can consider its action on certain sets, in particular on varieties (or schemes, or manifolds). A particularly natural choice would be to consider its action on those varieties which are obtained as a quotient of G by certain subgroups. In this light, the following subgroups will appear quite frequently throughout our discussion.

Definition 2.1.7. A maximal, Zariski-closed, connected solvable subgroup of G is called a *Borel subgroup.*

Definition 2.1.8. A square matrix A is called *unipotent* if $M - I$ is nilpotent, i.e. there exists some $n \in \mathbb{N}$ such that $(M - I)^n = 0$, where I is the identity matrix.

Definition 2.1.9. Let G be a linear algebraic group (i.e. a closed subgroup of $\mathrm{GL}_n(k)$, so all the elements of G are matrices). A subgroup $N \subset G$ is called a *unipotent subgroup* if all of its elements are unipotent. If N is not properly contained in any unipotent subgroup of G, then N is called a *maximal unipotent subgroup.*

Definition 2.1.10. A *torus* H in G is an algebraic subgroup which is isomorphic to a direct product of finitely many copies of the multiplicative group k^*. If H is not properly contained in any torus, then H is called a *maximal torus.*

Example 2.1.11. i) Let $G = \mathrm{GL}_n(\mathbb{C})$. The subgroup T of matrices containing nonzero entries only along the diagonal is a torus. Moreover, $(\mathbb{C}^*)^{n+1}$ is not a subgroup of $\mathrm{GL}_n(\mathbb{C})$, so T is a maximal torus. If $P \in \mathrm{GL}_n(C)$ and $A, B \in T$, then

$$P^{-1}APP^{-1}BP = P^{-1}ABP \in P^{-1}TP,$$

and so we can see that any subgroup of the form $P^{-1}TP$ is a maximal torus. Moreover, every maximal torus of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to the aforementioned one [Hum75] (p.135).

ii) For $G = \mathrm{GL}_n(\mathbb{C})$, an example of a nilpotent matrix is one which has all its elements with zeros along and below the main diagonal. An example of a unipotent subgroup N would then be one consisting of matrices which have 1's along the main diagonal and 0's below the main diagonal. One can verify by matrix multiplication that this is indeed a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Since for any $A \in \mathrm{GL}_n(\mathbb{C})$ and $B \in N$ $1 - ABA^{-1} = A(1 - B)A^{-1}$, we see that any conjugate of a unipotent subgroup is unipotent. It turns out that the maximal unipotent subgroup of $\mathrm{GL}_n(\mathbb{C})$ is unique up to conjugation [Hum75] (p.135). For the remainder of this paper, unless specified otherwise, when we refer to a maximal unipotent subgroup, we refer to the group of matrices of the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

iii) Similarly to the example (ii), however we consider e.g. the subgroup N' of matrices of the form

$$\begin{pmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & 0 & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

This will also form a unipotent subgroup, however since N' is a proper subgroup of N in the example above, N' is not a maximal unipotent subgroup.

iv) Let $G = \mathrm{GL}_n(\mathbb{C})$. There is a Borel subgroup consisting of upper triangular matrices, i.e.

$$B = \left\{ \begin{pmatrix} \alpha_1 & \beta_{12} & \cdots & \beta_{1n} \\ 0 & \alpha_2 & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \,\middle|\, \alpha_i \in \mathbb{C}^*,\, \beta_{ij} \in \mathbb{C},\, 1 \leq i \leq n,\, i < j \leq n \right\}$$

To see that B is indeed a Borel subgroup, write $B_0 = B$, let B_1 be the maximal unipotent subgroup as in (ii), and for $i \geq 2$, let B_i be the subgroup of B_{i-1} whose entries $i-1$ places above the main diagonal are zero. Then B_n is just the trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$. One can directly verify that each B_i is a normal subgroup of B_{i-1}, since for $A \in \mathrm{GL}_n(\mathbb{C})$ and $C \in B_i$, $1 - ACA^{-1} = A(1-C)A^{-1}$, so if $(1-C)^k = 0$, then $(1-ACA^{-1})^k = 0$. Additionally, the quotient B_i/B_{i-1} is abelian, since for any matrices $P, Q \in B_{i-1}$, the commutator $PQ - QP \in B_i$. That B is maximal follows from the fact that $\mathrm{SL}_2(\mathbb{C})$ (more generally, $\mathrm{SL}_n(\mathbb{C})$) is perfect, and any subgroup of $\mathrm{GL}_n(\mathbb{C})$ which properly contains B as a subgroup contains $\mathrm{SL}_2(\mathbb{C})$ as a subgroup. Hence, the group described above is a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$. Note that as in the case of maximal unipotent subgroups, any conjugate of a Borel subgroup is a Borel subgroup, since if $P \in \mathrm{GL}_n(\mathbb{C})$, then $[PB_iP^{-1}, PB_iP^{-1}] = P[B_i, B_i]P^{-1}$. The Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ is unique up to conjugation [Hum75] (p.134).

v) All of the above examples can be worked to suit $\mathrm{SL}_n(\mathbb{C})$. In fact, simply taking the intersection with $\mathrm{SL}_n(\mathbb{C})$ of a (maximal) torus, unipotent and Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ gives a (maximal) torus, unipotent and Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$.

2.2 Lie algebras

Definition 2.2.1. A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a bilinear product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* which is alternating (i.e. $[x,x] = 0$) and satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$$

for all $x, y, z \in \mathfrak{g}$.

Remark 2.2.2. *Alternativity and bilinearity of the Lie bracket imply* $[x,y] = -[y,x]$, *since*

$$\begin{aligned} 0 &= [x+y, x+y] \\ &= [x, x+y] + [y, x+y] \\ &= [x,x] + [x,y] + [y,x] + [y,y] \\ &= [x,y] + [y,x]. \end{aligned}$$

Definition 2.2.3. A *Lie algebra morphism* $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a vector space morphism which commutes with the Lie Bracket, i.e. $\varphi([x,y]) = [\varphi(x), \varphi(y)]$, for all $x, y \in \mathfrak{g}$.

Lie algebras over a field k form a category k-**Lie**

Example 2.2.4. i) Given a vector space V, to the set of endomorphisms of V, we may place a binary relation given by the commutator $[A,B] = AB - BA$, where composition of endomorphisms A and

B here is denoted by juxtaposition. We can check that if $A, B, C \in \mathrm{End}(V)$, then

$$\begin{aligned}
[A,[B,C]]+[B,[C,A]] &= A(BC-CB)-(BC-CB)A+B(CA-AC)-(CA-AC)B \\
&= ABC+CBA-BAC-CAB \\
&= C(BA-AB)-(BA-AB)C \\
&= [C,[B,A]] \\
&= -[C,[A,B]],
\end{aligned}$$

so indeed, the commutator bracket is a Lie bracket, and therefore the vector space $\mathrm{End}(V)$ equipped with commutator bracket forms a Lie algebra. We denote this Lie algebra by $\mathfrak{gl}(V)$. If $\dim V = n$ is finite, then this is the same as the Lie algebra of $n \times n$ matrices equipped with the commutator bracket, and we denote it $\mathfrak{gl}_n(k)$.

ii) Specifically, for the case that $n = 1$ in the above example, we have $[A, B] = 0$, for all $A, B \in k$. A Lie algebra $\mathfrak{g}$ satisfying this property that $[A, B] = 0$, $\forall A, B \in \mathfrak{g}$ is called an *abelian Lie algebra*. More generally, given any vector space, we may equip a Lie bracket which vanishes for any two inputs, thus yielding an abelian Lie algebra.

iii) Given a field k, the vector space k^3 equipped with the familiar cross product is a Lie algebra. The conditions of bilinearity and alternativity are straightforward to see, and a (somewhat tedious) computation can be done to convince the reader that the Jacobi identity holds.

iv) Given a k-algebra A, we define the set of derivations on A to be the subset of vector space endomorphisms $D : A \to A$ satisfying Leibniz's rule, $D(ab) = aD(b) + D(a)b$. While the composition of derivations may not yield a derivation, the following computation shows that the set of derivations are closed under commutation

$$\begin{aligned}
(\theta\varphi-\varphi\theta)(ab) &= \theta(a\varphi(b)+\varphi(a)b)-\varphi(a\theta(b)+\theta(a)b) \\
&= a\theta\varphi(b)+\theta(a)\varphi(b)+\theta\varphi(a)b+\varphi(a)\theta(b) \\
&\quad -a\varphi\theta(b)-\varphi(a)\theta(b)-\varphi\theta(a)b-\theta(a)\varphi(b) \\
&= a(\theta\varphi-\varphi\theta)(b)+(\theta\varphi-\varphi\theta)(a)b,
\end{aligned}$$

where θ, φ are derivations of A and $a, b \in A$. Thus, as the commutator is a Lie bracket, the derivations form a Lie algebra. We denote the Lie algebra of derivations of A by $\mathrm{Der}\, A$.

v) The set of $n \times n$ matrices with entries in k with zero trace equipped with the commutator forms a Lie algebra. Such matrices form a vector space, and since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for $n \times n$ matrices A and B, the trace zero condition is preserved by the commutator. We denote this Lie algebra by $\mathfrak{sl}_n(k)$.

vi) In the prior example, when $n = 2$, is the Lie algebra $\mathfrak{sl}_2(k)$, which finds particular importance in describing the structure of more complicated Lie algebras. We mention here that $\mathfrak{sl}_2(k)$ is spanned by the matrices

$$\mathrm{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and these satisfy the commutation relations

$$[\mathrm{E},\mathrm{F}] = \mathrm{H}, \quad [\mathrm{H},\mathrm{E}] = 2\,\mathrm{E}, \quad \text{and} \quad [\mathrm{H},\mathrm{F}] = -2\,\mathrm{F}.$$

vii) We saw above that $\mathfrak{gl}_1(k)$ is abelian, and in fact every 1-dimensional Lie algebra is abelian, since the Lie bracket is alternating. We can also easily characterize all the 2-dimensional Lie algebras. Suppose $\mathfrak{g}$ is a 2-dimensional Lie algebra over k. Then $\mathfrak{g}$ is generated by two vectors, say x and y, and we obtain the Lie algebra structure by defining the bracket on x and y. If $[x, y] = 0$, then $\mathfrak{g}$ is abelian. On the other hand, if $[x, y] \neq 0$, then we may assume without loss of generality that $[x, y] = tx$, for some $t \in k \setminus \{0\}$. If $\mathfrak{l}$ is another 2-dimensional non-abelian Lie algebra over k, generated by vectors z and w, then we may assume that $[z, w] = sz$, for some $s \in k$. The linear map $\varphi : \mathfrak{l} \to \mathfrak{g}$ which sends $z \mapsto x$ and $w \mapsto \frac{s}{t}y$ defines a Lie algebra isomorphism, by linearity of the Lie bracket and since

$$\begin{aligned}
[\varphi(z),\varphi(w)] &= \left[x, \frac{s}{t}y\right] = \frac{s}{t}[x,y] \\
&= sx = s\varphi(z) = \varphi(sz) \\
&= \varphi([z,w]).
\end{aligned}$$

It follows that any 2-dimensional Lie algebra must be isomorphic to one of these two Lie algebras depending upon if it is abelian.

In general, given any (associative) algebra A, the commutator bracket is a Lie bracket - one verifies this with essentially the same computation as in Example 2.2.4 (i). This provides a way to construct a Lie algebra $\mathcal{L}(A)$ from A, viz. by replacing the usual product with the commutator bracket. Suppose $\eta : A \to B$ is a morphism of algebras. Then given $a_1, a_2 \in A$, we have $\eta(a_1a_2 - a_2a_1) = \eta(a_1)\eta(a_2) - \eta(a_2)\eta(a_1)$. It follows that η naturally induces a Lie algebra morphism $\mathcal{L}(\eta) : \mathcal{L}(A) \to \mathcal{L}(B)$. Hence, we have a functor $\mathcal{L}$ from the category of algebras to the category of Lie algebras. On the other hand, given any Lie algebra $\mathfrak{g}$ over a field k, we can construct an algebra $\mathcal{U}(\mathfrak{g})$ which is universal in the sense that there is a natural morphism $u : \mathfrak{g} \to \mathcal{L}(\mathcal{U}(\mathfrak{g}))$ of Lie algebras such that for any algebra A and any Lie algebra morphism $f : \mathfrak{g} \to \mathcal{L}(A)$, there exists a unique algebra morphism $\tilde{f} : \mathcal{U}(g) \to A$ which makes the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{u} & \mathcal{L}(\mathcal{U}(\mathfrak{g})) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \mathcal{L}(\tilde{f})} \\ & & \mathcal{L}(A) \end{array}$$

commute. This universal property makes it clear that $\mathcal{U}(\mathfrak{g})$ is unique up to unique isomorphism (this can be seen by letting $\mathcal{U}'$ be another such algebra and $u' : \mathfrak{g} \to \mathcal{L}(\mathcal{U}')$ the corresponding natural Lie algebra morphism, and then replacing A with $\mathcal{U}'$ and f with u' in the above diagram). We call the algebra $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and it can be constructed in the following way. Let

$$T(\mathfrak{g}) = k \oplus \bigoplus_{j \geq 1} \mathfrak{g}^{\otimes j}$$

be the tensor algebra of $\mathfrak{g}$, and $I \lhd T(\mathfrak{g})$ the ideal generated by all the elements in $T(\mathfrak{g})$ of the form $[x, y] - x \otimes y + y \otimes x$. Then we shall see that $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$, satisfies the desired universal property. There is a natural homomorphism from $T(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g})$, and the restriction to $\mathfrak{g} = \mathfrak{g}^{\otimes 1}$ induces a Lie algebra morphism $u : \mathfrak{g} \to \mathcal{L}(\mathcal{U}(\mathfrak{g}))$. Now we need to show that for any algebra A and any Lie algebra morphism $f : \mathfrak{g} \to \mathcal{L}(A)$, there exists a unique algebra morphism $\tilde{f} : \mathcal{U}(\mathfrak{g}) \to A$ such that $f = \mathcal{L}(\tilde{f}) \circ u$. Using f, we can define a natural algebra morphism $f^* : T(\mathfrak{g}) \to A$, sending $x_1 \otimes x_2 \otimes ... \otimes x_n \mapsto f(x_1)f(x_2) \cdots f(x_n)$. Since

$$f^*([x, y] - x \otimes y + y \otimes x) = f([x, y]) - f(x)f(y) + f(y)f(x) = 0,$$

the ideal I is in the kernel of f^*, and so we have a well-defined morphism $\tilde{f} : \mathcal{U}(\mathfrak{g}) \to A$ satisfying $\tilde{f}(x + I) = f(x)$, which implies that $\mathcal{L}(\tilde{f}) \circ u(x) = f(x)$, for all $x \in \mathfrak{g}$. By construction, u is epic (i.e. right cancellable), so $\tilde{f}$ is unique; thus, we have shown that $(\mathcal{U}(\mathfrak{g}), u)$ is indeed a universal from $\mathfrak{g}$ to $(\cdot)^-$. What we have shown is equivalent to the statement that the functor $\mathcal{U}(-)$ from k-**Lie** to k-**alg** is the left adjoint of the functor $\mathcal{L}(-)$ from k-**alg** to k-**Lie**. The following theorem, due to Poincare-Birkhoff-Witt provides a nice way to write the elements of the universal enveloping algebra.

Theorem 2.2.5 (PBW Theorem). *Let $\mathfrak{g}$ be a Lie algebra over k. If $(x_1, \ldots, x_n)$ is a basis for $\mathfrak{g}$, then the set of monomials of the form $x_1^{a_1} \ldots x_n^{a_n}$ forms a basis for $\mathcal{U}(\mathfrak{g})$.*

The proof is given in [Hum72]

Definition 2.2.6. i) We call a Lie algebra morphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ a *representation* of the Lie algebra $\mathfrak{g}$. If V is finite dimensional, we call it a *finite dimensional representation.*

ii) Given a k-Lie algebra $\mathfrak{g}$, a (left) $\mathfrak{g}$-*module* M is a k-vector space equipped with a scalar multiplication $\mathfrak{g} \times M \to M$, such that for any $x, y \in \mathfrak{g}$, and any $m \in M$, the relation

$$[x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m)$$

is satisfied. If the underlying k-vector space of M is finite dimensional, we call M a finite dimensional module.

Just as how representations of a group are in one-to-one correspondence with modules over the group algebra, representations of a Lie algebra are in one-to-one correspondence with modules over the Lie algebra. Moreover by the universal property above, modules over a Lie algebra are in one-to-one correspondence with modules over the universal enveloping algebra. One has a natural notion of morphisms of representations of $\mathfrak{g}$ and morphisms of $\mathfrak{g}$-modules.

Definition 2.2.7. i) Given a Lie algebra $\mathfrak{g}$ and representations $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$, a morphism $\varphi \to \rho$ of representations is a Lie algebra morphism $\eta : \mathfrak{gl}(V) \to \mathfrak{gl}(W)$ such that $\rho = \eta \circ \varphi$.
ii) Given $\mathfrak{g}$-modules M and N, a morphism $M \to N$ of $\mathfrak{g}$-modules is a vector space morphism which respects the scalar multiplication by $\mathfrak{g}$.

Identifying each representation of the Lie algebra $\mathfrak{g}$ with the corresponding $\mathfrak{g}$-module, one sees that Definition 2.2.6 (i) and (ii) coincide and Definition 2.2.7 (i) and (ii) coincide. Thus we may speak of the category of representations of $\mathfrak{g}$ and of the category of $\mathfrak{g}$-modules. Moreover, by the foregoing discussion, these two categories are both isomorphic to the category of $\mathcal{U}(\mathfrak{g})$-modules. Henceforth we shall refer to these isomorphic categories as **$\mathfrak{g}$-mod**, and interchangeably refer to the objects as either $\mathfrak{g}$-modules, $\mathcal{U}(\mathfrak{g})$-modules, or $\mathfrak{g}$-representations. A key example of a Lie algebra representation is the adjoint representation.

Definition 2.2.8. Let $\mathfrak{g}$ be a Lie algebra, and for any $x \in \mathfrak{g}$, denote by ad_x the morphism from $\mathfrak{g}$ to $\mathfrak{g}$ defined by $\mathrm{ad}_x(y) = [x, y]$. The morphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, sending $x \mapsto \mathrm{ad}_x$ is called the *adjoint representation.*

To check that this is indeed a representation, just see that for $x, y, z \in \mathfrak{g}$,

$$\mathrm{ad}_{[x,y]}(z) = [[x, y], z] = [x, [y, z]] - [y, [x, z]] = \mathrm{ad}_x(\mathrm{ad}_y(z)) - \mathrm{ad}_y(\mathrm{ad}_x(z))$$

by the Jacobi identity. Assuming that $\mathfrak{g}$ is finite dimensional, then the adjoint representation is finite dimensional, and for any $x \in \mathfrak{g}$, we may consider the trace $\mathrm{Tr}(\mathrm{ad}_x)$ of ad_x. With this, we have the following definition.

Definition 2.2.9. Given a Lie algebra $\mathfrak{g}$, the *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form defined by

$$(x, y) = \kappa(x, y) := \mathrm{Tr}(\mathrm{ad}_x \, \mathrm{ad}_y),$$

for all $x, y \in \mathfrak{g}$.

The Killing form finds various use in the study of Lie algebras, and we shall soon see an example of this in an important criterion (Proposition 2.2.18).

There are various definitions pertaining to (associative) algebras which have analogous notions for Lie algebras.

Definition 2.2.10. i) A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a vector subspace which is closed under the Lie bracket.
ii) An *ideal* of $\mathfrak{g}$ is a Lie subalgebra I such that for any $a \in L$, and any $m \in I$, $[a, m] \in I$.
iii) Given an ideal I of $\mathfrak{g}$, we may consider the *quotient Lie algebra* $\mathfrak{g}/I$, consisting of equivalence classes $x + I$, where $x \in \mathfrak{g}$. It is easy to check that addition and the Lie bracket are well defined on $\mathfrak{g}/I$.
iv) For a Lie algebra $\mathfrak{g}$, define the following ideals

$$\mathcal{C}^1(\mathfrak{g}) = \mathcal{D}^0(\mathfrak{g}) = \mathfrak{g}, \quad \mathcal{C}^{i+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^i(\mathfrak{g})], \quad \mathcal{D}^i(\mathfrak{g}) = [\mathcal{D}^{i-1}(\mathfrak{g}), \mathcal{D}^{i-1}(\mathfrak{g})],$$

for $i \geq 1$. Call the $\mathcal{C}^i(\mathfrak{g})$ the *central descending series* of $\mathfrak{g}$ and call $\mathcal{D}^i(\mathfrak{g})$ the *derived series* of $\mathfrak{g}$.
v) A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\{0\}$ is an element of the central descending series of $\mathfrak{g}$.
vi) A Lie algebra $\mathfrak{g}$ is called *solvable* if $\{0\}$ is an element of the derived series of $\mathfrak{g}$.
vii) Given a subset S of a Lie algebra $\mathfrak{g}$, the *normalizer* of S is given by

$$N(S) = \{x \in \mathfrak{g} \mid [x, s] \in S, \text{ for all } s \in S\}.$$

If S is closed under addition, then by the Jacobi identity, $N(S)$ is a Lie subalgebra of $\mathfrak{g}$.

Lemma 2.2.11. *The elements $\mathcal{D}^i(I)$ of the derived series and $\mathcal{C}^i(I)$ of the central descending series of an ideal I in $\mathfrak{g}$ are all ideals in $\mathfrak{g}$.*

Proof. For $a, x \in \mathfrak{g}$, and $y \in I$, we have, by the Jacobi identity,

$$[a, [x, y]] = -[x, [y, a]] - [y, [a, x]],$$

and since $[y, a] \in I$ and $[a, x] \in \mathfrak{g}$, we see that $[a, [x, y]] \in [\mathfrak{g}, I]$. If $x \in I$, then $[a, x] \in I$, so we have $[a, [x, y]] \in [I, I]$. The result then follows by induction. □

We have the following criteria for nilpotency and solvability of a Lie algebra. The proofs can be found in [Hum72] (p.12 and p.20 respectively).

Theorem 2.2.12 (Engel's Theorem)**.** *Let* $\mathfrak{g}$ *be a Lie algebra. If* ad_x *is nilpotent for every* $x \in \mathfrak{g}$*, then* $\mathfrak{g}$ *is nilpotent.*

Theorem 2.2.13 (Cartan's criterion for solvability)**.** *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{gl}(V)$*, where* V *is a finite dimensional vector space. Then* $\mathfrak{g}$ *is solvable if and only if* $\mathrm{Tr}(ab) = 0$ *for every* $a \in [\mathfrak{g}, \mathfrak{g}]$ *and* $b \in \mathfrak{g}$*.*

Since $\mathcal{D}^i \subset \mathcal{C}^{i+1}$, any nilpotent Lie algebra is solvable. We have the following proposition regarding solvable ideals in $\mathfrak{g}$.

Proposition 2.2.14. *There exists a unique maximal solvable ideal in* $\mathfrak{g}$*.*

Proof. It suffices to see that if I_1 and I_2 are two solvable ideals of $\mathfrak{g}$, then $I_1 + I_2$ is solvable, for then the maximal solvable ideal will be the sum of all solvable ideals. Obviously $I_1 \cap I_2$ is solvable, and one can plainly see that quotients of solvable ideals are solvable, so in particular $I_1/(I_1 \cap I_2)$ is solvable. Applying the second isomorphism theorem (for Lie algebras) shows that $(I_1 + I_2)/I_2$ is solvable. Hence, $\mathcal{D}^i(I_1 + I_2) \subset I_2$, for some $i \geq 0$, and $\mathcal{D}^j(I_2) = \{0\}$ for some $j \geq 0$. Hence, $\mathcal{D}^{i+j}(I_1 + I_2) \subset \mathcal{D}^j(I_2) = \{0\}$, so indeed, $I_1 + I_2$ is solvable, and the proposition follows immediately. □

Definition 2.2.15. Call the unique maximal solvable ideal of the Lie algebra $\mathfrak{g}$ the *radical* of $\mathfrak{g}$, and denote it by $\mathfrak{rad}(\mathfrak{g})$.

Definition 2.2.16. The largest nilpotent ideal of a Lie algebra $\mathfrak{g}$ is called the *nilpotent radical* of $\mathfrak{g}$, which we denote $\mathfrak{nil}(\mathfrak{g})$.

It can be seen that the nilpotent radical of a Lie algebra $\mathfrak{g}$ coincides with $[\mathfrak{g}, \mathfrak{rad}(\mathfrak{g})] = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{rad}(\mathfrak{g})$.

We call a Lie algebra simple if it is not abelian and has no non-zero proper ideals. We define a semisimple Lie algebra as follows.

Definition 2.2.17. A Lie algebra $\mathfrak{g}$ is called semisimple if $\{0\}$ is the only abelian ideal of $\mathfrak{g}$.

If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field of characteristic 0, then its adjoint representation is a subalgebra of $\mathfrak{gl}(V)$, for some finite dimensional V. We may apply Cartan's criterion to this adjoint representation to conclude that $\kappa(a, b) = \mathrm{tr}(\mathrm{ad}_a \,\mathrm{ad}_b) = 0$ for every $a \in [\mathfrak{g}, \mathfrak{g}]$ and every $b \in \mathfrak{g}$, if and only if $\mathfrak{g}$ is semisimple.

We have the following proposition which provides some equivalent characterizations of a semisimple Lie algebra.

Proposition 2.2.18. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. The following are equivalent.*

i) $\mathfrak{g}$ *is semisimple;*
ii) $\mathrm{rad}\,\mathfrak{g} = \{0\}$*;*
iii) The Killing form $\kappa(-, -)$ *on* $\mathfrak{g}$ *is non-degenerate;*
iv) $\mathfrak{g}$ *is a direct sum of simple Lie algebras*

Proof. This proof follows [TY05] (p. 299-300).
(i) $\Longrightarrow$ (ii): If $\mathfrak{g}$ is semisimple and I is a solvable ideal of $\mathfrak{g}$, then for some element $\mathcal{D}^i(I)$ of the derived series of I, we have $[\mathcal{D}^i(I), \mathcal{D}^i(I)] = \{0\}$. By lemma 1.1, $\mathcal{D}^i(I)$ is an ideal, and since the only abelian ideal of $\mathfrak{g}$ is $\{0\}$, we conclude that $\mathcal{D}^i(I) = \{0\}$ for every $i \geq 0$. in particular, $I = \{0\}$, and thus we conclude that the radical $\mathrm{rad}\,\mathfrak{g} = \{0\}$.
(ii) $\Longrightarrow$ (iii): Now the kernel of $\kappa(-, -)$ is an ideal of $\mathfrak{g}$. Since $\kappa(-, -)$ restricted to the kernel is vanishing, by Cartan's criterion for solvability, $\ker \kappa(-, -)$ is a solvable ideal, and therefore $\{0\}$. Hence, κ is non-degenerate.
(iii) $\Longrightarrow$ (i): Now suppose that the Killing form $\kappa(-, -)$ is non-degenerate. Then it follows from Cartan's criterion that the only solvable ideal of $\mathfrak{g}$ is $\{0\}$, and hence the only abelian ideal of $\mathfrak{g}$ is $\{0\}$.
(i) $\Longrightarrow$ (iv): If $\mathfrak{g}$ is semisimple, and I is a nonzero ideal of $\mathfrak{g}$, then $1 < \dim I < \dim \mathfrak{g}$. If we could show that $\mathfrak{g}/I$ and I are both semisimple, then the implication would follow by induction and the fact that $\mathfrak{g} = I \oplus \mathfrak{g}/I$. See [TY05] (p. 300) for details.
(iv) $\Longrightarrow$ (iii): Since simple algebras are obviously semisimple, the Killing form on a direct sum of simple algebras will be non-degenerate. □

Definition 2.2.19. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which equals its normalizer, i.e. $\mathfrak{h} = \{x \in \mathfrak{g} \,|\, [x, h] \in \mathfrak{h}, \forall h \in \mathfrak{h}\}$.

It follows from Theorem 2 (Chap. VII, §2, no. 4) in [Bou75] that if $\mathfrak{g}$ finite dimensional and semisimple over a field k of characteristic 0, then a Cartan subalgebra $\mathfrak{h}$ is the maximally abelian subalgebra such that for any $h \in \mathfrak{h}$, the adjoint ad_h is diagonalizable. The existence and uniqueness up to isomorphism of a Cartan subalgebra for such a Lie algebra $\mathfrak{g}$ then follows easily - just construct the maximal abelian subalgebra whose elements are such that their adjoints are diagonalizable.

Definition 2.2.20. A *Borel subalgebra* $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is a maximal solvable Lie algebra.

Since any Borel subalgebra $\mathfrak{b}$ is solvable, its nilpotent radical is just $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. Assuming that $\mathfrak{g}$ is semisimple, then we can write $\mathfrak{b} \simeq \mathfrak{n} \oplus \mathfrak{h}$, and so the quotient $\mathfrak{b}/\mathfrak{n}$ is isomorphic to a Cartan subalgebra.

2.3 Root Systems

For a general finite dimensional vector space, loosely speaking, a root system provides a (small) finite spanning set of vectors which is very symmetric. Being a spanning set, the root system can fully describe the structure of the vector space. The symmetries of the set then provide additional information, and this can be seen to pertain to the structure of a Lie algebra. More precisely, we have the following theorem (which we will not prove) due to Serre [Ser66] (Chapter VI):

Theorem 2.3.1. *To each isomorphism class of semisimple Lie algebras, there is a unique associated root system which determines that isomorphism class.*

Now we begin with the general definition of a root system for a vector space. Let V be a vector space with a symmetric bilinear form $(-,-)$. For any element $\alpha \in V$, define s_α to be the involution $s_\alpha(x) = x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha$. We see that $s_\alpha(\alpha) = -\alpha$, and if $\beta \perp \alpha$, then $s_\alpha(\beta) = \beta$. Denote by P_α the hyperplane consisting of all $\beta \in V$ such that $\beta \perp \alpha$. We shall denote by $\alpha^\vee$ the element $2\frac{(\alpha,-)}{(\alpha,\alpha)}$ in V^*.

Definition 2.3.2. Let V be a finite dimensional vector space. A subset $\Phi \subset V$ is called a *root system* in V if the following hold:

i) $|\Phi| < \infty$;
ii) $0 \notin \Phi$;
iii) $\operatorname{Span} \Phi = V$;
iv) If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\Phi$.
v) Φ is invariant under s_α for every $\alpha \in \Phi$.
vi) For all $\alpha, \beta \in \Phi$, $\alpha^\vee(\beta) \in \mathbb{Z}$.

Definition 2.3.3. The *rank* of a root system is $\dim V$.

Definition 2.3.4. Given a root system Φ in the vector space V, the *Weyl group* W is the subgroup of $\mathrm{GL}(V)$ generated by all the s_α, for $\alpha \in \Phi$.

Definition 2.3.5. Given a root system Φ, we define the *dual root system* $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$.

Proposition 2.3.6. *If Φ is a root system of V, then the dual root system is a root system of V^*.*

Proof. That $\Phi^\vee$ satisfies (i) and (ii) in 2.3.2 is immediate from the definition of $\Phi^\vee$. If $\{\alpha_1, \dots, \alpha_n\}$, forms a basis for V, where $n = \dim V$ and the $\alpha_i \in \Phi$, then another basis $\{\beta_1, \dots \beta_n\}$ for V may constructed such that $\alpha_i^\vee(\beta_j) \neq 0$ if and only if $i = j$. Since every element of V^* is fully defined by how it acts on a basis of V, it follows that any element of V^* is a linear combination of the $\alpha_i^\vee$, so $\operatorname{Span} \Phi^\vee = V^*$, therefore $\Phi^\vee$ satisfies condition (iii) of Definition 2.3.2

The bilinear form on V^* induced by the bilinear form on V satisfies $((x,-),(y,-)) = (x,y)$, for $x, y \in V$, and so for $\alpha, \beta \in \Phi$, we have

$$\begin{aligned}(\alpha^\vee)^\vee(\beta^\vee) &= 2\frac{(\alpha^\vee,\beta^\vee)}{(\alpha^\vee,\alpha^\vee)} \\ &= 2\frac{\left(\frac{4(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)}\right)}{\left(\frac{4(\alpha,\alpha)}{(\alpha,\alpha)^2}\right)} \\ &= 2\frac{(\alpha,\beta)}{(\beta,\beta)} \\ &= \beta^\vee(\alpha).\end{aligned}$$

Since $\Phi^\vee$ spans V^*, we may identify $(\alpha^\vee)^\vee$ with α, and then identify $(\Phi^\vee)^\vee$ with Φ. Now if $\alpha \in \Phi$, and $\lambda\alpha^\vee \in \Phi^\vee$ for some $\lambda \in k$, then $(\lambda\alpha^\vee)^\vee \in \Phi$. Since The only multiples of α in Φ are $\pm\alpha$, it follows that $\lambda = \pm 1$, so $\Phi^\vee$ satisfies condition (iv) of Definition 2.3.2.

To see that (v) is satisfied, let $\alpha, \beta \in V$ and note that

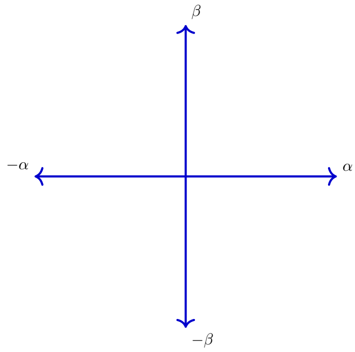
$$s_{\alpha^\vee}(\beta^\vee) = \beta^\vee - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha^\vee = \left(\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha\right)^\vee.$$

Finally, that $\Phi^\vee$ satisfies (vi) follows from the identification of Φ with $(\Phi^\vee)^\vee$. □

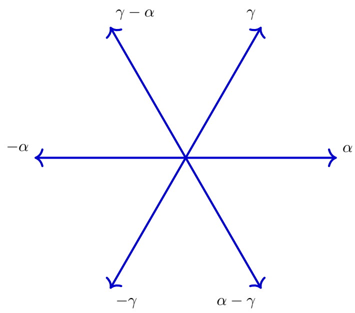
Example 2.3.7. i) A_1 : if $\dim V = 1$, and Φ is a root system in V containing α, then necessarily $\Phi = \{\alpha, -\alpha\}$. This is the only root system of a 1-dimensional vector space.

For the next four examples, assume $\dim V = 2$, and Φ is a root system in V.

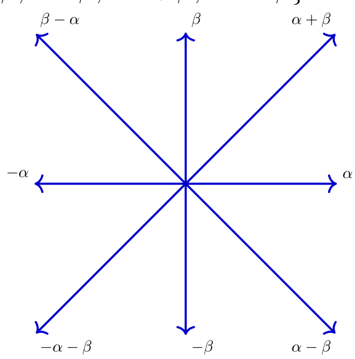
ii) $A_1 \times A_1$: if α is some element of Φ, then there exists $\beta \perp \alpha$, and so we may form the root system consisting of $\{\alpha, -\alpha, \beta, -\beta\}$.



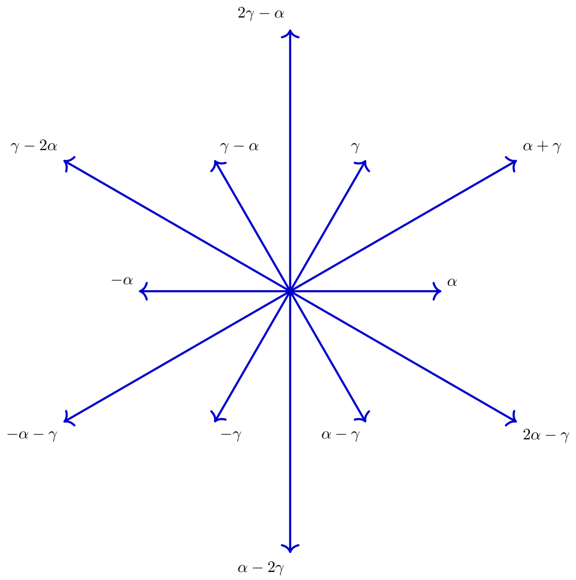
iii) A_2 : Now let $\alpha \in \Phi$ and β be as in the previous example, and stipulate now that β has the same length as α. Here, however rather than choosing $\beta \in \Phi$, here we shall take $\gamma = \frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta \in \Phi$. Then applying e.g. s_α shows that $\gamma - \alpha \in \Phi$. It is straightforward then to see that $\Phi = \{\alpha, -\alpha, \gamma, -\gamma, \alpha - \gamma, \gamma - \alpha\}$.



iv) B_2 : returning to the case where $\alpha, \beta \in \Phi$ are as in (ii), we may additionally add the vector $\alpha + \beta$ to the root system, and applying the various reflections, we see that in this case $\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha + \beta, \alpha - \beta, -\alpha + \beta, -\alpha - \beta\}$.



v) G_2 : finally, given the root system in example (iii), we may adjoin the vector $\alpha + \gamma$. Then this yields the root system $\Phi = \{\alpha, -\alpha, \gamma, -\gamma, \alpha - \gamma, \gamma - \alpha, 2\beta, -2\beta, \gamma + \alpha, \gamma - 2\alpha, -\gamma - \alpha, 2\alpha - \gamma\}$.



The above examples are all the possible rank 1 and rank 2 roots systems.

Definition 2.3.8. A subset $\Delta \subset \Phi$ of a root system in the vector space V is called a *base* if it forms a basis for V and moreover every root can be written as either a $\mathbb{Z}_{\geq 0}$-linear or a $\mathbb{Z}_{\leq 0}$-linear combination of elements in Δ. The roots in a chosen base of $\Delta \subset \Phi$ are called *simple roots.*

Definition 2.3.9. We call the $\mathbb{Z}$-span Λ_r of a root system Φ the *root lattice* of Φ.

Definition 2.3.10. If we fix a base Δ, if $\alpha \in \Phi$ is a $\mathbb{Z}_{\geq 0}$-linear combination of roots in Δ, then we say that α is a *positive root.* Similarly, if α is a $\mathbb{Z}_{\leq 0}$-linear combination, then we say it is a *negative root.* We denote by Φ^+ and Φ^- the set of all positive and negative roots respectively.

Theorem 2.3.11. *Every root system has a base.*

For the proof, see [Hum72] (p.48).

Let Φ be a root system in V. We remark that the reflecting hyperplanes $P_\alpha = \{\beta \in V \,|\, \beta \perp \alpha\}$ for $\alpha \in \Phi$ provide a partition of the subset $V \setminus \bigcup P_\alpha$.

Definition 2.3.12. Call each connected component of $V \setminus \bigcup P_\alpha$ a *Weyl chamber.* Call $v \in V$ *regular* if v lies in a Weyl chamber.

Given a finite dimensional semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, if $\alpha \in \mathfrak{h}^*$, define the vector subspace

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,|\, [h,x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}.$$

Definition 2.3.13. We say that $\alpha \in \mathfrak{h}^*$ is a *root of* $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$.

The following proposition is from [Ser66] (p.43).

Proposition 2.3.14. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

i) The set Φ of all the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ form a root system of $\mathfrak{h}$.
ii) for any $\alpha \in \Phi$, the subspace $\mathfrak{g}_\alpha$ is 1-dimensional.
iii) $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a 1-dimensional subspace of $\mathfrak{h}$.
iv) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.

Given a root $\alpha \in \Phi$, by multiplying a nonzero element in $\mathfrak{h}_\alpha$ by a sufficient scalar, one finds an element $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$. Moreover, there are elements $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha, \quad \text{and} \quad [h_\alpha, y_\alpha] = -2y_\alpha.$$

Hence, for every $\alpha \in \Phi$, there is a Lie subalgebra $\mathfrak{sl}_2(\alpha) := \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \subset \mathfrak{g}$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Now suppose we are given a root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then there is a natural choice for a Borel subalgebra of $\mathfrak{g}$,

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha,$$

with nilpotent radical

$$\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

which is shown in [Ser66] (p. 47-48).

2.4 The correspondence between algebraic groups and Lie algebras

One of the key realizations in Lie theory is to see that Lie groups may be better understood by studying the corresponding Lie algebra, which is the tangent space at the identity of a given Lie group. To each Lie group, one can of course take the tangent space at the identity and this space may naturally be given the structure of a Lie algebra. On the other hand, there is an exponential map which associates a Lie group to any given Lie algebra. We shall see that a similar correspondence exists between algebraic groups and Lie algebras. While the material in this section is important to give the reader an understanding of the connection between algebraic groups and Lie algebras, it is not so vital for the main premise of this thesis, and so we will refer the reader to [TY05] for more details on this correspondence.

Let G be an algebraic group over the $\mathbb{C}$, and $\mathscr{O}(G)$ its algebra of regular functions. The set of derivations (Example 2.2.4 (iv)) on $\mathscr{O}(G)$ form a Lie algebra. For $f \in \mathscr{O}(G)$, and $\alpha \in G$, define the left (respectively right) translation of f by α as

$$(\lambda_\alpha f)(\beta) = f(\alpha^{-1}\beta), \quad \text{(respectively } (\rho_\alpha f)(\beta) = f(\beta\alpha)),$$

for all $\beta \in G$.

Definition 2.4.1. Call a derivation X of $\mathscr{O}_G$ *left invariant* if $X\lambda_\alpha = \lambda_\alpha X$.

Definition 2.4.2. We define the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ corresponding to the algebraic group G to be the subalgebra of derivations of $\mathscr{O}_G$ which are left invariant.

This is indeed a Lie algebra since if X and Y are left invariant derivations of $\mathscr{O}_G$, $c \in \mathbb{C}$ and $\alpha \in G$, then $cX + Y$ is left invariant, and

$$\begin{aligned}\lambda_\alpha[X, Y] &= \lambda_\alpha XY - \lambda_\alpha YX \\ &= X\lambda_\alpha Y - Y\lambda_\alpha X \\ &= XY\lambda_\alpha - YX\lambda_\alpha \\ &= [X, Y]\lambda_\alpha,\end{aligned}$$

so $[X, Y]$ is left invariant. It is shown in [TY05] (p.349) that $\mathfrak{g} = \mathfrak{L}(G)$ is isomorphic to $T_e(G)$, the tangent space of G at the identity e, via $\theta : X \mapsto \chi_e \circ X$, where χ_e is the evaluation at e. Thus we shall identify these Lie algebras, and observe that this definition for the Lie algebra of an algebraic group is analogous to that of a Lie group. In what follows, we shall show that for any morphism u of algebraic groups, we may define a morphism $\mathfrak{L}(u)$ of the corresponding Lie algebras so that $\mathfrak{L}$ forms a functor from the category of algebraic groups to the category of Lie algebras.

Definition 2.4.3. If $u : G \to H$ is a morphism of algebraic groups, then the differential of u at the identity e is the map $du : \mathfrak{L}(G) \to \mathfrak{L}(H)$, defined as the dual map of the induced map $u^x : \mathfrak{m}_{H,u(e)}/\mathfrak{m}^2_{H,u(e)} \to \mathfrak{m}_{G,e}/\mathfrak{m}^2_{G,e}$.

With u being a morphism of algebraic groups, hence of varieties, we have a morphism of the rings of regular functions $u^* : \mathscr{O}_H \to \mathscr{O}_G$. We shall observe that the differential du at the identity is a Lie algebra homomorphism. Indeed for any $x, y \in \mathfrak{L}(G)$, and $f \in H$, we have $du(x)(f) = xu^*(f)$, so $du(x) = xu^*$. Then

$$\begin{aligned}d[x, y] &= [x, y]u^* \\ &= (x \cdot y)u^* - (y \cdot x)u^* \\ &= (xu^*) \cdot (yu^*) - (yu^*) \cdot (xu^*) \\ &= [du(x), du(y)],\end{aligned}$$

where the third equality is proved in [TY05]. We shall define $\mathfrak{L}(u) := du$. One can check that for algebraic group morphisms $u : G \to H$, and $v : H \to K$, then $d(vu) = dv\, du$. A nice consequence of this is that given an algebraic group G a Borel subgroup B and a maximal torus H, if $\mathfrak{g}$ denotes the Lie algebra of G, then $\mathfrak{b} = \mathfrak{L}(B)$ will be the Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{h} = \mathfrak{L}(H)$ will be a Cartan subalgebra of $\mathfrak{g}$.

On the other hand, there is a map, exp, which takes a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(k)$ to a corresponding linear algebraic group $G \subset \mathrm{GL}_n(k)$. For such Lie algebras, this map is given by the exponential of the matrix,

$$\exp(A) = \sum_{j \geq 0} \frac{1}{j!} A^j, \quad A \in \mathfrak{g}.$$

Given an action of G on a variety M, we can define an action of G on functions on M by $g \cdot f(m) = f(g^{-1} \cdot m)$, where $m \in M$, $g \in G$, and f is a function whose domain is M. Then there exists a corresponding action of the Lie algebra $\mathfrak{g}$ on functions on M given by

$$A \cdot f(m) = \frac{d}{dt}\Big|_{t=0} \exp(tA) \cdot f(m), \tag{2.1}$$

where f and m are as above, and $A \in \mathfrak{g}$.

2.5 $\mathfrak{sl}_2(\mathbb{C})$

First, we shall show, via purely algebraic methods, that $\mathfrak{sl}_2(\mathbb{C})$ is the corresponding Lie algebra of the algebraic group $\mathrm{SL}_2(\mathbb{C})$. Since $\mathrm{SL}_2(\mathbb{C})$ is the affine variety in $\mathbb{C}^4$ given by the vanishing of the polynomial $ad - bc - 1$, where a, b, c, and d are the coordinates of $\mathbb{C}^4$, the ring of regular functions $\mathscr{O}(\mathrm{SL}_2(\mathbb{C}))$ of $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to $\mathbb{C}[x_1, x_2, x_3, x_4]/I$, where I is the ideal $(x_1x_4 - x_2x_3 - 1)$. Let X be a left invariant derivation of $\mathscr{O}(\mathrm{SL}_2(\mathbb{C}))$ (Definition 2.4.1). If $f, g \in \mathbb{C}[x_1, x_2, x_3, x_4]$, then

$$\begin{aligned} X(fg + I) &= X((f + I)(g + I)) \\ &= (f + I)X(g + I) + X(f + I)(g + I) \\ &= fX(g) + X(f)g + fX(I) + X(I)g + I \\ &= fX(g) + X(f)g + I. \end{aligned}$$

We can consider how X acts on monomials of the form $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4} + I$. Moreover, using the fact that X is a derivation, we may simply consider how it acts on each monomial x_i, $1 \leq i \leq 4$, and use the above equality to deduce the action on $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4} + I$. In order to satisfy the condition of being a derivation, $X(x_i)$ must be a polynomial, so we may write $X(x_i) = s_i$, where $s_i \in \mathscr{O}(\mathrm{SL}_2(\mathbb{C}))$. Now if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, then the left translation by A of $f(x_1, x_2, x_3, x_4)$ is

$$\lambda_A f(x_1, x_2, x_3, x_4) = f(dx_1 - bx_3, dx_2 - bx_4, ax_3 - cx_1, ax_4 - cx_2).$$

Since X is left invariant, we have

$$\lambda_A s_1 = \lambda_A(X(x_1)) = X(\lambda_A(x_1)) = X(dx_1 - bx_3) = ds_1 - bs_3,$$

$$\lambda_A s_2 = \lambda_A(X(x_2)) = X(\lambda_A(x_2)) = X(dx_2 - bx_4) = ds_2 - bs_4,$$

$$\lambda_A s_3 = \lambda_A(X(x_3)) = X(\lambda_A(x_3)) = X(ax_3 - cx_1) = as_3 - cs_1,$$

$$\lambda_A s_4 = \lambda_A(X(x_4)) = X(\lambda_A(x_4)) = X(ax_4 - cx_2) = as_4 - cs_2.$$

Since these equations hold for every $A \in \mathrm{SL}_2(\mathbb{C})$, the s_i cannot contain any constant term. Similarly, by an argument on the order of the degree, one can check that the s_i cannot contain any terms of degree greater than 1. Considering the action of λ_A on each of the x_i, we can conclude, by the above equations, that

$$s_1 = m_1x_1 + m_2x_2,$$

$$s_2 = m_3x_1 + m_4x_2,$$

$$s_3 = m_1x_3 + m_2x_4,$$

$$s_4 = m_3x_3 + m_4x_4,$$

where the $m_i \in \mathbb{C}$, for $1 \leq i \leq 4$.
Finally, since $x_1x_4 - x_2x_3$ is identified with 1 in the ring of regular functions, we have

$$\begin{aligned}
0 &= X(1) = X(x_1x_4 - x_2x_3) \\
&= x_1X(x_4) + x_4X(x_1) - x_2X(x_3) - x_3X(x_2) \\
&= x_1s_4 + x_4s_1 - x_2s_3 - x_3s_2 \\
&= m_3x_1x_3 + m_4x_1x_4 + m_1x_1x_4 + m_2x_2x_4 \\
&\quad - m_1x_2x_3 - m_2x_2x_4 - m_3x_1x_3 - m_4x_2x_3 \\
&= (m_1 + m_4)(x_1x_4 - x_2x_3) \\
&= m_1 + m_4.
\end{aligned}$$

Hence, X is fully determined by the $m_1, m_2, m_3 \in \mathbb{C}$, and using the characterization of derivations above, we may write

$$X = (m_1x_1 + m_2x_2)\partial_1 + (m_3x_1 - m_1x_2)\partial_2 + (m_1x_3 + m_2x_4)\partial_3 + (m_3x_3 - m_1x_4)\partial_4,$$

where ∂_i denotes the partial differentiation operator on x_i, $1 \leq i \leq 4$. It is a straightforward, though tedious, computation to check that if we represent X as a 2×2 matrix

$$(X) = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

and if Y is another left invariant derivation represented by the matrix

$$(Y) = \begin{pmatrix} k_1 & k_2 \\ k_3 & -k_1 \end{pmatrix},$$

then the commutator $[X, Y]$ is represented by the matrix $(X)(Y) - (Y)(X)$. It then follows that the Lie algebra $\mathfrak{L}(\mathrm{SL}_2(\mathbb{C}))$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, the Lie algebra of 2×2 matrices with trace zero, and thus we shall identify these two Lie algebras.

As mentioned in Example 2.2.4 (vi), the standard basis of $\mathfrak{sl}_2(\mathbb{C})$ is $\{\mathrm{E}, \mathrm{F}, \mathrm{H}\}$, where

$$\mathrm{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These satisfy the commutation relations

$$\begin{aligned}
[\mathrm{E}, \mathrm{F}] &= \mathrm{H}, \\
[\mathrm{H}, \mathrm{E}] &= 2\,\mathrm{E}, \\
[\mathrm{H}, \mathrm{F}] &= -2\,\mathrm{F}.
\end{aligned}$$

Here, $\mathbb{C}\,\mathrm{H} =: \mathfrak{h}$ is a Cartan subalgebra - in particular it is isomorphic to (and hence can be identified with) $\mathbb{C}$. Since $\mathfrak{h}$ is 1-dimensional, the root system of $\mathfrak{sl}_2(\mathbb{C})$ relative to $\mathfrak{h}$ can be taken to be $\Phi = \{2, -2\}$, and we can decompose $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\,\mathrm{F} \oplus \mathbb{C}H \oplus \mathbb{C}\,\mathrm{E}$, where E is the element in the root space which corresponds to the root 2, and F is the element in the root space which corresponds to the root -2.

From our discussion in section 1.2, one can see that the flag variety for $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to $\mathbb{P}^1$, and the base affine space is isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Recall that we denote a point in $\mathbb{P}^1$ with homogeneous coordinates $(x_1 : x_2)$, where $x_1 = 0 \implies x_2 \neq 0$ and $(x_1 : x_2) = (\lambda x_1 : \lambda x_2)$, for all $\lambda \in \mathbb{C}^*$. We have the natural group actions of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{P}^1$ and $\mathbb{C}^2 \setminus \{0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x_1 : x_2) = (ax_1 + bx_2 : cx_1 + dx_2),$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$$

respectively. There is then a corresponding Lie algebra action of $\mathfrak{sl}_2(\mathbb{C})$ on regular functions on $\mathbb{P}^1$ and on $\mathbb{C}^2 \setminus \{0\}$ (see (2.1)). Since E, F, and H form a basis for $\mathfrak{sl}_2(\mathbb{C})$, it suffices to determine this action for those elements. On $\mathbb{C}^2 \setminus \{0\}$, we have

$$\begin{aligned}
E \cdot f(x_1, x_2) &= \frac{d}{dt}\Big|_{t=0} \exp(tE) \cdot f(x_1, x_2) \\
&= \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \cdot (x_1, x_2)\right) \\
&= \frac{d}{dt}\Big|_{t=0} f(x_1 - tx_2, x_2) \\
&= -x_2 \partial_1 f(x_1, x_2), \qquad (2.2)
\end{aligned}$$

$$\begin{aligned}
F \cdot f(x_1, x_2) &= \frac{d}{dt}\Big|_{t=0} \exp(tF) \cdot f(x_1, x_2) \\
&= \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \cdot (x_1, x_2)\right) \\
&= \frac{d}{dt}\Big|_{t=0} f(x_1, x_2 - tx_1) \\
&= -x_1 \partial_2 f(x_1, x_2), \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
H \cdot f(x_1, x_2) &= \frac{d}{dt}\Big|_{t=0} \exp(tH) \cdot f(x_1, x_2) \\
&= \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \cdot (x_1, x_2)\right) \\
&= \frac{d}{dt}\Big|_{t=0} f(e^{-t}x_1, e^t x_2) \\
&= (-x_1 \partial_1 + x_2 \partial_2) f(x_1, x_2), \qquad (2.4)
\end{aligned}$$

Verification of the commutation relations will show that the Lie subalgebra of derivations on $\mathbb{C}^2 \setminus \{0\}$ generated by $-x_2\partial_1$, $-x_1\partial_2$ and $-x_1\partial_1 + x_2\partial_2$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. At the cost of notational abuse, we shall henceforth refer to these derivations respectively as E, F, and H. One may similarly compute the actions of E, F, and H on $\mathscr{O}_{\mathbb{P}^1}$, or alternatively just notice that given the natural projection of $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ via $(x_1, x_2) \mapsto (x_1 : x_2)$, the regular functions of $\mathbb{P}^1$ on the open affine sets $U_0 = \mathbb{P}^1 \setminus \{(1:0)\}$ and $U_\infty = \mathbb{P}^1 \setminus \{(0:1)\}$, are respectively $\mathbb{C}[z]$ and $\mathbb{C}[w]$, where $z = x_1 x_2^{-1}$, and $w = x_1^{-1} x_2$. Then the elements E, F, and H are going to correspond with the differential operators $-\partial_z$, $z^2\partial_z$, and $-2z\partial_z$. For details, see [Rom20].

2.6 The flag variety and base affine space

We now proceed to define two particularly important spaces, which will provide the setting for our D-modules of interest. In the cases where G is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, we will always take a maximal unipotent subgroup N and a Borel subgroup B to consist of upper triangular matrices, as described in Example 2.1.11 (ii) and (iv).

Definition 2.6.1. Let G be an algebraic group, N a maximal unipotent subgroup and B a Borel subgroup, with $N \subset B \subset G$. Both N and B act on G by right multiplication.

i) The set G/B of cosets of B is called the *flag variety*.
ii) The quotient G/N of cosets of N is called the *base affine space* (or *enhanced affine space*, or *enhanced flag variety*).

Notice that neither B nor N is normal in G, so neither G/B nor G/N have the structure of a group. They do, however, have the structure of a variety. In the following, we describe these structures for $G = \mathrm{SL}_n(\mathbb{C})$ and for $G = \mathrm{GL}_n(\mathbb{C})$.

Definition 2.6.2. Let V be a vector space. A *flag* in V is a sequence of subspaces

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_m = V.$$

We call a flag *complete* if $m = \dim V$, otherwise we call it a *partial flag.* We call the sequence

$$(\dim V_1, \dim V_2, ..., \dim V_m)$$

the *signature* of the flag.

We shall show that the set X of complete flags in $\mathbb{C}^n$ can be identified with the flag variety of $\mathrm{GL}_n(\mathbb{C})$. First, the group $\mathrm{GL}_n(\mathbb{C})$ acts transitively on X. Indeed, choosing a vector from each of $V_i \backslash V_{i-1}$, $1 \leq i \leq n$ yields a basis for $\mathbb{C}^n$. Since $\mathrm{GL}_n(\mathbb{C})$ acts transitively on the set of bases for $\mathbb{C}^n$, it follows that $\mathrm{GL}_n(\mathbb{C})$ acts transitively on X. Next, it suffices to see that the stabilizer of some element of X is B (since $\mathrm{GL}_n(\mathbb{C})$ acts transitively, it doesn't matter which element). Consider the flag $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$, given by $V_i = \mathrm{Span}\{\mathbf{e}_1, ..., \mathbf{e}_i\}$, $1 \leq i \leq n$, where the $\mathbf{e}_i$ form the standard basis for $\mathbb{C}^n$. The multiplication of an element in $\mathbb{C}^n$ whose last $n-i$ entries are 0 by an element in B yields an element in $\mathbb{C}^n$ whose last $n-i$ entries are 0. Therefore, B stabilizes the flag given above. Moreover, B is the largest subgroup which stabilizes this flag, and therefore is the stabilizer of it. It follows that there is a one-to-one correspondence between elements of X and elements of the flag variety G/B for $G = \mathrm{GL}_n(\mathbb{C})$.

Since the points in X are complete flags, we may view X as being embedded in

$$\mathbb{G}(n, 1) \times \mathbb{G}(n, 2) \times \ldots \mathbb{G}(n, n-1),$$

where $\mathbb{G}(n, r)$ is the Grassmannian, consisting of all the r-dimensional linear subspaces of $\mathbb{C}^n$. With this view, we shall show that the flag variety is a projective variety. To do so, we must show that it is a Zariski closed and irreducible subset of $\mathbb{P}^N$, for some $N \in \mathbb{N}$. First we show that $\mathbb{G}(n, r) \subset \mathbb{P}^L$, where $L = \binom{n}{r} - 1$ via use of Plücker coordinates. It is clear to see that elements of $\mathbb{G}(n, r)$ may be written as $n \times r$ matrices whose columns are linearly independent, and two such matrices represent the same element of $\mathbb{G}(n, r)$ if and only if they differ by a factor of $\mathrm{GL}_r(\mathbb{C})$.

The Plücker coordinates are defined by the map ψ from $M_{n,r}$ the set of $n \times r$ matrices to $\mathbb{P}^L$, which takes an $n \times r$ matrix to the homogeneous coordinates consisting of all the possible $r \times r$ determinants within the matrix. Since the determinant of a product is equal to the product of the determinants, it follows that the Plücker coordinates of a given matrix are invariant in $\mathbb{P}^L$ under the action of $\mathrm{GL}_r(\mathbb{C})$. Thus, to each element of $\mathbb{G}(n, r)$, we may associate an element in $\mathbb{P}^L$. To see that this element is unique, we shall show that any two elements of $\mathbb{G}(n, r)$ with the same Plücker coordinates differ by a factor of $\mathrm{GL}_r(\mathbb{C})$. Indeed, by applying a change of basis - i.e. multiplying by some particular element of $\mathrm{GL}_r(\mathbb{C})-$ to a matrix A representing an element of $\mathbb{G}(n, r)$, we may assume that we can select r rows of the newly formed matrix, and upon rearranging, have them form the $r \times r$ identity matrix, since A has rank r. Without loss of generality, we may then assume that A is a matrix of the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

and show that any matrix B whose first r rows are the same as A and with a nonzero entry in one of the last $n-r$ rows will be sent to a different element of $\mathbb{P}^L$ than A by ψ. But this is immediately clear, since all but 1 of the entries in $\psi(A)$ are nonzero, while $\psi(B)$ will clearly have more than one nonzero entry. In the same vein, it is straightforward then to see that if C is another matrix whose first r rows are the same as those of A, then C and B represent the same element of $\mathbb{G}(n, r)$ if and only if $B = C$, and then one easily concludes that indeed $\mathbb{G}(n, r)$ can be embedded into $\mathbb{P}^L$. For details on the closedness of $\mathbb{G}(n, r)$ inside $\mathbb{P}^L$, see e.g. [Hud07] or [Hum75], where the idea is to show that it is closed on a set of affine open subsets of $\mathbb{P}^L$ which cover $\mathbb{P}^L$.
Now the Segre embedding allows us to then embed $\mathbb{G}(n, 1) \times \mathbb{G}(n, 2) \times \ldots \mathbb{G}(n, n-1)$ in some $\mathbb{P}^N$, where $N \in \mathbb{N}$. Hence, to see that the flag variety is a projective variety, one must show that the set of all pairs $(V_i, V_{i+1}) \in \mathbb{G}(n, i) \times \mathbb{G}(n, i+1)$ with $V_i \subset V_{i+1}$ is closed. For details, see [Hum72].

Having found the flag variety for $\mathrm{GL}_n(\mathbb{C})$, we can quite easily determine the flag variety for $\mathrm{SL}_n(\mathbb{C})$. In fact, the flag variety for $\mathrm{SL}_n(\mathbb{C})$ is the same as the flag variety for $\mathrm{GL}_n(\mathbb{C})$. This can be seen by noting that for every element b in the Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$, for every complex number λ, the matrix λb is in the Borel subgroup of $\mathrm{GL}_2(\mathbb{C})$

Now using a similar idea to the Plücker coordinates above, we can get a handle on what the base affine space - which we denote by $\widetilde{X}$ - looks like for $\mathrm{GL}_n(\mathbb{C})$ and for $\mathrm{SL}_n(\mathbb{C})$. In fact, we have an open embedding of $\mathrm{GL}_n(\mathbb{C})/N$ into some $\binom{n+1}{2}$-dimensional affine Y of $\mathbb{C}^{2^n-1}$. Indeed, we can take each of the coordinates of $\mathbb{C}^{2^n-1}$ to correspond with exactly one of the $r \times r$ minors, $1 \leq r \leq n$ consisting of elements in the r-leftmost columns of a choice of representative of a coset in $\widetilde{X}$. By augmenting any matrix $A \in \mathrm{GL}_n(\mathbb{C})$ by adjoining another copy of the first column of A to the left of the matrix A (so that the first two columns of this augmented matrix are identical), it is obvious that for $3 \leq q \leq n$, any $q \times q$ minor involving the first q columns of this augmented matrix is zero. Applying this statement for every such minor gives exactly

$$\sum_{j=3}^{n} \binom{n}{j}$$

equalities with 0 on one side and a (quadratic) polynomial in the aforementioned minors of A on the other side. From this we can conclude that differential operators on $\widetilde{X}$ must be those differential operators on $\mathbb{C}^{2^n-1}$ which stabilize each of the ideals generated by each one of those polynomials. Hence, the image of $\widetilde{X}$ lies in the intersection of the vanishing of these polynomials (identifying each relevant minor of A with its corresponding coordinate in $\mathbb{C}^{2^n-1}$). We denote this intersection Y, and observe that assuming some independence conditions, the dimension of Y is indeed

$$2^n - 1 - \sum_{j=3}^{n} \binom{n}{j} = \sum_{j=1}^{n-1} \binom{n}{j} - \sum_{j=3}^{n} \binom{n}{j} + 1 = \binom{n}{1} + \binom{n}{2} = \binom{n+1}{2},$$

as we should expect.

The approach in the case of $\mathrm{SL}_n(\mathbb{C})$ is almost identical, except that since the determinant of any matrix in $\mathrm{SL}_n(\mathbb{C})$ is 1, the coordinate corresponding to the $n \times n$ minor will be fixed, and so the dimension of everything above will be reduced by 1.

Remark 2.6.3. *From the above characterizations, we immediately see that in the case of* $\mathrm{SL}_2(\mathbb{C})$, *the flag variety is* $\mathbb{P}^1$ *and the base affine space is* $\mathbb{C}^2 \setminus \{(0,0)\}$.

Thus, we have shown the following more general theorem in the case that $G = \mathrm{SL}_n(\mathbb{C})$ and $G = \mathrm{GL}_n(\mathbb{C})$.

Theorem 2.6.4. *Let G be a finite dimensional algebraic variety.*

i) The flag variety G/B is projective;

ii) The base affine space G/N is quasi-affine.

Chapter 3

Category $\mathcal{O}$, Verma modules, and representations of infinitesimal character

In the following, $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$, denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Throughout, we identify the categories of $\mathfrak{g}$-modules, $\mathcal{U}(\mathfrak{g})$-modules and representations of $\mathfrak{g}$, and we denote these categories by $\mathcal{M}od(\mathfrak{g})$. There is a Borel subalgebra $\mathfrak{b} = \mathfrak{b}^+$ and an opposite Borel subalgebra $\mathfrak{b}^-$ so that $\mathfrak{h} = \mathfrak{b}^+ \cap \mathfrak{b}^-$ is a Cartan sub-algebra. Let $\mathfrak{n}^\pm = [\mathfrak{b}^\pm, \mathfrak{b}^\pm]$ be the nilpotent radical of $\mathfrak{b}^\pm$. We will often write $\mathfrak{n} = \mathfrak{n}^+$. We can decompose $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. It is well-known that $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$. Let $\mathfrak{h}^*$ denote the dual of $\mathfrak{h}$, and fix a root system Φ of $\mathfrak{h}^*$, with positive roots $\Phi^+ \subset \Phi$, and simple roots $\Delta \subset \Phi^+$. For each root $\alpha \in \Phi$, there is the corresponding 1-dimensional subspace of $\mathfrak{g}$,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}.$$

Given a root system, we can choose the Borel subalgebra $\mathfrak{b}$ so that its nilpotent radical $\mathfrak{n}$ decomposes as

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

and $\mathfrak{n}^-$ hence decomposes as

$$\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

For any positive root $\alpha \in \Phi^+$, we can find $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] \in \mathfrak{h}$. Indeed, this follows from the Jacobi identity and the fact that for $g \in \mathfrak{g}$, $[h, g] = 0$ for all $h \in \mathfrak{h}$ implies $g \in \mathfrak{h}$ (this follows from our decomposition of $\mathfrak{g}$), as

$$\begin{aligned} [h, [x_\alpha, y_\alpha]] &= [x_\alpha, [h, y_\alpha]] - [y_\alpha, [h, x_\alpha]] \\ &= [x, -\alpha(h)y] - [y, \alpha(h)x] \\ &= -\alpha(h)([x, y] + [y, x]) \\ &= 0. \end{aligned}$$

Moreover, by scaling x_α and y_α, we may take the element $h_\alpha := [x_\alpha, y_\alpha]$ to be such that $\alpha(h_\alpha) = 2$. Evidently, $(x_\alpha, y_\alpha, h_\alpha)$ form an $\mathfrak{sl}_2$-triple.

Let Γ be the set of $\mathbb{Z}_+$-linear combinations of simple roots, and $\Gamma_\mathbb{R}$ be the set of $\mathbb{R}_+$-linear combinations of simple roots. There is a partial order on $\mathfrak{h}^*$ defined by $\lambda \leq \mu$ if and only if $\mu - \lambda \in \Gamma_\mathbb{R}$. Let W denote the Weyl group of $\mathfrak{g}$, i.e. the group generated by the reflections with respect to each of the roots in Φ, and let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

We call ρ the *Weyl vector*. The Weyl group W has a natural action $\lambda \mapsto w(\lambda)$ on $\mathfrak{h}^*$, and we can also define the so-called dot-action of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$, for $\lambda \in \mathfrak{h}^*$, $w \in W$. It is straightforward to see that this dot-action has a unique fixed point $-\rho$. For a fixed $\lambda \in \mathfrak{h}^*$, we denote by $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$, the isotropy group with respect to the dot action.

In case $\mathfrak{g} = \mathfrak{sl}_2$, we may take $\mathfrak{n} = \text{Span}\{E\}$, $\mathfrak{n}^- = \text{Span}\{F\}$, $\mathfrak{h} = \text{Span}\{H\}$, where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, $\mathfrak{h}^*$ is a 1-dimensional vector space so can be identified with $\mathbb{C}$, and we can take $\Phi = \{-2, 2\}$ and $\Delta = \Phi^+ = \{2\}$. The Weyl vector is therefore $\rho = 1$. The partial order on $\mathfrak{h}^*$ described above is that for which $a + bi \leq c + di$ if and only if $a \leq c$ and $b = d$, where $a, b, c, d \in \mathbb{R}$ and i is the imaginary unit. The Weyl group W here is the 2-element group $\{\text{id}, s_2\}$, where s_2 denotes the reflection across 0 - i.e. for any $\lambda \in \mathfrak{h}^*$, $s_2(\lambda) = -\lambda$. The dot-action of W therefore corresponds with the identity and a reflection across $-\rho = -1$. The isotropy group $W_{-1} = W$, and for every $\lambda \in \mathfrak{h}^* \setminus \{-1\}$, the isotropy group W_λ is trivial.

3.1 Category $\mathcal{O}$

Definition 3.1.1. Let M be a $\mathfrak{g}$-module, and $\lambda \in \mathfrak{h}^*$. The *weight space* corresponding to λ is the vector space

$$M_\lambda := \{v \in M \mid h \cdot v = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}.$$

If $M_\lambda \neq 0$, then we say that λ is a *weight* of M. We call M a *weight module* if it decomposes into a direct sum of weight spaces (i.e. $\mathfrak{h}$ acts semisimply on M).

Definition 3.1.2. We call a weight $\lambda \in \mathfrak{h}^*$

(i) *Regular* if W_λ is trivial, and otherwise we call it *singular*.
(ii) *Integral* if $\alpha^\vee(\lambda) \in \mathbb{Z}$ for any root $\alpha \in \Phi$, where $\alpha^\vee$ is the corresponding coroot of α.
(iii) *Dominant* if $\alpha^\vee(\lambda + \rho) \geq 0$ for every positive root $\alpha \in \Phi^+$.

In the case of $\mathfrak{g} = \mathfrak{sl}_2$, the weight -1 is a singular weight, and every other weight is regular. The integral weights are precisely the integers.

Definition 3.1.3. Say that a $\mathfrak{g}$-module M is locally $\mathfrak{n}$-finite if for every $v \in M$, the vector subspace $\mathcal{U}(\mathfrak{n}) \cdot v$ is finite dimensional.

Definition 3.1.4. The *BGG category* $\mathcal{O}$ is the full subcategory of $\mathcal{M}od(\mathfrak{g})$ consisting of objects M satisfying the following:

1. M is finitely generated over $\mathcal{U}(\mathfrak{g})$;
2. M is a weight module.
3. M is locally $\mathfrak{n}$-finite

Definition 3.1.5. Let M be a $\mathfrak{g}$-module. Say that $v_m \in M$ is a *maximal vector of weight* λ if $v_m \in M_\lambda$ and $\mathfrak{n} \cdot v_m = 0$.

It is clear from the definition of the category $\mathcal{O}$ that every object in $\mathcal{O}$ contains a maximal vector.

Definition 3.1.6. We call M a *highest weight module (of weight λ)* if it is generated over $\mathcal{U}(\mathfrak{g})$ by a maximal vector v_m for which $h \cdot v_m = \lambda(h)v_m$, for any $h \in \mathfrak{h}$.

Remark 3.1.7. *If M is a highest weight module generated by v_m, i.e. $M = \mathcal{U}(\mathfrak{g}) \cdot v_m$, then it follows from the PBW theorem that $M = \mathcal{U}(\mathfrak{n}^-) \cdot v_m$.*

Proposition 3.1.8. *Let M be a highest weight module of weight λ.*

(i) The dimension of any weight space M_μ is finite and, in particular, the dimension of M_λ is 1.
(ii) Any submodule of M is a weight module. In particular, M is a weight module;
(iii) Any nonzero quotient of M is a highest weight module;
(iv) M has a unique maximal submodule;
(v) M has a unique simple quotient;
(vi) If M is simple, then any highest weight module of weight λ is isomorphic to M.

The proof can be found in [Hum75] (p. 16-17). From this proposition, we see the following nice property of modules in the category $\mathcal{O}$.

Proposition 3.1.9. *Every module M in the category $\mathcal{O}$ has a finite filtration whose nonzero quotients are all highest weight modules.*

Proof. Since $M \in \text{ob}\,\mathcal{O}$, it can be seen to be generated by finitely many weight vectors $v_1, v_2, \ldots, v_n$ and the $\mathfrak{n}$-submodule generated by those vectors is finite dimensional. If this $\mathfrak{n}$-submodule has dimension 1, then M is generated by a single highest weight vector, and the result is immediate, so assume the dimension is greater than 1. Since M is generated by finitely many weight vectors, we can find one - in fact we can assume without loss of generality that it is v_1 - whose weight is greater than all those with which it is comparable. Then the $\mathfrak{g}$-module M_1 generated by v_1 is a highest weight module. The quotient module M/M_1 is isomorphic to a module contained in the $\mathfrak{g}$-module generated by the weight vectors $v_2, \ldots, v_n$. Since v_1 had maximal weight among the $v_1, v_2, \ldots, v_n$, the $\mathfrak{n}$-submodule generated by the $v_2, \ldots, v_n$ has strictly smaller dimension that the one generated by the $v_1, v_2, \ldots, v_n$. The result then follows by induction. □

Definition 3.1.10. For any $\lambda \in \mathfrak{h}^*$, denote by $\mathbb{C}_\lambda$ the 1-dimensional left $\mathfrak{b}$-module on which the $\mathfrak{b}$-action factors through $\mathfrak{b} \to \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$. We define the *Verma module* to be the $\mathfrak{g}$-module

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

This is a highest weight module of weight λ, and we shall denote the unique maximal submodule of $M(\lambda)$ by $N(\lambda)$ and the unique simple quotient of $M(\lambda)$ by $L(\lambda)$.

We may alternatively describe Verma modules as follows. Let $I \subset \mathcal{U}(\mathfrak{g})$ denote the left ideal which annihilates the maximal vector $v_m \in M(\lambda)$. By definition, we have $\mathfrak{n} \subset I$ and $h - \lambda(h) \in I$ for every $h \in \mathfrak{h}$. On the other hand, if $a \in \mathcal{U}(\mathfrak{g})$ does not annihilate v_m, then by the PBW theorem, we may write a as a polynomial in the generators of $\mathcal{U}(\mathfrak{n}^-)$ left multiplying a polynomial in the generators of $\mathcal{U}(\mathfrak{h})$ containing no factors of $h - \lambda(h)$ for $h \in \mathfrak{h}$. Hence, I is precisely the left ideal generated by $\mathfrak{n}$ and elements $h - \lambda(h)$, $h \in \mathfrak{h}$.

One can check, by the PBW theorem, that as a $\mathcal{U}(\mathfrak{n}^-)$ module, the Verma module $M(\lambda)$ is free of rank 1. Hence, the foregoing discussion implies that $M(\lambda) \simeq \mathcal{U}(\mathfrak{g})/I$. Since I annihilates any maximal vector of weight λ in an arbitrary highest weight module, this shows that the Verma module $M(\lambda)$ maps surjectively onto any highest weight module of weight λ. By Proposition 3.1.8 (vi), and since every object in category $\mathcal{O}$ contains a maximal weight vector, it follows that every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$, for some $\lambda \in \mathfrak{h}^*$. We can also restate Proposition 3.1.9 to say that modules in category $\mathcal{O}$ have a finite filtration whose composition factors are quoteints of Verma modules.

Next, we would like to define a notion of duality in the category $\mathcal{O}$, so that it is closed under taking duals. Suppose M is a $\mathfrak{g}$-module in $\mathcal{O}$, and let $M^* := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ be the dual vector space. We would like to define a $\mathfrak{g}$-action on M^* so that as a $\mathfrak{g}$-module, M^* is an object in $\mathcal{O}$. The naïve approach of taking the natural action $(x \cdot f)(v) = -f(x \cdot v)$, for $x \in \mathfrak{g}$, $f \in M^*$ and $v \in M$ fails if M is infinite dimensional. Indeed, one can see that in this case M^* would not be locally $\mathfrak{n}$-finite, nor would it necessarily be a weight module. Instead, we proceed as follows: for each simple root $\alpha \in \Delta$, we can choose an $\mathfrak{sl}_2$-triple $(x_\alpha, y_\alpha, h_\alpha)$, where $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ with $\alpha(h_\alpha) = 2$. There exists an anti-involution τ on $\mathfrak{g}$ so that for every simple root α,

$$\tau(x_\alpha) = y_\alpha, \quad \tau(y_\alpha) = x_\alpha, \quad \tau(h_\alpha) = h_\alpha,$$

and this anti-involution extends to one on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Hence, we have an action of $\mathcal{U}(\mathfrak{g})$ on M^* via

$$(x \cdot f)(v) = f(\tau(x) \cdot v),$$

and in the sequel, we shall view M^* as a $\mathfrak{g}$-module with this action. Given $\lambda \in \mathfrak{h}^*$, the dual $(M_\lambda)^*$ of M_λ can be embedded into the M^* by identifying each $f \in (M_\lambda)^*$ with the element in M^* which agrees with f on M_λ and is 0 elsewhere. The image of this embedding is then $(M^*)_\lambda$, and henceforth we shall write both $(M_\lambda)^*$ and $(M^*)_\lambda$ simply as M^*_λ. By considering the weight spaces of M^*, we can then obtain a weight module

$$M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^*} M^*_\lambda.$$

It is a routine check, using the $\mathfrak{g}$-action as defined above, to see that this module is locally $\mathfrak{n}$-finite. It turns out to be true also that $M^\vee$ is finitely generated - this follows from the fact that we can define an exact functor on the subcategory of $\mathfrak{g}$-modules consisting of weight modules whose weight spaces are finite dimensional which sends $M \mapsto M^\vee$; exactness of this functor implies that since if $M \in \mathrm{ob}\,\mathcal{O}$, then $M^\vee$ is finitely generated. Hence $M^\vee$ is in category $\mathcal{O}$, and we call $M^\vee$ the dual of M in $\mathcal{O}$. We see that

$$M^{\vee\vee} = \bigoplus_{\lambda \in \mathfrak{h}^*} M^{**}_\lambda \simeq \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda = M,$$

and so we may identify $M^{\vee\vee}$ with M.

In particular, given any Verma module $M(\lambda)$ of maximal weight λ, there is a corresponding *dual Verma module* $M(\lambda)^\vee$. We state, without proof, the following proposition which is dual to part of Proposition 3.1.8

Proposition 3.1.11. *The dual Verma module $M(\lambda)^\vee$ has a unique maximal quotient and a unique simple submodule.*

By our characterization of simple modules in $\mathcal{O}$, we see that the unique simple submodule of $M(\lambda)^\vee$ is $L(\lambda)$. The unique maximal quotient is therefore isomorphic to $N(\lambda)$.

3.2 Modules of infinitesimal (central) character

Definition 3.2.1. An algebra morphism $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is called an *central character.*

Since this central character is associated with the Lie algebra (as opposed to a Lie group or an algebraic group), it is often referred to as an *infinitesimal character.*

For a highest weight module $M \in \operatorname{ob}\mathcal{O}$ of weight λ, the dimension of the maximal weight space M_λ is 1, so we shall write $M_\lambda = \operatorname{Span}(v_m)$. If $z \in \mathcal{Z}(\mathfrak{g})$, then

$$h \cdot z \cdot v_m = z \cdot h \cdot v_m = \lambda(h) z \cdot v_m.$$

Hence, $z \cdot v_m \in \operatorname{Span}(v_m)$, so we may write $z \cdot v_m = \chi_\lambda(z) v_m$, for some $\chi_\lambda(z) \in \mathbb{C}$. Moreover, since z commutes with every element of $\mathcal{U}(\mathfrak{g})$ and M is generated by v_m, we have $z \cdot v = \chi_\lambda(z) v$, for every $v \in M$. One can check that $\chi_\lambda : z \mapsto \chi_\lambda(z)$ defines an algebra morphism from $\mathcal{Z}(\mathfrak{g})$ to $\mathbb{C}$, and hence is a central character. Noting that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n})$, and since monomials in $\mathfrak{n}$ will kill v_m and monomials in $\mathfrak{n}^-$ will lower the weight of v_m, we conclude that the value of $\chi_\lambda(z)$ depends upon only the monomials in $\mathfrak{h}$. In fact, letting pr denote the projection from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{h})$ defined by setting non-constant monomials in $\mathcal{U}(\mathfrak{n})$ and $\mathcal{U}(\mathfrak{n}^-)$ to zero, it can be seen that $\chi_\lambda(z) = \lambda(\operatorname{pr}(z))$, for $z \in \mathcal{Z}(\mathfrak{g})$.

Definition 3.2.2. Let $\lambda \in \mathfrak{h}^*$ and let M be a $\mathfrak{g}$-module. We say that M is a *module of infinitesimal character* χ_λ if for every $z \in \mathcal{Z}(\mathfrak{g})$ and for every $v \in M$, we have $z \cdot v = \chi_\lambda(z) v$, where χ_λ is as defined above.

In what follows, we identify $\mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, and let $\mathbb{C}[\mathfrak{h}^*]^W$ denote the dot-invariant polynomials in $\mathfrak{h}^*$. For now, we shall take the following theorem for granted. The proof can be found in [Hum75] (p. 26).

Theorem 3.2.3.
1. *There is an isomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^W$, called the* Harish-Chandra isomorphism*, obtained by restricting the projection* $\operatorname{pr} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$;
2. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then $\lambda = w \cdot \mu$ for some $w \in W$ if and only if $\chi_\lambda = \chi_\mu$;*
3. *Every central character is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$.*

Definition 3.2.4. We say a character is *dominant, regular, integral, etc.* if the corresponding weight is respectively dominant, regular, integral, etc.

Henceforth, we shall identify $\mathcal{Z}(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{h}^*]^W$. The Harish-Chandra isomorphism then allows us to describe a character as an algebra morphism

$$\mathbb{C}[\mathfrak{h}^*]^W = S(\mathfrak{h})^W \xrightarrow{\sim} \mathcal{Z}(\mathfrak{g}) \to \mathbb{C},$$

where $\mathbb{C}[\mathfrak{h}^*]^W$ are the dot-invariant polynomials on $\mathfrak{h}^*$. Now we may identify $\operatorname{Specm}(\mathbb{C}[\mathfrak{h}^*]^W)$ with $\mathfrak{h}^*/W$ the space of W-orbits of $\mathfrak{h}^*$ (under the dot-action), and so choosing an infinitesimal character amounts to choosing a W-orbit of $\mathfrak{h}^*$, equivalently choosing a maximal ideal in $\mathbb{C}[\mathfrak{h}^*]^W$.

Now for any $\lambda \in \mathfrak{h}^*$, we shall denote by $|\lambda|$ the dot-orbit of λ, and let $I_{|\lambda|}$ denote the maximal ideal of polynomials in $\mathbb{C}[\mathfrak{h}^*]^W$ vanishing at $|\lambda|$. Then a module M has infinitesimal character χ_λ if and only if $I_{|\lambda|} M = 0$. This motivates the following definition.

Definition 3.2.5. Let $\lambda \in \mathfrak{h}^*$ and let M be a $\mathfrak{g}$-module. We say that M is a *module of generalized infinitesimal character* χ_λ if $I^n_{|\lambda|} M = 0$, for some $n \geq 1$. Denote by $\mathcal{M}od_{|\lambda|}(\mathcal{U}(\mathfrak{g}))$ the category of $\mathfrak{g}$-modules of generalized infinitesimal character χ_λ.

There is a similar category of modules which we may define as follows. First we remark that the injection $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^*]$ induces a projection $\pi : \mathfrak{h}^* = \operatorname{Specm}(\mathbb{C}[\mathfrak{h}^*]) \to \operatorname{Specm}(\mathcal{Z}(\mathfrak{g})) = \mathfrak{h}^*/W$. Denoting the maximal ideal in $\mathbb{C}[\mathfrak{h}^*]$ of polynomials vanishing at some fixed $\lambda \in \mathfrak{h}^*$, by $\mathcal{J}_\lambda$. It is clear that $\pi^{-1}(\mathcal{J}_\lambda) = I_{|\lambda|}$. It is also clear to see that $\mathbb{C}[h^*]$ has the structure of a $\mathcal{Z}(\mathfrak{g})$-module.

Definition 3.2.6. The *extended enveloping algebra* is the algebra $\widetilde{\mathcal{U}} := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathbb{C}[h^*]$.

We shall see that this definition is rather nice in the view of differential operators of the base affine space. We remark that the center of $\widetilde{\mathcal{U}}$ is $\mathbb{C}[\mathfrak{h}^*]$, since $\mathcal{Z}(\mathfrak{g}) \subset \mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{C}[\mathfrak{h}^*]$ is commutative. Moreover, since we have identified $\mathbb{C}[\mathfrak{h}^*]^W$ with $\mathcal{Z}(\mathfrak{g})$, we see that $\widetilde{\mathcal{U}}^W = \mathcal{U}(\mathfrak{g})$, where the W-action on $\widetilde{\mathcal{U}}$ is induced by the W-action on $\mathbb{C}[\mathfrak{h}^*]$. Now let $\mathcal{M}od_\lambda(\widetilde{\mathcal{U}})$ denote the category of $\widetilde{\mathcal{U}}$-modules M such that $\mathcal{J}^n_\lambda M = 0$ for some $n \geq 1$. Any $\widetilde{\mathcal{U}}$-module can be given the structure of a $\mathcal{U}(\mathfrak{g})$-module by restriction of the action.

Additionally, since $\pi^{-1}(\mathcal{J}_\lambda) = I_{|\lambda|}$, if M is a $\widetilde{\mathcal{U}}$-module with $\mathcal{J}_\lambda^n M = 0$ and $\operatorname{Res} M$ is its restriction to a $\mathcal{U}(\mathfrak{g})$-module, then $I_{|\lambda|}^n(\operatorname{Res} M) = 0$. Thus, we have an exact functor $\operatorname{Res}_\lambda : \mathcal{M}od_\lambda(\widetilde{\mathcal{U}}) \to \mathcal{M}od_{|\lambda|}(\mathcal{U}(\mathfrak{g}))$.
So far, we have the inclusion $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^*]$ of W-invariant polynomials in $\mathfrak{h}^*$ into the algebra of all polynomials in $\mathfrak{h}^*$. For a fixed $\lambda \in \mathfrak{h}^*$, the isotropy group $W_\lambda \subset W$, and so any polynomial which is W-invariant is necessarily W_λ-invariant. We may then factor the above inclusion as

$$\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \hookrightarrow \mathbb{C}[\mathfrak{h}^*],$$

and the corresponding projection $\pi : \mathfrak{h}^* \to \mathfrak{h}^*/W$ on the maximal spectra may be factored through

$$\mathfrak{h}^* \to \mathfrak{h}^*/W_\lambda \to \mathfrak{h}^*/W.$$

The first arrow sends the maximal ideal $\mathcal{J}_\lambda$ to $\mathcal{J}_\lambda^{W_\lambda} := \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \cap \mathcal{J}_\lambda$, which can be seen to be the maximal ideal W_λ-invariant polynomials which vanish at $w \cdot \lambda$ for every $w \in W_\lambda$. Indeed, this is clear, since any W_λ-invariant polynomial vanishing at λ must vanish at $w \cdot \lambda$ whenever $w \in W_\lambda$. We may consider the intermediate algebra $\widetilde{\mathcal{U}}^{W_\lambda} \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$, and in particular, we have the category $\mathcal{M}od_\lambda(\widetilde{\mathcal{U}}^{W_\lambda})$ of $\widetilde{\mathcal{U}}^{W_\lambda}$-modules M for which $(\mathcal{J}_\lambda^{W_\lambda})^n M = 0$ for some $n \geq 1$. Naturally, we have an exact functor

$$\mathcal{M}od_\lambda(\widetilde{\mathcal{U}}^{W_\lambda}) \to \mathcal{M}od_{|\lambda|}(\mathcal{U}(\mathfrak{g}))$$

via restriction of the module structure. According to Beilinson and Ginzburg's paper [BG97] (p. 4), by considering the completions of the algebras $\mathcal{Z}(\mathfrak{g})$ and $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$ with respect to the ideals $I_{|\lambda|}$ and $\mathcal{J}_\lambda^{W_\lambda}$ respectively, this functor can be seen to be an equivalence of categories. Hence, if λ is a regular weight (i.e. W_λ is trivial), the category $\mathcal{M}od_\lambda(\widetilde{\mathcal{U}})$ is equivalent to the category $\mathcal{M}od_{|\lambda|}(\mathcal{U}(\mathfrak{g}))$, and so we may study modules with generalized infinitesimal character χ_λ by studying the category $\mathcal{M}od_\lambda(\widetilde{\mathcal{U}})$.

Chapter 4

D-modules

4.1 Rings of Differential Operators

Here, we shall build up to a general definition of a (sheaf of) rings (or algebras) of differential operators via some concrete and more intuitive examples. In this first section, we closely follow [Sch19]. To begin, let k be a field, and consider the vector space $k[x_1, ..., x_n]$ of polynomials in n indeterminates over k. We have the vector space endomorphisms of $k[x_1, \ldots, x_n]$ given by $x_i : f \mapsto x_i f$ and $\partial_i : f \mapsto \partial_i f$, $1 \leq i \leq n$, where $\partial_i f$ is the formal derivative of $f \in k[x_1, ..., x_n]$ with respect to x_i. These endomorphisms can be added, scalar multiplied and composed. The composition follows the commutation relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$, and $[\partial_i, x_j] = \delta_{ij}$.

Definition 4.1.1. The subset of $\mathrm{End}_k(k[x_1, ..., x_n])$ generated by k, x_i and ∂_i form a noncommutative algebra, $A_n = k[x_1, ..., x_n, \partial_1, ..., \partial_n]$, called the *Weyl algebra*.

Remark 4.1.2. *It can be seen that the Weyl algebra is the subalgebra of* $\mathrm{End}_k(k[x_1, \ldots, x_n])$ *generated by* $k[x_1, \ldots, x_n]$ *and* $\mathrm{Der}\, k[x_1, \ldots, x_n]$. *This follows since any derivation* δ *is uniquely determined by an n-tuple of elements* $(s_1, \ldots, s_n)$ *in* $k[x_1, \ldots, x_n]$ *such that* $\delta(x_i) = s_i$, $i = 1, \ldots, n$, *and so we may write* $\delta = \sum s_i \frac{d}{dx_i}$.

In the same way that vector spaces are of use in solving systems of linear equations, one might guess that the modules over the Weyl algebra will have use in solving systems of linear partial differential equations. In the following, we shall describe this more precisely. Recall that associated to the system of linear equations

$$\sum_{j=1}^{m} a_{ij} v_j = 0, \quad i = 1, ..., n \tag{4.1}$$

is the $n \times m$ matrix (a_{ij}), which defines a linear map $a : k^n \to k^m$ by right multiplication on the transpose. Writing $V = \operatorname{coker} a = k^m / \operatorname{im} a$, then $\mathrm{Hom}_k(V, k)$ gives a space of solutions to (1). Indeed, if $f : k^m \to k$ such that $f \circ a = 0$, then f uniquely determines an element in $\mathrm{Hom}_k(V, k)$, and moreover, since f is fully determined by how it maps basis elements of k^m, we may write

$$f(y_1, ..., y_m) = \sum_{j=1}^{m} y_j f_j, \quad y_j, f_j \in k, \quad 1 \leq j \leq m.$$

Then $f \circ a = 0$ implies that for every $(x_1, ..., x_n) \in k^n$,

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} x_i a_{ij} f_j = 0,$$

which holds if and only if $\sum_{j=1}^{m} a_{ij} f_j = 0$ for $1 \leq i \leq n$.

Now consider in equation (1), we replace the $a_{ij} \in k$ by elements $P_{ij} \in A_r$, and assume the elements v_j, rather than being elements of k are elements of some suitable A_r-module, giving the equation

$$\sum_{j=1}^{m} P_{ij} v_j = 0, \quad 1 \leq i \leq n.$$

Then we can essentially take the same approach as above; the matrix (P_{ij}) defines a map $P : A_r^n \to A_r^m$ by right multiplication, and then given an A_r-module S, the space of solutions in S will be given by $\mathrm{Hom}_{A_r}(\operatorname{coker} P, S)$. In such a way, one can nicely study the solutions to linear partial differential equations by studying (finitely generated) A_r-modules.

While being more complicated than, for instance, the commutative algebra $k[x_1, ..., x_n]$ due to its noncommutativity, there are quite a few properties that allow us to get more of a handle on the Weyl algebra A_n. Before we remark upon these, we must first introduce some notation.

Definition 4.1.3. Given an object S in a category $\mathcal{C}$, an *increasing (resp. decreasing) filtration* on S is a collection of subobjects $(S_i)_{i\in\mathcal{I}}$ of S indexed over a totally ordered index set $\mathcal{I}$ such that if $i \leq j$, then $S_i \subset S_j$ (resp. $S_j \subset S_i$).

Definition 4.1.4. A *filtered k-algebra* is an algebra R such that there is an increasing filtration (of vector subspaces) of R,

$$\{0\} = R_{-1} \subset R_0 \subset R_1 \subset R_2 \subset \cdots \subset R$$

such that $R = \bigcup_i R_i$, and for any $i, j \in \mathbb{N}$, $R_i \cdot R_j \subset R_{i+j}$.

Remark 4.1.5. *In particular, in the definition above, note that R_0 forms an algebra since $R_0 \cdot R_0 \subset R_0$. Moreover every R_i is an R_0-module.*

Definition 4.1.6. Let R be a commutative k-algebra. The *algebra of differential operators* $\mathrm{Diff}(R)$ on R is the subalgebra of $\mathrm{End}_k(R)$ defined inductively as follows: Let $D_0 = R$, and for $i \geq 1$, put

$$D_i = \{d \in \mathrm{End}_k(R) \,|\, rd - dr \in D_{i-1}, \text{ for all } r \in R\}.$$

Then $\mathrm{Diff}(R) = \bigcup_i D_i$.

Remark 4.1.7. *It can readily be seen that this definition gives $\mathrm{Diff}(R)$ the structure of a filtered algebra, since*

$$\{0\} \subset R = D_0 \subset D_1 \subset D_2 \subset \cdots \subset \bigcup_i D_i = \mathrm{Diff}(R),$$

and it can be shown by induction that $D_i \cdot D_j \subset D_{i+j}$, for if $d_i \in D_i$, $d_j \in D_j$, then

$$\begin{aligned} rd_id_j - d_id_jr &= rd_id_j - d_ird_j + d_ir_j - d_id_jr \\ &= (rd_i - d_ir)d_j + d_i(rd_j - d_jr). \end{aligned}$$

By replacing $\mathrm{End}_k(R)$ by a more general k-algebra in Definition 4.1.6, we have the following more general definition.

Definition 4.1.8. Let R be a commutative k-algebra, and A a k-algebra with an R-bimodule structure and a morphism $\iota : R \to A$. Define a filtration $A^\vee_\bullet$, where $A^\vee_{-1} = \{0\}$, and for $i \geq 1$,

$$A^\vee_i := \{m \in A \,|\, rm - mr \in A^\vee_{i-1}, \text{ for all } r \in R\}.$$

We say that A is an *R-differential algebra* if $A = \bigcup_i A^\vee_i$.

Next, we give some examples of filtered algebras.

Example 4.1.9. i) The polynomial algebra $R = k[x_1, ..., x_n]$, is a filtered algebra by taking

$$A_j = \left\{ \sum x_1^{\alpha_1} \cdots x_n^{\alpha_n} \,\middle|\, \sum_{i=1}^{n} \alpha_i \leq j \right\}.$$

In a similar vein to above, if $R = A_n$, then there are the following three filtrations:

ii) $R_j = \{\sum x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \,|\, \sum_{i=1}^n \alpha_i \leq j\}$.

iii) $R_j = \{\sum x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \,|\, \sum_{i=1}^n \beta_i \leq j\}$.

iv) $R_j = \{\sum x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \,|\, \sum_{i=1}^n (\alpha_i + \beta_i) \leq j\}$.

We remark that (iii) above coincides with the filtration obtained for $A_n = \mathrm{Diff}(k[x_1, \ldots, x_n])$ from Definition 4.1.6. In the above examples, the filtrations are defined so that the elements in R_j have degree (in some sense of the word) less than or equal j. If we are interested in distinguishing elements by degree, then we ought to consider quotients, e.g. $\mathrm{gr}_j R := R_j/R_{j-1}$ would be isomorphic to the subspace containing only the elements of R with degree equal to j (as well as 0 of course). We see that if $a \in R_i$, $b \in R_j$ then $aR_{j-1} \subset R_{i+j-1}$, and $bR_{i-1} \subset R_{i+j-1}$, so $(a+R_{i-1})(b+R_{j-1}) = ab+R_{i+j-1}$ is well defined and hence $\mathrm{gr}_j R \cdot \mathrm{gr}_i R \subset \mathrm{gr}_{i+j} R$.

Definition 4.1.10. We may take the direct sum of all of the $\mathrm{gr}_j R$ to obtain an algebra

$$\mathrm{gr}\, R = \bigoplus \mathrm{gr}_j R$$

called the *associated graded algebra* (to the filtration $R_\bullet$).

The following proposition suggests an immediate interest in the associated graded algebra for the Weyl algebra.

Proposition 4.1.11. *Given any of the filtrations (ii), (iii), (iv) from Example 4.1.9 of the Weyl algebra A_n, the associated graded algebra is commutative, and in fact isomorphic to the polynomial ring in $2n$ indeterminates over the field k.*

Proof. The proof follows from the relation $[x_i, \partial_j] = \delta_{ij} \in F_0$, and from the fact that in the case of (iii) (respectively (iv), (v)) the elements of any grade $\mathrm{gr}_j A_n$ are of the form $f + F_{j-1}$, where f is a degree j homogeneous polynomial in the x_i (respectively the ∂_i, the x_i and ∂_i) with coefficients in $k[\partial_1, ..., \partial_n]$ (respectively $k[x_1, ..., x_n]$, k). In particular, the generators for $\mathrm{gr}\, A_n$ commute. □

Thus, by considering the associated graded algebra of A_n with respect to a well-chosen grading as in the examples, we can hope to be able to utilize tools from commutative algebra in order to get a better handle on the noncommutative Weyl algebra. Now just as we may define a filtration and an associated graded algebra for a k-algebra R, we may define a compatible filtration and an associated graded module for an R-module M.

Definition 4.1.12. Let R be a filtered k-algebra, with filtration $R_\bullet$, and M a (left) R-module[1]. A *compatible filtration* $M_\bullet$ of M is an increasing filtration (of vector subspaces) of M

$$\{0\} = M_{-1} \subset M_0 \subset M_1 \subset ...$$

such that $R_i \cdot M_j \subset M_{i+j}$, and $M = \bigcup M_i$.

Definition 4.1.13. Let M and R be as in the above definition. Defining $\mathrm{gr}_j M = M_j/M_{j-1}$, we may form the *associated graded module*

$$\mathrm{gr}\, M = \bigoplus \mathrm{gr}_j M.$$

Given any $r \in R_i$, $m \in M_j$, the product $(r + R_{i-1})(m + M_{j-1}) = rm + M_{i+j-1}$ is well-defined and gives $\mathrm{gr}\, M$ the structure of a $\mathrm{gr}\, R$-module.

Definition 4.1.14. We call the filtration $M_\bullet$ *good* if $\mathrm{gr}\, M$ is finitely generated over $\mathrm{gr}\, R$.

Observe that just as the R_i in the filtration of a ring are all R_0-modules, all the M_i in the compatible filtration of the R-module M are R_0-modules.

Proposition 4.1.15. *The Weyl algebra $A_n = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ is simple and noetherian.*

Proof. Let $0 \neq P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta$, $\alpha, \beta \in \mathbb{N}^n$, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $c_{\alpha,\beta} \in k$ and the sum is taken over the α_i, β_j, $1 \leq i, j \leq n$. Denote by $e_j \in \mathbb{N}^n$ the element whose jth entry is 1 and all other elements 0. We have the commutator relations

$$[\partial_j, x^\alpha \partial^\beta] = \alpha_j x^{\alpha - e_j} \partial^\beta,$$

$$[x_j, x^\alpha \partial^\beta] = -\beta_j x^\alpha \partial^{\beta - e_j}.$$

The elements $[\partial_j, P]$ and $[x_j, P]$ are in the two-sided ideal generated by P, and so it follows immediately that 1 is in the two-sided ideal generated by P, and hence the two-sided ideal generated by P is A_n. Since $P \neq 0$ was arbitrarily chosen, it follows that A_n has no proper nonzero ideals.
The detailed proof that A_n is noetherian may be found in [Sch19] (lecture 2), and here we just give a rough outline. The idea is to show that given any finitely generated A_n-module M, every submodule $N \subset M$ is finitely generated. Considering either of the filtrations (iv) or (v) of A_n in the above example, M admits a good filtration since it is finitely generated. One can check that the filtration $N_j = N \cap M_j$ defines a good filtration of N, and therefore N is finitely generated over A_n. □

[1] we can just as easily work a definition for right R-modules

4.2 Modules over the Weyl Algebra

The discussion in the previous section shows that solutions of partial differential equations can be seen as modules over the Weyl algebra A_n, and such modules provide our initial examples of D-modules. Naturally, the Weyl algebra itself is an A_n-module, and since A_n is a subring of the ring of endomorphisms of $k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]$ is an A_n-module, where the x_i act by multiplication and ∂_i act by partial differentiation on x_i. There are other examples of A_n-modules such as $k[x_1]$, where x_1 and ∂_1 act as above and for $i \geq 2$, the x_i and ∂_i annihilate everything. We see that in this case,

$$k[x_1] \simeq \frac{k[x_1, \ldots, x_n]}{\langle x_2, \ldots, x_n \rangle} \simeq \frac{A_n}{\langle x_2 \ldots x_n, \partial_1, \ldots, \partial_n \rangle}.$$

In fact, by considering quotients of A_n by ideals of polynomials (in both the x_i and the ∂_i), we are provided with various interesting A_n-modules. We may also consider Laurent polynomials and quotients of those for more examples of A_n-modules. Some interesting A_n-modules are detailed below.

Example 4.2.1. i) We have already observed above that $k[x_1, \ldots, x_n]$ is an A_n-module with the usual actions of x_i and ∂_i.

ii) We may also consider the module $k[\partial_1, \ldots, \partial_n] \simeq \frac{A_n}{\langle x_1, \ldots, x_n \rangle}$, in which the ∂_i act by multiplication and the x_i act by negative differentiation - i.e. differentiation followed by multiplication by -1.

iii) The vector space of Laurent polynomials $k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, with the usual actions of x_i and ∂_i also forms an A_n-module, and one may note that this module is generated the x_i^{-1}.

iv) The vector space of polynomials $k[\sqrt{x_1}, \sqrt{x_1}^{-1}, \ldots, \sqrt{x_n}, \sqrt{x_n}^{-1}]$ forms an A_n-module. We may view this module instead as $k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, where x_i acts as multiplication by t_i^2, and $\partial_i t_i^m = \frac{m}{2} t_i^{m-2}$, and $\partial_i t_j = 0$, for $i \neq j$, with the ∂_i still adhering to the usual product rule.

v) We can generalize the previous example by instead considering m^{th} roots, obtaining the A_n-module $k[x_1^{1/m}, x_1^{-1/m}, \ldots, x_n^{1/m}, x_n^{-1/m}]$.

vi) Rather than only polynomials, we may consider the ring of formal power series $k[[x_1, \ldots, x_n]]$, and see that it is an A_n-module.

vii) Here we shall assume, for simplicity, that $n = 1$. We may consider the A_1-module $A_1/A_1(\partial - a)$, for $a \in k^*$. Then x acts as the usual multiplication by x, and the action of ∂ is defined by $\partial \cdot f = \frac{df}{dx} + af$, where $f \in A_1/A_1(\partial - a)$. This follows from the relation

$$\partial x^n = nx^{n-1} + x^n \partial = nx^{n-1} + ax^n.$$

Given a polynomial $f \in k[x]$, applying the product rule to $\frac{d}{dx}(f(x)\exp(ax))$ shows that this A_n-module is isomorphic to the A_n-module $k[x] \cdot \exp(ax)$, consisting of polynomials multiplied by $\exp(ax)$.

viii) Similar to above, we may take $A_1/A_1(x - a)$, and here, ∂ will act as multiplication and x will act by $x \cdot g(\partial) = -\frac{d}{d\partial} g(\partial) + ag(\partial)$, where $g(\partial) \in A_1/A_1(\partial - a)$.

ix) We may also consider the A_1-module $A_1/A_1(\partial^2 - a^2)$, where $a \in k = \mathbb{C}$. We may write any element of this module as a linear combination of elements of the form $x^n \partial^i$, where $n \in \mathbb{N}$ and $i \in \{0, 1\}$. The action of x will be the usual multiplication, and we deduce the action of ∂ from the relations

$$\partial x^n = nx^{n-1} + x^n \partial,$$

$$\partial x^n \partial = nx^{n-1} \partial + a^2 x^n.$$

Comparing this action to the actions of ∂ on $x^n \cosh(ax)$ and on $x^n \sinh(ax)$, we see that this A_1-module is isomorphic to $\mathbb{C}[x] \cdot \text{Span}\{\cosh(ax), \sinh(ax)\}$, consisting of polynomials multiplied by linear combinations of $\cosh(ax)$ and $\sinh(ax)$. Since $\cosh(ax) \pm \sinh(ax) = e^{\pm ax}$, this A_1-module is isomorphic to $\mathbb{C}[x] \cdot \text{Span}\{e^x, e^{-x}\}$.

x) More generally, $A_1/A_1(\partial^m - a^m) \simeq \mathbb{C}[x] \cdot \text{Span}\{e^{2i\pi ajm}\}_{j=0}^{m-1}$ is an A_1-module.

xi) We may further generalize (vii), (viii), (ix) and (x) as follows. Given a (linear) differential operator $\mathcal{L}$, we may form the A_1-module $A_1/A_1\mathcal{L}$. Denote by V the vector space of solutions f to the ordinary differential equation $\mathcal{L}f = 0$. Then $A_1/A_1\mathcal{L} \simeq k[x] \cdot V$, the module consisting of polynomials multiplied by elements of V.

xii) Various examples may be made by generalizing the previous five examples, for instance considering A_n-modules for $n \geq 2$, we may take $A_n/A_n(\partial_1 - a, \partial_2, \partial_3, \ldots, \partial_n)$, or $A_n/A_n(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$, where the $a_i \in k$. We may also consider, for instance, $k[x_1, x_1^{-1}] \cdot V$, where V is as in the previous example.

One common fact the reader might notice about all of the above examples is that they are all infinite dimensional (as a vector space over k). In fact, we have the following proposition.

Proposition 4.2.2. *The only finite dimensional A_n-module is $\{0\}$.*

Proof. Since we can always restrict the scalars to an algebra isomorphic to A_1, it suffices to just prove it in the case of $A_1 = k[x, \partial]$. If M is a nonzero finite dimensional A_1-module, then we can write $M = k[v_1, \dots, v_r]$ for some finite basis $(v_1, \dots, v_r)$ of M. We have $[\partial, x]m = m$, for all $m \in M$. Additionally, for each $i \in \{1, \dots, r\}$, we can write

$$x \cdot v_i = \sum_j x_{ij} v_j,$$

$$\partial \cdot v_i = \sum_j d_{ij} v_j,$$

where the x_{ij} and d_{ij} are some elements of k. Note that the (x_{ij}) and (d_{ij}) form matrices which we denote A and B respectively. Then we have

$$\begin{aligned} v_i &= [\partial, x]v_i \\ &= (\partial x - x\partial)v_i \\ &= \sum_j x_{ij} \sum_l d_{jl} v_l - \sum_j d_{ij} \sum_l x_{jl} v_l \\ &= \sum_l \sum_j (x_{ij} d_{jl} - d_{ij} x_{jl}) v_l \end{aligned}$$

In particular, this implies that $\sum_j (x_{ij} d_{ji} - d_{ij} x_{ji}) = 1$. But $\sum_j x_{ij} d_{ji} = \operatorname{tr}(AB)$ and $\sum_j d_{ij} d_{ji} = \operatorname{tr}(BA)$, implying $\operatorname{tr}(AB) - \operatorname{tr}(BA) = 1$, which is a contradiction. Therefore, $M = \{0\}$. □

4.3 Sheaves of Rings of Differential Operators

The notions in the previous section can be made to work in a global context, using sheaves. Throughout this section, let X be a smooth (i.e. nonsingular) variety over $\mathbb{C}$, with $\mathscr{O}_X$ its structure sheaf (i.e. the sheaf whose sections on any open $U \subset X$ form the ring of regular functions on U). We may define an $\mathscr{O}_X$-differential algebra (or simply D-algebra) as follows.

Definition 4.3.1. Given a smooth variety X, an *$\mathscr{O}_X$-differential algebra* (or *D-algebra*) $\mathscr{A}$ on X is a sheaf of (associative) algebras on X along with an sheaf morphism $i : \mathscr{O}_X \to \mathscr{A}$ such that for every open $U \subset X$, $\mathscr{A}(U)$ is a differential $\mathscr{O}_X(U)$-algebra (Definition 4.1.8).

We shall look at some examples to make more concrete this fairly abstract definition. Just as in the local case, where we considered the subalgebra of $\operatorname{End}_k(k[x_1, \dots, x_n]$ generated by $k[x_1, \dots, x_n]$ and $\partial_1, \dots, \partial_n$, we may, in a similar fashion, construct a sheaf of rings (or algebras) of differential operators. First, however, we shall require some definitions.

Definition 4.3.2. Define the *tangent sheaf* Θ_X on X to be the sheaf $\mathcal{D}er(\mathscr{O}_X)$ whose sections on any open $U \subset X$ are derivations of $\mathscr{O}_X(U)$.

Definition 4.3.3. Let $\mathcal{E}nd_k(\mathscr{O}_X)$ denote the sheaf of k-linear endomorphisms of $\mathscr{O}_X$, that is, for any open set $U \subset X$, $\mathcal{E}nd_k(\mathscr{O}_X)(U) = \operatorname{End}_k(\mathscr{O}_X(U))$. The *sheaf of differential operators* on X, denoted by $\mathscr{D}_X$, is the subsheaf (of algebras) of $\mathcal{E}nd_k(\mathscr{O}_X)$ whose sections on any open $U \subset X$ are those generated by $\mathscr{O}_X(U)$ and $\Theta_X(U)$.

Now it is straightforward to see that this adheres to the definition of a D-algebra, as for any Zariski open subset $U \subset X$, $\mathscr{D}_X(U)$ consists of endomorphisms generated by $\mathscr{O}_X(U)$ and derivations of $\mathscr{O}_X(U)$, and we have already seen that these form the differential $\mathscr{O}_X(U)$-algebra $\operatorname{Diff}(\mathscr{O}_X(U))$.

Since X is nonsingular at every point, for any $p \in X$, the local ring $\mathscr{O}_{X,p}$ is regular of dimension $n = \dim X$, and hence there exist elements $x_1, \dots, x_n \in \mathscr{O}_{X,p}$ which generate the maximal ideal $\mathfrak{m}_p \lhd \mathscr{O}_{X,p}$. If U is an affine (i.e. isomorphic to an affine variety) open neighborhood of p, then it has a local coordinate system [Sch19] (lecture 9) $\{x_i, \partial_i\}$, for $1 \le i \le n$, and we can explicitly write

$$\mathscr{D}_X(U) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathscr{O}_X(U) \partial^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. We see how this nicely lines up with the rings of differential operators in the previous section. When U is affine, it is clear from the previous section that we can define a filtration akin to that in Example 4.1.9 (iii) i.e. filtering by the degree in the ∂_i. Using this definition, we may define a filtration on an arbitrary open subset U by considering the maximal degree on the restriction to any affine open subset of U. More precisely, we have the following:

Definition 4.3.4. The *order filtration* $F_\bullet$ on $\mathscr{D}_X(U)$ is defined by

$$F_i(U) = \{P \in \mathscr{D}_X(U) \,|\, P_V \in F_i(V)\,,\ \text{for all } V \subset U \text{ open and affine}\},$$

where P_V denotes the restriction of P to V.

We may do similarly with the filtrations from Example 4.1.9 (ii) and (iv), however these are not necessary for our discussion. We saw that in the local case, if $F_\bullet$ is the order filtration on the Weyl algebra, then F_i consists of all those elements P for which $[P, f] \in F_{i-1}$, for every $f \in \mathscr{O}_X$. The definition of the order filtration on $\mathscr{D}_X$ makes it clear that such a characterization holds in the global case. We remark that now that we have an explicit filtration on $\mathscr{D}_X$, we may consider the sheaf of associated graded algebras, which we shall simply refer to as the associated graded algebra. This is a sheaf of commutative algebras, and on open affine $U \subset X$, it will be isomorphic to the commutative algebra $\mathscr{O}_X(U)[\xi_1, \ldots, \xi_n] \simeq \mathrm{Sym}(\Theta_X(U))$. The differential algebra $\mathscr{D}_X$ defined above actually fits into the definition of a twisted differential operator.

Definition 4.3.5. An *algebra of twisted differential operators (TDO)* on a variety X is a sheaf of algebras $\mathscr{D}$, equipped with a morphism $\iota : \mathscr{O}_X \to \mathscr{D}$ such that there is a filtration $D_\bullet$ on $\mathscr{D}$ with $\iota(\mathscr{O}_X) = D_0$, and there are isomorphisms of sheaves $\mathrm{Sym}(D_1/D_0) \xrightarrow{\sim} \mathrm{gr}(\mathscr{D})$ and $\sigma : D_1/D_0 \xrightarrow{\sim} \Theta_X$, the latter defined by $\sigma(\theta)(f) = \theta\iota(f) - \iota(f)\theta$, for $\theta \in D_1$ and $f \in \mathscr{O}_X$.

Comparing to our definition for a differential algebra, a TDO has the additional features that $\mathscr{O}_X$ may be identified with D_0, and D_1/D_0 is isomorphic to Θ_X. We have seen here that the tangent sheaf finds use in the above discussion. The cotangent sheaf will also be a useful sheaf to consider.

Definition 4.3.6. Let X be an algebraic variety, $\delta : X \to X \times X$ the embedding into the diagonal of $X \times X$, and $\mathscr{J}$ be the ideal sheaf defined by

$$\mathscr{J}(V) = \{f \in \mathscr{O}_{X\times X}(V) \,|\, f(V \cap \delta(X)) = \{0\}\},$$

for any open set V in $X \times X$. Then the cotangent sheaf of X is $\Omega^1_X = \delta^{-1}(\mathscr{J}/\mathscr{J}^2)$, where δ^{-1} is the sheaf theoretic inverse image functor. (See Section 4.6)

Note that the idea behind such a definition is that it puts into a global point of view the idea of the cotangent space at a point. Recall that for $x \in X$, we have the ideal $\mathfrak{m}_{X,x}$ consisting of regular functions vanishing at x, and then the cotangent space at x is $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$. Since X is nonsingular, the cotangent sheaf is locally free [Sch19] (lecture 9). Naturally then, we may take wedge products of the elements in Ω^1_X, giving sections on any open set $U \subset X$ as elements of the wedge product of elements in $\Omega^1_X(U)$.

Definition 4.3.7. If $\dim X = n$, the *canonical sheaf* is $\omega_X := \bigwedge^n \Omega^1_X$.

Since X is smooth, on any affine open subset $U \subset X$, there exist local coordinates $\{x_i, \partial_i\}$, such that elements of Ω^1_X will be of the form $f_1 dx_1 + \cdots + f_n dx_n$, where the $f_i \in \mathscr{O}_X(U)$, so we see that $\Omega^1_X = \bigoplus_{i=1}^n \mathscr{O}_X(U) dx_i$. It is clear then that in this case, the canonical sheaf restricted to U is $\omega_X = \mathscr{O}_X dx = \mathscr{O}_X(dx_1 \wedge \cdots \wedge dx_n)$.

Example 4.3.8. In the case that $X = \mathbb{P}^1$, the complex projective line, we shall look at the sheaf of regular functions, the tangent sheaf, and the cotangent sheaf (which in this case is the same as the canonical sheaf). Let $U_0 = X \setminus \{(0:1)\} = \{(1:z) \,|\, z \in \mathbb{C}\}$, and $U_\infty = X \setminus \{(1:0)\} = \{(w:1) \,|\, w \in \mathbb{C}\}$, and let $V = U_0 \cap U_\infty$. It is clear that these form an affine open cover of $\mathbb{P}^1$, and that $U_0 \simeq \mathbb{C}^1 \simeq U_\infty$, as varieties. Hence,

$$\Theta_X(U_0) \simeq \Omega^1_X(U_0) \simeq \mathscr{O}_X(U_0) = \mathbb{C}[z],$$

and

$$\Theta_X(U_\infty) \simeq \Omega^1_X(U_\infty) \simeq \mathscr{O}_X(U_\infty) = \mathbb{C}[w],$$

where it should be noted that $w = z^{-1}$. Since U_0 and U_∞ are affine, the sections over any open subset of either U_0 or U_∞ can be deduced. Now in order to be a sheaf, the cocycle condition must be satisfied, i.e. in the case of $\mathbb{P}^1$ this means that if $f \in \mathscr{O}_X(X)$, then

$$\mathrm{res}^{U_0}_V \circ \mathrm{res}^X_{U_0} f = \mathrm{res}^{U_\infty}_V \circ \mathrm{res}^X_{U_\infty} f,$$

where res^A_B is the restriction morphism, where $B \subset A$. Importantly, a global section of $\mathscr{O}_X$ can be fully described by a section over U_0 and a section over U_∞ such that they "glue together", i.e. their restrictions to V are equal. Since $V \simeq \mathbb{C} \setminus \{0\}$, it can be readily verified that $\mathscr{O}_X(V) = \mathbb{C}[z, z^{-1}]$, so the restriction morphisms from U_0 to V and from U_∞ to V are simply inclusions. Hence, if $g_0(z) \in \mathscr{O}_X(U_0)$ and $g_1(w) = g_1(z^{-1}) \in \mathscr{O}_X(U_\infty)$ together describe a global section of $\mathscr{O}_X$, then $g_0(z) = g_1(z^{-1})$. Since $g_0(z)$ and $g_1(w)$ are both polynomials, it follows that they must be constants, and hence $\mathscr{O}_X(X) = \mathbb{C}$.

Next, we shall consider the global sections of Θ_X. Any element of $\Theta_X(U_0)$ can be expressed as $f(z)\partial_z$, where $f(z)$ is a polynomial in z, and any element of $\Theta_X(U_\infty)$ can be similarly expressed as $g(w)\partial_w$, where $w = z^{-1}$. Now on the intersection $V = U_0 \cap U_\infty$, one can see that

$$\partial_w = \frac{d}{dw} = \frac{dz}{dw}\frac{d}{dz} = -w^{-2}\frac{d}{dz} = -z^2\partial_z,$$

and with this, elements in $\Theta_X(V)$ can be written as $h(z, z^{-1})\partial_z$. Suppose now that $f(z)\partial_z \in \Theta_X(U_0)$ and $g(w)\partial_w \in \Theta_X(U_\infty)$ together describe a global section of Θ_X. As previously, the restriction morphisms are simply inclusion, so we have

$$f(z)\partial_z = g(w)\partial_w = -g(z^{-1})z^2\partial_z,$$

and therefore $f(z) = -g(z^{-1})z^2$. Then $f(z)$ and $g(w)$ must both be polynomials of degree at most 2, and it is easy to confirm that for any choice of a degree at most 2 polynomial $f(z)$, there is a unique $g(w)$ of degree at most 2 such that $f(z) = g(z^{-1})z^2$. Therefore, the global sections of Θ_X are isomorphic to the polynomials in z of degree at most 2. The sheaf of differential operators can then be deduced as the sheaf of rings generated by Θ_X and $\mathscr{O}_X$. Now for the sheaf Ω^1_X, the elements of $\Omega^1_X(U_0)$ may be written as $f(z)dz$ and the elements of $\Omega^1_X(U_\infty)$ can be written as $g(w)dw$, where $w = z^{-1}$. Again, the restrictions to V are going to be inclusions, and we observe that $dw = d(z^{-1}) = -z^{-2}dz$. This implies then that Ω^1_X has no global sections. We will speak more on this sheaf when we discuss twisted differential operators.

4.4 Modules over sheaves of differential operators

Similar to the previous section, here we look to globalize the notions of modules over rings of differential operators.

Definition 4.4.1. Let $\mathscr{A}$ be an $\mathscr{O}_X$-differential algebra on the variety X. An *$\mathscr{A}$-module M* is a quasi-coherent sheaf such that for any open set $U \subset X$, $M(U)$ is an $\mathscr{A}(U)$-module. Moreover, we say that M is a coherent sheaf if every $M(U)$ is finitely generated over $\mathscr{A}(U)$.

If $\mathscr{A}$ is any $\mathscr{O}_X$-differential algebra on the variety X, then any $\mathscr{A}$-module naturally has the structure of an $\mathscr{O}_X$-module by restriction of the action. We have the following lemma providing a way to obtain $\mathscr{A}$-modules from $\mathscr{O}_X$-modules.

Lemma 4.4.2. [HTT08] *Let M be a left $\mathscr{O}_X$-module and $\mathscr{D}_X$ the $\mathscr{O}_X$-differential algebra generated by $\mathscr{O}_X$ and Θ_X. The module structure on M may be extended to a left $\mathscr{D}_X$-module structure by providing a vector space morphism $\Delta : \Theta_X \to \mathscr{E}nd_{\mathbb{C}}(M)$; $\theta \mapsto \Delta_\theta$, such that for any $f \in \mathscr{O}_X$, and $\theta, \varphi \in \Theta_X$,*

a) $\Delta_{f\theta} = f\Delta_\theta$;
b) $\Delta_\theta f = \theta(f) + f\Delta_\theta$;
c) $\Delta_{[\theta,\varphi]} = [\Delta_\theta, \Delta_\varphi]$.

Proof. The Leibniz rule for elements $\theta \in \Theta_X$, and $f \in \mathscr{O}_X$ shows that $\theta(f) = \theta f - f\theta$. Since $\mathscr{D}_X$ is generated by $\mathscr{O}_X$ and Θ_X, the result follows. □

Definition 4.4.3. We call a morphism Δ satisfying the conditions in Lemma 4.4.2 a *flat connection.* If Δ satisfies only the conditions (a) and (b), but non necessarily (c), then we call it a *connection.*

An analogous result holds for right modules.

Lemma 4.4.4. [HTT08] *Let M be a right $\mathscr{O}_X$-module and sd_X the $\mathscr{O}_X$-differential algebra generated by $\mathscr{O}_X$ and Θ_X. The module structure on M may be extended to a right $\mathscr{D}_X$-module structure by providing a vector space morphism $\Delta' : \Theta_X \to \mathscr{E}nd_{\mathbb{C}}(M)$; $\theta \mapsto \Delta'_\theta$, such that for any $f \in \mathscr{O}_X$, and $\theta, \varphi \in \Theta_X$,*

a) $\Delta'_{f\theta} = \Delta'_\theta f$;
b) $\Delta'_\theta f = \theta(f) + f\Delta'_\theta$;
c) $\Delta'_{[\theta,\varphi]} = [\Delta'_\theta, \Delta'_\varphi]$.

Hence, these lemmas allow us to view a $\mathscr{D}_X$-module as a quasi-coherent $\mathscr{O}_X$-module with a flat connection. One should expect a correspondence between left and right $\mathscr{D}_X$-modules. Indeed, we shall show that this is obtained via the canonical sheaf ω_X. Since Ω^1_X may be regarded as an $\mathscr{O}_X$-linear map from Θ_X to $\mathscr{O}_X$, we may view ω_X as an $\mathscr{O}_X$-multilinear map from $\bigoplus_{i=1}^n \Theta_X$ to $\mathscr{O}_X$, i.e. for $\rho \in \omega_X$, and

$\theta_1, \ldots, \theta_n \in \Theta_X$, we have $\rho(\theta_1, \ldots, \theta_n) \in \mathscr{O}_X$. If $\theta \in \Theta_X$ is some other element, then we have an action on ρ via the so-called Lie derivative, $\mathfrak{Lie}(\theta)$. This action is defined by

$$(\mathfrak{Lie}(\theta)\rho)(\theta_1, \ldots, \theta_n) := \theta(\rho(\theta_1, \ldots, \theta_n)) - \sum_{i=1}^{n} \rho(\theta_1, \ldots, [\theta, \theta_i], \ldots, \theta_n).$$

Observe that if $f \in \mathscr{O}_X$, then we have the following:

$$\begin{aligned}(\mathfrak{Lie}(f\theta)(\rho))(\theta_1 \ldots, \theta_n) &= f\theta(\rho(\theta_1, \ldots, \theta_n)) - \sum_{i=1}^{n} \rho(\theta_1, \ldots, [f\theta, \theta_i], \ldots, \theta_n) \\ &= f\theta(\rho)(\theta_1, \ldots, \theta_n)) - \sum_{i=1}^{n} f\rho(\theta_1, \ldots, [\theta, \theta_i], \ldots, \theta_n) + \sum_{i=1}^{n} \theta_i(f)\rho(\theta_1, \ldots, \theta, \ldots, \theta_n) \\ &= f((\mathfrak{Lie}(\theta)(\rho))(\theta_1, \ldots, \theta_n)) + \theta(f)\rho(\theta_1, \ldots, \theta_n),\end{aligned}$$

where the last equality comes from the fact that given $k \in \mathbb{N}$, $f \in \mathscr{O}_X$, and a k-form τ, then

$$\sum_{i=1}^{k+1} \theta_i(f)\tau(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_k)$$

defines a $k+1$ from, and the only $n+1$-form on a space of dimension n is zero.
Furthermore, we have the equalities,

$$\begin{aligned}(\mathfrak{Lie}(\theta)(f\rho))(\theta_1, \ldots, \theta_n) &= \theta(f\rho(\theta_1, \ldots, \theta_n)) - \sum_{i=1}^{n} f\rho(\theta_1, \ldots [\theta, \theta_i], \ldots, \theta_n) \\ &= \theta(f)(\rho(\theta_1, \ldots, \theta_n)) + f\theta(\rho(\theta_1, \ldots, \theta_n)) - \sum_{i=1}^{n} f\rho(\theta_1, \ldots [\theta, \theta_i], \ldots, \theta_n) \\ &= \theta(f)(\rho(\theta_1, \ldots, \theta_n)) + f((\mathfrak{Lie}(\theta)(\rho))(\theta_1, \ldots, \theta_n)),\end{aligned}$$

and if θ' is another element of Θ_X, then

$$\begin{aligned}([\mathfrak{Lie}(\theta), \mathfrak{Lie}(\theta')](\rho))(\theta_1, \ldots, \theta_n) = \;& (\mathfrak{Lie}(\theta)\mathfrak{Lie}(\theta')(\rho))(\theta_1, \ldots, \theta_n) - (\mathfrak{Lie}(\theta')\mathfrak{Lie}(\theta)(\rho))(\theta_1, \ldots, \theta_n) \\ = \;& \theta((\mathfrak{Lie}(\theta')(\rho))(\theta_1 \ldots, \theta_n)) - \sum_{i=1}^{n}(\mathfrak{Lie}(\theta')\rho)(\theta_1 \ldots, [\theta, \theta_i], \ldots, \theta_n) \\ & - \theta'(\mathfrak{Lie}(\theta)(\rho))(\theta_1, \ldots, \theta_n) + \sum_{i=1}^{n}(\mathfrak{Lie}(\theta)\rho)(\theta_1 \ldots, [\theta', \theta_i], \ldots, \theta_n) \\ = \;& (\theta\theta'(\rho))(\theta_1, \ldots, \theta_n) - (\theta'\theta(\rho))(\theta_1, \ldots, \theta_n) \\ & - \sum_{i=1}^{n} \rho(\theta_1, \ldots, [\theta'[\theta, \theta_i]], \ldots, \theta_n) + \sum_{i=1}^{n} \rho(\theta_1, \ldots, [\theta[\theta', \theta_i]], \ldots, \theta_n) \\ = \;& ([\theta, \theta'](\rho))(\theta_1, \ldots, \theta_n) - \sum_{i=1}^{n} \rho(\theta_1, \ldots, [[\theta, \theta'], \theta_i], \ldots, \theta_n) \\ = \;& (\mathfrak{Lie}([\theta, \theta'])(\rho))(\theta_1, \ldots, \theta_n).\end{aligned}$$

Hence, it is clear that the action of Θ_X on ω_X via the Lie derivative gives ω_X a right $\mathscr{D}_X$-module structure by Lemma 4.4.4. This right $\mathscr{D}_X$-module structure is seen more easily in the affine case, where we have local coordinates $\{x_i, \partial_i\}$ and for a differential operator $P(x, \partial) = \sum_\alpha f_\alpha(x)\partial^\alpha$, the right action of $P(x, \partial)$ on ω_X is given by the left operation on $\rho = g(x)dx$, by the formal adjoint, i.e.

$$\rho P(x, \partial) = (g(x)dx)P(x, \partial) = (P^t(x, \partial)g(x))dx,$$

where $P^t - \sum_\alpha(-\partial)^\alpha f_\alpha(x)$. It can be readily verified that the partial differential operators P and S, satisfy the equation $(PS)^t = S^t P^t$, and that the right $\mathscr{D}_X$-action via the formal adjoint coincides with the right action via the Lie derivative.

Proposition 4.4.5. *[HTT08] Let M and N be left $\mathscr{D}_X$-modules and M' and N' be right $\mathscr{D}_X$-modules. Note that since $\mathscr{O}_X$ is commutative, all of these modules naturally have the structure of left and right $\mathscr{O}_X$-modules. Then,*

i) $M \otimes_{\mathscr{O}_X} N$ is a left $\mathscr{D}_x$ module via $\theta \cdot (m \otimes n) = \theta m \otimes n + m \otimes \theta n$;
ii) $M' \otimes_{\mathscr{O}_X} N$ is a right $\mathscr{D}_X$-module via $(m' \otimes n) \cdot \theta = m'\theta \otimes n - m' \otimes \theta n$;
iii) $\mathcal{H}om_{\mathscr{O}_X}(M, N)$ is a left $\mathscr{D}_X$-module via $(\theta \cdot \psi)(m) = \theta(\psi(m)) - \psi(\theta(m))$;
iv) $\mathcal{H}om_{\mathscr{O}_X}(M', N')$ is a left $\mathscr{D}_X$-module via $(\theta \cdot \psi)(m) = \psi(m)\theta) - \psi(m)\theta$;
v) $\mathcal{H}om_{\mathscr{O}_X}(M, N')$ is a right $\mathscr{D}_X$-module via $(\psi \cdot \theta)(m) = \psi(m)\theta - \psi(\theta m)$,

where in all of the above, $\theta \in \Theta_X$, $m \in M$, $n \in N$, $M' \in M'$, and ψ is a morphism from M (or M') to N (or N').

The proof of this proposition is just a matter of verifying that the proposed modules satisfy the conditions of the relevant one of the above two lemmas. Now with this proposition, along with the right $\mathscr{D}_X$-module structure on ω_X given above, we have a functor

$$\omega_X \otimes_{\mathscr{O}_X} - \tag{4.2}$$

from the category of left $\mathscr{D}_X$-modules to the category of right $\mathscr{D}_X$-modules. Similarly, we have a functor

$$\mathcal{H}om_{\mathscr{O}_X}(\omega_X, -) \tag{4.3}$$

from the category of right $\mathscr{D}_X$-modules to the category of left $\mathscr{D}_X$-modules. Note that we also have a functor

$$\mathcal{H}om_{\mathscr{O}_X}(-, \omega_X),$$

and this still takes us from the category of right $\mathscr{D}_X$-modules to the category of left $\mathscr{D}_X$-modules.

Definition 4.4.6. We call the functors (6) and (7) *side-changing functors*

The side changing functors provide an equivalence of categories between left and right $\mathscr{D}_X$-modules. If X is affine, then $\omega_X \simeq \mathscr{O}_X$, and so for any left $\mathscr{D}_X$-module, M, the elements of the right $\mathscr{D}_X$-module $\omega_X \otimes_{\mathscr{O}_X} M$ may simply be viewed as elements of M with the right action of $P \in \mathscr{D}_X$ being left action by the adjoint P^t. Similarly, for any right $\mathscr{D}_X$-module, N, the elements of the left $\mathscr{D}_X$-module $\mathcal{H}om_{\mathscr{O}_X}(\omega_X, N)$ may be viewed as the elements in N with left action of $P \in \mathscr{D}_X$ given by right action of the adjoint P^t.

Example 4.4.7. i) Let $X = \mathbb{C}^n$. Naturally, here one can obtain $\mathscr{D}_X$-modules in a similar way to modules over the Weyl algebra. Let $\mathscr{I}$ be a constant ideal sheaf of $\mathscr{O}_X$. Then $\mathscr{O}_X/\mathscr{I}$ is a $\mathscr{D}_X$-module, with the natural action of $\mathscr{D}_X$. The sections of this sheaf are regular functions modulo the ideal.

ii) Let $X = \mathbb{C}^n \setminus \{0\}$. Here, the space X is not affine, however it is quasi-affine, so we must consider its sections on an open affine cover, say $\{U_i = X \setminus V(x_i) \,|\, 1 \le i \le n\}$, where $V(x_i)$ denotes the vanishing of the ideal generated by the polynomial x_i. Each of these U_i is isomorphic to an affine variety in $\mathbb{C}^{n-1}$. A $\mathscr{D}_X(U_i)$-module can be constructed on each of the U_i in the same way as in the above example, and then in order to give a $\mathscr{D}_X$-module, they must coincide on the intersections of the U_i.

iii) Let $X = \mathbb{P}^1$. The sheaves $\mathscr{O}_X$ and $\Omega^1_X = \omega_X$, which we found in the previous section, naturally have the structure of a right $\mathscr{D}_X$-module. $\mathscr{O}_X$ also has the structure of a left $\mathscr{D}_X$-module.

4.5 Twisted D-modules on $\mathbb{P}^1$

Let us return to the sheaves of $\mathscr{O}_X$-modules, Θ_X and Ω^1_X, for $X = \mathbb{P}^1$. The structure for the global sections of these sheaves came from the relations between the $\partial_z = \frac{d}{dz}$ and $\partial_w = \frac{d}{dw}$, and between the dz and dw, where $w = z^{-1}$. We saw that the sections over the subsets U_0, U_∞ and V were isomorphic to the regular functions over those subsets. Recall that we could write elements of $\Theta_X(U_0)$ as $f(z)\partial_z$, elements of $\Theta_X(U_\infty)$ as $g(w)\partial_w$, and elements of $\Theta_X(V)$ as $h(z, z^{-1})\partial_z$, where f, g and h are polynomials. The restriction morphism from U_0 to V is then just $f(z)\partial_z \mapsto f(z)\partial_z$, however the restriction morphism from U_∞ to V is $g(w)\partial_w \mapsto -z^2 g(z^{-1})\partial_z$. It is clear to see then that the sheaf Θ_X is isomorphic to the (twisted) sheaf $\mathscr{O}_{\mathbb{P}^1}(2)$ of $\mathscr{O}_{\mathbb{P}^1}$-modules, which is defined by $\mathscr{O}_{\mathbb{P}^1}(2)(U_0) \simeq \mathscr{O}_{\mathbb{P}^1}(U_0)$ and $\mathscr{O}_{\mathbb{P}^1}(2)(U_\infty) \simeq \mathscr{O}_{\mathbb{P}^1}(U_\infty)$, with the restriction morphisms $\mathrm{res}^{U_0}_V : f(z) \mapsto f(z)$, and $\mathrm{res}^{U_\infty}_V : g(w) \mapsto z^2 g(z^{-1})$. These restriction morphisms make clear that the global sections of $\mathscr{O}_{\mathbb{P}^1}(2)$ may be regarded as polynomials of degree at most 2. In a similar way, we can see that Ω^1_X is isomorphic to the sheaf $\mathscr{O}_{\mathbb{P}^1}(-2)$, which is defined similarly, except that the restriction morphisms are $\mathrm{res}_V^{U_0} : f(z) \mapsto f(z)$, and $\mathrm{res}_V^{U_\infty} : g(w) \mapsto z^{-2}g(z^{-1})$. This construction generalizes to any $n \in \mathbb{Z}$.

Definition 4.5.1. For any $n \in \mathbb{Z}$, the twisted $\mathscr{O}_{\mathbb{P}^1}$-module $\mathscr{O}_{\mathbb{P}^1}(n)$ is defined as the sheaf such that $\mathscr{O}_{\mathbb{P}^1}(n)(U_0) \simeq \mathscr{O}_{\mathbb{P}^1}(U_0)$ and $\mathscr{O}_{\mathbb{P}^1}(n)(U_\infty) \simeq \mathscr{O}_{\mathbb{P}^1}(U_\infty)$, and the restriction morphisms are $\mathrm{res}_V^{U_0} : f(z) \mapsto f(z)$, and $\mathrm{res}_V^{U_\infty} : g(w) \mapsto z^n g(z^{-1})$.

If $n \geq 0$, then the global sections of $\mathscr{O}_{\mathbb{P}^1}(n)$ are polynomials of degree at most n, and if $n < 0$, then the sheaf has no nonzero global sections. Now the $\mathscr{O}_{\mathbb{P}^1}(n)$ are $\mathscr{O}_{\mathbb{P}^1}$-modules, however they are not $\mathscr{D}_{\mathbb{P}^1}$-modules, since e.g. for $\partial_w \in \mathscr{D}_{\mathbb{P}^1}(U_1)$, and $1 \in \mathscr{O}_{\mathbb{P}^1}(n)(U_1)$, we have $\partial_w 1 = 0$, but

$$\mathrm{res}_{U_1}^V(\partial_w)\,\mathrm{res}_{U_1}^V(1) = -z^2\partial_z z^n,$$

which is not equal to $0 = \mathrm{res}_{U_1}^V(0)$ unless $n = 0$. We would like to put some differential structure on the modules $\mathscr{O}_{\mathbb{P}^1}(n)$, i.e. we want to define a TDO $\mathscr{D}_n$ on X so that $\mathscr{O}_{\mathbb{P}^1}(n)$ has the structure of a $\mathscr{D}_n$-module. This amounts to finding a $\mathscr{O}_X$-module $\mathscr{T}$ which is isomorphic to Θ_X such that the conditions in lemma 2.1 are satisfied. The idea is that we need to come up with some kind of gluing morphisms of the operators ∂_z and ∂_w such that

$$\mathrm{res}_{U_0}^V(\partial_z)z^m = \mathrm{res}_{U_0}^V(\partial_z)\,\mathrm{res}_{U_0}^V(z^m) = \mathrm{res}_{U_0}^V(\partial_z z^m) = \mathrm{res}_{U_0}^V(mz^{m-1}) = mz^{m-1},$$

and

$$\mathrm{res}_{U_1}^V(\partial_w)w^{m-n} = \mathrm{res}_{U_1}^V(\partial_w)\,\mathrm{res}_{U_0}^V(w^m) = \mathrm{res}_{U_0}^V(\partial_w w^m)\,\mathrm{res}_{U_1}^V(mw^{m-1}) = mw^{m-n-1}.$$

Then this implies that

$$\mathrm{res}_{U_0}^V(\partial_z) = \partial_z,$$

and

$$\mathrm{res}_{U_1}^V(\partial_w) = w^{-n}\partial_w w^n = \partial_w + nw^{-1} = -z^2\partial_z + nz.$$

We define $\mathscr{D}_{n+1}$ to be the TDO over X generated by ∂_z and ∂_w with the above restriction morphisms. By construction, it is clear that $\mathscr{O}_{\mathbb{P}^1}(n)$ is a $\mathscr{D}_{n+1}$-module. We call the modules $\mathscr{O}_{\mathbb{P}^1}(n)$ twisted D-modules over $\mathbb{P}^1$. We remark that TDO's and twisted D-modules may be constructed over $\mathbb{P}^m$ for any $m \geq 1$ in a similar manner.

4.6 Inverse Image and Direct Image

Given a morphism $X \to Y$, there is a notion of a direct image functor which allows us to push forward sheaves on X to sheaves on Y. Similarly, there is an inverse image functor which allows us to pull back sheaves on Y to sheaves on X. With some work, we can define these notions in the setting of D-modules, and this gives a rather nice formalism which gives one a way to construct D-modules in a natural way, and furthermore to relate these various D-modules. We would really prefer to work in the setting of derived categories for these definitions, but in the interest of simplicity, we will avoid this point of view, and just describe these functors in the abelian categories of D-modules.

Definition 4.6.1. Let $f : X \to Y$ be a morphism of topological spaces and let F be a sheaf on Y. The *sheaf-theoretic inverse image* (or *pullback*) $f^{-1}F$ is the sheaf on X defined as the sheafification of the presheaf $\widehat{f^{-1}F}$, defined by $\widehat{f^{-1}F}(U) = \lim_{V \supset f(U)} F(V)$, for all open sets $U \in X$, and where $\lim_{V \supset f(U)}$ denotes the direct limit over all the open sets in Y containing $f(U)$.

It follows from this definition that at the level of stalks we have $f^{-1}F_x = F_{f(x)}$, for all $x \in X$. Now let $f : X \to Y$ be a morphism of smooth algebraic varieties. If X and Y are affine, then morphisms are regular maps and so we see that this morphism induces a morphism of the structure sheafs, $\mathscr{O}_Y \to \mathscr{O}_X$ via $g \mapsto g \circ f$. This morphism gives $\mathscr{O}_X$ the structure of a left (or right) $\mathscr{O}_Y$-module.

In the more general case of quasi-projective varieties, however, we cannot be guaranteed such a morphism of the structure sheafs, and as such, there is no clear way to give an $\mathscr{O}_Y$-module structure on $\mathscr{O}_X$. However, from the definition of the inverse image functor, it is evident that we can obtain an $f^{-1}\mathscr{O}_Y$-module structure on $\mathscr{O}_X$. Indeed, at the level of stalks, we have that for $x \in X$, $(f^{-1}\mathscr{O}_Y)_x = (\mathscr{O}_Y)_{f(x)}$, so for an affine covering $\{U_i\}$ of X, we see that on any U_i, $\mathscr{O}_X(U_i)$ can be given the structure of an $f^{-1}\mathscr{O}_Y(U_i)$-module since a sheaf on a quasi-affine variety can be fully recovered from its stalks. Gluing conditions can be checked to verify that indeed, $\mathscr{O}_X$ has a $f^{-1}\mathscr{O}_Y$-module structure.

In general, if we are given a $\mathscr{D}_Y$-module M, then upon taking the sheaf-theoretic pullback, we are not guaranteed a $\mathscr{D}_X$-module structure. In order to obtain a D-module structure, we define the inverse image functor f^* from the category of left $\mathscr{D}_Y$-modules to the category of left $\mathscr{D}_X$-modules by

$$f^*M := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(M).$$

The $\mathscr{D}_X$-module structure is obtained by extending the natural $\mathscr{O}_X$-module via the action of $\theta \in \Theta_X$ the vector fields (or derivations) on X given by

$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi\tilde{\theta}(s), \quad \psi \in \mathscr{O}_X,\, s \in f^{-1}M,$$

where $\tilde{\theta}$ is obtained from the canonical $\mathscr{O}_X$-linear homomorphism, $\Theta_X \to f^*\Theta_Y$.

Definition 4.6.2. Applying this inverse image functor to $\mathscr{D}_Y$, we obtain the $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule

$$\mathscr{D}_{X\to Y} := f^*\mathscr{D}_Y,$$

which we call the *transfer bimodule.*

With this, one can check that

$$f^* = \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}(-).$$

If X and Y are affine, then we may identify $f^{-1}\mathscr{O}_Y$ with $\mathscr{O}_Y$ and furthermore, for any $\mathscr{D}_Y$-module M, we may identify $f^{-1}M$ with M. In this case we may simply write

$$f^*M = \mathscr{O}_X \otimes_{\mathscr{O}_Y} M = \mathscr{D}_{X\to Y} \otimes_{\mathscr{D}_Y} f^{-1}(M).$$

Given local coordinates, $\{x_i, \partial_{x_i}\}_{1\le i\le m}$ and $\{y_i, \partial_{y_i}\}_{1\le i\le n}$ for open affine subsets $U \subset X$ and $V \subset Y$ respectively, the left $\mathscr{D}_X$-module structure can be fully described as follows: by multiplication on the first factor by polynomials, and for $\psi \in \mathscr{O}_X$, $s \in M$,

$$\partial_{x_i}(\psi \otimes s) = \partial_{x_i}(\psi) \otimes s + \psi \otimes \partial_{y_i}(s),$$

where ∂_{y_i} is identified with zero if $i > n$.

Definition 4.6.3. Let $f : X \to Y$ be a morphism of topological spaces and let F be a sheaf on X. The *sheaf-theoretic direct image* (or *pushforward*) f_*F is the sheaf on Y defined by $f_*F(U) = F(f^{-1}(U))$, for every $U \subset Y$ open.

We remark that the sheaf theoretic direct pushforward has a more straightforward definition - not requiring limits nor sheafification - since $f^{-1}(U)$ is open for any open $U \subset Y$ by virtue of the continuity of f, while the same cannot be said for $f(V)$, for $V \subset X$ open. Unfortunately, to define the D-module direct image functor is a rather difficult task compared to the D-module inverse image functor. To define the D-module direct image functor, we require side-changing functors (Definition 4.4.6) to go from left D-modules to right D-modules, and from right D-modules to left D-modules. We may apply these side changing operations to both sides of the $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule $\mathscr{D}_{X\to Y}$ to obtain an $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule

$$\mathscr{D}_{Y\leftarrow X} := \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\omega_Y^{\otimes -1},$$

where $\omega_Y^{\otimes -1}$ is the sheaf inverse of ω_Y, and we may identify the functor $-\otimes_{\mathscr{O}_Y} \omega_Y^{-1}$ with $\mathcal{H}om_{\mathscr{O}_Y}(-, \omega_Y)$. We shall define the D-module direct image functor f_* as that which sends a $\mathscr{D}_X$-module M to

$$f_*M := f_*(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} M),$$

excusing the abuse of notation; the f_* on the right hand side denotes the sheaf-theoretic pushforward. Note that in general, when looking globally, we require the sheaf-theoretic direct image f_* to give the $\mathscr{D}_Y$-module structure, since as mentioned above, in general we can only guarantee an $f^{-1}\mathscr{D}_Y$-module structure on $\mathscr{D}_{Y\leftarrow X}$.

Restricting to case where X and Y are affine, then we may identify $f^{-1}\mathscr{D}_Y$ with $\mathscr{D}_Y$, and then we can simply write $f_*M = \mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} M$. In fact, by recognizing how the side-changing operations work in the affine case, we may further simplify this to

$$f_*M = M \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y} = M \otimes_{\mathscr{D}_X} \mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{D}_Y,$$

where the left $\mathscr{D}_Y$-action is given by right multiplication on the last factor by the formal adjoint.

Computing the direct image simplifies in the case of a closed embedding of an affine space $X = k^m$ into another affine space $Y = k^n$, with $n > m$. Let $\iota : X \to Y$ be an embedding. We have coordinates $\{x_i, \partial_{x_i}\}_{1\leq i\leq m}$ and $\{y_i, \partial_{y_i}\}_{1\leq i\leq n}$ for X and Y respectively, and we may identify the coordinates of the image of X with the first m coordinates of Y. Noting that any term of a differential operator in $\mathscr{D}_Y$ can be written uniquely in the form $g(y)(\partial_{y_1}^{\alpha_1}...\partial_{y_m}^{\alpha_m})(\partial_{y_{m+1}}^{\alpha_{m+1}}...\partial_{y_n}^{\alpha_n})$, with $g(y) \in \mathscr{O}_Y$, we see that as a $\mathscr{D}_Y$-module,

$$\mathscr{D}_Y \simeq \mathscr{O}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_1}, ..., \partial_{y_m}] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}],$$

where the action of $\mathscr{D}_Y$ is clear (just think about the usual action on $\mathscr{D}_Y$ followed by the necessary rearrangement of factors within each term). Now the transfer bimodule is then given by

$$\begin{aligned}
\mathscr{D}_{X\to Y} &= \mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \\
&\simeq \mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_1}, ..., \partial_{y_m}] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \\
&= \mathscr{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_1}, ..., \partial_{y_m}] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \\
&\simeq \mathscr{D}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \\
&= \mathscr{D}_X[\partial_{y_{m+1}}, ..., \partial_{y_n}]
\end{aligned}$$

This gives a $(\mathscr{D}_X, \mathscr{D}_Y)$-bimodule. As mentioned above, since X and Y are affine, we may easily construct from this a $(\mathscr{D}_Y, \mathscr{D}_X)$-bimodule via multiplication by the formal adjoint. Perhaps a more natural way to think about this is as the bimodule

$$\mathscr{D}_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}_X.$$

Here, the right $\mathscr{D}_X$ action is given by the usual right multiplication on the second factor, and the left $\mathscr{D}_Y$ action is given as follows. For $m + 1 \leq i \leq n$, ∂_i acts as left multiplication on the first factor. For $1 \leq i \leq m$, ∂_i acts as left multiplication on the second factor. For $P \in \mathscr{O}_Y$, use the commutation relations to bring P to the right hand side of the first factor, and then allow the resulting polynomial to act on the left of $\mathscr{D}_X$; explicitly, if $F \in \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}]$, then $PF = \sum F_i P_i$, for some $F_i \in \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}]$ and $P_i \in \mathscr{O}_Y$. Then the P_i have a left action on $\mathscr{D}_X$ as given above.

Now given a $\mathscr{D}_X$-module M, the direct image of M may now be expressed as

$$\iota_*M = \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}_X \otimes_{\mathscr{D}_X} M = \mathbb{C}[\partial_{y_{m+1}}, ..., \partial_{y_n}] \otimes_{\mathbb{C}} M,$$

where the left $\mathscr{D}_Y$ action is given as follows: left multiplication on the first factor by ∂_{y_i} if $i > m$, and $P \in \mathscr{O}_Y[\partial_{y_1}, ..., \partial_{y_m}]$ acts on the left of the second factor M.

Another standard example for pushforwards and pullbacks is in the case where $f : X \to Y$ is an open affine embedding. In this case, the pullback is obtained by simply extending the $\mathscr{D}_Y$-module to have the structure of a $\mathscr{D}_X$-module which is clear from the definition $f^*M = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}M$, noting that Y is affine and X is open in Y. Along similar lines of reasoning, the pushforward along the open embedding is the restriction of the $\mathscr{D}_X$-module structure to $\mathscr{D}_Y$,

The inverse image and direct image functors provide one way to pull back and push forward D-modules. There are additionally two other functors, which we can define in terms of these pushforward and pullback functors involving a certain duality functor. This is a situation where it is really more sensible to be dealing with the bounded derived category of coherent D-modules, so in our avoidance of such concepts but the necessity to have the following definitions, these concepts may appear unmotivated. We refer the reader to [HTT08] (ch. 1-3) for a more detailed discussion. Given a coherent $\mathscr{D}_X$ module M, we may consider a projective resolution of M, and apply the $\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X)$ functor to the resulting complex. For a general coherent $\mathscr{D}_X$-module, none of the nth cohomology modules for $n < 2 \dim X$ need vanish, and so we fail to obtain a canonical duality functor from coherent $\mathscr{D}_X$-modules to coherent $\mathscr{D}_X$-modules. We can, however, restrict ourselves to certain kinds of $\mathscr{D}_X$-modules - called holonomic $\mathscr{D}_X$-modules - for which we are able to canonically define a duality functor. Typically, a holonomic $\mathscr{D}_X$-module is defined in terms of its characteristic variety, however we shall define it here as follows.

Definition 4.6.4. A *holonomic* $\mathscr{D}_X$-module M is a coherent $\mathscr{D}_X$-module for which $\mathcal{E}xt^n_{\mathscr{D}_X}(M, \mathscr{D}_X) \neq 0$ only if $n = \dim X$.

This affords us the following.

Definition 4.6.5. The *duality functor* is the functor defined by $\mathbb{D}_X := \mathcal{E}xt^{\dim X}_{\mathscr{D}_X}(-, \mathscr{D}_X)$ from the category of holonomic left (resp. right) $\mathscr{D}_X$-modules to the category of holonomic right (resp. left) $\mathscr{D}_X$-modules.

Returning to the setting of a continuous map $f : X \to Y$, we can define two more functors $f^!$ and $f_!$ on holonomic $\mathscr{D}_X$-modules and holonomic $\mathscr{D}_Y$-modules as

$$f_! := \mathbb{D}_Y f_* \mathbb{D}_X,$$

and

$$f^! := \mathbb{D}_X f^* \mathbb{D}_Y.$$

Commonly, we refer to f_* as the $*$-pushforward (star pushforward), $f_!$ as the !-pushforward (shriek pushforward), f^* as the $*$-pullback (star pullback), and $f^!$ as the !-pullback (shriek pullback).

4.7 Monodromic D-modules on the flag variety

We use this section to very briefly discuss this concept of monodromic D-modules, which we will ultimately use to obtain our Lie algebra representations. We recall the following general definition.

Definition 4.7.1. Let G be a group which acts on spaces X and Y, and let $f : X \to Y$. We say that f is a G-equivariant map if $g \cdot f(x) = f(g \cdot x)$, for any $g \in G$ and $x \in X$.

Now let G be an algebraic group with corresponding Lie algebra $\mathfrak{g}$, and H a maximal torus with corresponding Lie algebra $\mathfrak{h}$. Also, let $B \subset G$ be a Borel subgroup, $N \subset G$ and maximal unipotent subgroup as usual, and let $X = G/B$ be the flag variety and $\widetilde{X} = G/N$ be the base affine space, which we may regard as an H-torsor $\pi : \widetilde{X} \to X$. We recall that H acts on $\widetilde{X}$ from the right, which induces a left action of H on the differential operators $\mathscr{D}_{\widetilde{X}}$. We also remark that the H-invariants of $\pi_* \mathscr{D}_{\widetilde{X}}$ form an algebra $\widetilde{\mathscr{D}}$ of differential operators on X.

Definition 4.7.2. A monodromic D-module on X is a $\mathscr{D}_{\widetilde{X}}$-module M with an action of H such that

$$h \cdot (\theta m) = (h \cdot \theta)(h \cdot m).$$

Remark 4.7.3. *Though it is defined as a monodromic D-module on X, since the module M is a $\mathscr{D}_{\widetilde{X}}$-module, we will often refer to it as a monodromic $\mathscr{D}_{\widetilde{X}}$-module, or a monodromic D-module on $\widetilde{X}$.*

In [BB93] (2.5), it is shown that the category of monodromic D-modules on X is equivalent to the category of $\widetilde{\mathscr{D}}$-modules.

The action of H $\mathscr{D}_{\widetilde{X}}$-module M induces an action of $\mathfrak{h}$ on M, which we denote by $\alpha^h_M : \mathfrak{h} \to \mathrm{End}(M)$. We also have the action $\alpha^0_M : \mathfrak{h} \to \mathrm{End}(M)$ where every $h \in \mathfrak{h}$ acts as 0 acts. Then M is monodromic if and only if $\alpha^h_M - \alpha^0_M$ acts by a constant on M, and we call this $\alpha^h_M - \alpha^0_M$ the monodromy.

Chapter 5

Beilinson-Bernstein localization on the flag variety G/B

5.1 The Borel-Weil theorem

We would be remiss to not mention the Borel-Weil theorem and its generalization the Borel-Weil-Bott theorem, as these theorems provide the initial ideas of how studying the geometric structure of the flag variety can yield results in the representation theory of an algebraic group. Throughout this section, G is a simply-connected, semisimple, complex algebraic group, we fix a Borel subgroup $B \subset G$ with a maximal torus $H \subset B$, unipotent radical $N \subset B$, and associated is the Weyl group W. By a character of H, we mean an element of $\mathrm{Hom}_{\mathcal{G}rp/\mathbb{C}}(H, \mathbb{C}^*)$. Each character of H corresponds to an integral weight of $\mathfrak{h}$, and so we identify the characters of H with elements in the weight lattice of $\mathfrak{h}$. We say that a character χ is dominant if $\alpha^\vee(\chi) \geq 0$ for any positive root $\alpha \in \Phi^+$. Any character of H uniquely determines a 1-dimensional B-module where the unipotent radical N acts trivially. Conversely, since any B-module on which N acts non-trivially has dimension strictly greater than 1, each 1-dimensional B-module uniquely determines a character of H. Hence, there is a one-to-one correspondence between 1-dimensional B-modules and characters of H.

Definition 5.1.1. Let X be a topological space equipped with a continuous G-action, and let $\pi : V \to X$ be a vector bundle. For $x \in X$, denote by V_x the fiber of V at x. We say that V is *G-equivariant* if there is a continuous G-action on V so that for any $g \in G$, we have $g \cdot V_x = V_{g \cdot x}$ and the map $g \cdot : V_x \to V_{g\cdot x}$ is a linear isomorphism.

We can extend this definition to arbitrary sheaves as follows:

Definition 5.1.2. Let X be a topological space equipped with a continuous G-action, and $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $m : G \times G \to G$ be the multiplication map, $a : G \times X \to X$ the map defined by the action of G, and let $p_2 : G \times X \to X$ and $p_{23} : G \times G \times X \to G \times X$ be the projection maps onto respectively the last coordinate and the last two coordinates. We say that $\mathcal{F}$ is G-equivariant if there exists an isomorphism $\Phi : a^*\mathcal{F} \to p^*\mathcal{F}$ such that

$$(p_{23}^*\Phi) \circ ((1 \times a)^*\Phi) = (m \times 1)^*\Phi.$$

Proposition 5.1.3. *G-equivariant vector bundles on $X := G/B$ are in one-to-one correspondence with B-modules.*

Proof (sketch). If V is a B-module, then the fiber product[1] $G \times_B V$ is the resulting G-equivariant vector bundle. On the other hand, given a G-equivariant vector bundle $\pi : E \to X$, the corresponding B-module is the fiber E_{eB}, where $e \in G$ is the identity. The remainder of the proof is just a matter of showing that $(G \times_B V)_{eB} \simeq V$ as B-modules and that $G \times_B E_{eB} \simeq E$ such that the diagram

$$\begin{array}{ccc} G \times_B E_{eB} & \xrightarrow{\sim} & E \\ & \searrow \quad \swarrow & \\ & G/B & \end{array}$$

commutes. □

The vector bundles corresponding to 1-dimensional B-modules are just line bundles. Since 1-dimensional B-modules correspond to characters of H, we have the following corollary.

Corollary 5.1.4. *G-equivariant line bundles on $X = G/B$ are in one-to-one correspondence with characters of H.*

[1]Here, one can explicitly construct this as the quotient of the trivial bundle $G \times V \to G$ by the B-action given by $b \cdot (g, v) = (gb^{-1}, b \cdot v)$ for $g \in G$, $b \in B$, and $v \in V$. In particular, the quotient identifies elements (gb, v) with $(g, b \cdot v)$.

Given a character $\lambda \in \mathrm{Hom}_{\mathcal{G}rp/\mathbb{C}}(H, \mathbb{C}^*)$, the corresponding G-equivariant line bundle on $X = G/B$ is $\pi : G \times_B \mathbb{C}_{-\lambda} \to X$, which we shall denote more commonly by $\mathscr{L}(\lambda)$.

Example 5.1.5. If $G = \mathrm{SL}_2(\mathbb{C})$, then $X \simeq \mathbb{P}^1$, and since $H \simeq \mathbb{C}^*$, we can identify the characters of $H \subset G$ with integers. The G-equivariant line bundle $\mathscr{L}(\lambda)$ corresponding to λ then is precisely the twisted line bundle $\mathscr{O}(\lambda)$.

Theorem 5.1.6 (Borel-Weil)**.** *Let $B \subset G$ be a Borel subgroup and $H \subset B$ a maximal torus. If λ is a dominant character of H, then $H^0(G/B, \mathscr{L}(\lambda)) \simeq V(\lambda)^*$, where $V(\lambda)^*$ is the dual of the G-module $V(\lambda)$ of highest weight λ.*

Proof (sketch). We give a sketch of the proof given in [Kum12] (p. 7-8). Fixing a dominant character λ of H, we pull back $\mathscr{L}(\lambda)$ along $G \to G/B$ to give a line bundle $\widehat{\mathscr{L}}(\lambda)$. We see this in the commutative diagram

$$\begin{array}{ccc} G \times \mathbb{C}_{-\lambda} & \longrightarrow & G \times_B \mathbb{C}_{-\lambda} \\ {\scriptstyle \hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ G & \longrightarrow & G/B \end{array}$$

Some staring will convince oneself that a section σ of $\widehat{\mathscr{L}}(\lambda)$ has the form $\sigma(g) := (g, f(g))$, where $g \in G$, and $f : G \to \mathbb{C}_{-\lambda}$. Hence we may identify $H^0(G, \widehat{\mathscr{L}}(\lambda))$ with $\mathbb{C}[G] \otimes \mathbb{C}_{-\lambda}$, since a global section of $\widehat{\mathscr{L}}(\lambda)$ will be determined entirely by map $f : G \to \mathbb{C}_{-\lambda}$. Now there is a natural B-action on $\mathbb{C}[G]$ given by $(b \cdot f)(g) = f(gb)$, and so B acts on $\mathbb{C}[G] \otimes \mathbb{C}_{-\lambda}$ with the diagonal action given by $b \cdot (f \otimes v) = (b \cdot f) \otimes (b \cdot v) = (b \cdot f) \otimes \lambda(b)^{-1} v$. We can then identify the B-module $H^0(G, \widehat{\mathscr{L}}(\lambda)) = \mathbb{C}[G] \otimes \mathbb{C}_{-\lambda}$ with $\mathbb{C}[G]$, where the action of B is given by

$$(b \cdot f)(g) = \lambda(b)^{-1} f(gb).$$

On the other hand, a section σ of $\mathscr{L}(\lambda)$ has the form $\sigma(gB) = [g, f(g)]$, where $[g, f(g)]$ denotes the equivalence class of the pair $(g, f(g))$ under the equivalence relation defined by $(gb, f(g)) \sim (g, b \cdot (f(g)))$ for any $b \in B$, $g \in G$, $f : G \to \mathbb{C}_{-\lambda}$. Such a section σ is well defined provided that for any $b \in B$,

$$[g, f(g)] = [gb, f(gb)] = [g, b \cdot (f(gb))] = [g, \lambda(b)^{-1} f(gb)].$$

This is true precisely when $f : G \to \mathbb{C}_{-\lambda}$ satisfies $f(g) = \lambda(b)^{-1} f(gb)$. But then $H^0(G/B, \mathscr{L}(\lambda))$ are the B-invariants of $H^0(G, \widehat{\mathscr{L}}(\lambda))$.

The final part of the proof is to show that $[H^0(G, \widehat{\mathscr{L}}(\lambda))]^B \simeq V(\lambda)^*$, which we omit. $\square$

By looking at global sections, the Borel-Weil theorem provides an analogy between line bundles on the flag variety corresponding to a character λ and the dual of the G-module of highest weight λ, provided λ is dominant. If the character λ is not dominant, then there are no global sections of $\mathscr{L}(\lambda)$, i.e. $H^0(G/B, \mathscr{L}(\lambda)) = 0$. The theorem was extended by Rauol Bott to account for characters which do not lie in the dominant Weyl chamber, and the key is to look at the higher cohomology groups. We recall that the Weyl group acts on $\mathfrak{h}^*$ via the dot-action

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

where ρ is the Weyl vector (the half sum of positive roots) in $\mathfrak{h}^*$. Recall that we identify any character of H with an integral weight in $\mathfrak{h}^*$, and any integral weight can be written as $w \cdot \lambda$, where $w \in W$ and λ is a dominant integral weight. Since $w \in W$, we can write it as a product of n simple reflections $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_n}$. The length $\ell(w)$ of w is the minimal n required to write w as such a product.

Theorem 5.1.7 (Borel-Weil-Bott)**.** *If λ is a dominant weight of H and $w \in W$, then*

$$H^{\ell(w)}(G/B, \mathscr{L}(w \cdot \lambda)) \simeq V(\lambda)^*,$$

and $H^i(G/B, \mathscr{L}(w \cdot \lambda)) = 0$ if $i \neq \ell(w)$.

We shall not prove this theorem, and instead refer the reader to [Kum12].

5.2 Twisted differential operators on the flag variety

While one may first be enticed to study G-modules explicitly, just as how it is unavoidable to consider the Lie algebra $\mathfrak{g}$ in the study of G, it is also the case that one finds a good deal of richness of G-module theory in the somewhat simpler study of $\mathfrak{g}$-modules. In their 1981 paper [BB81], Beilinson and Bernstein generalize the theorem of Borel-Weil by showing an equivalence of categories between $\mathfrak{g}$-modules of dominant, regular, infinitesimal character λ, and of a certain type of twisted D-modules on the flag variety G/B. More precisely, we recall that for the sheaf of differential operators on an arbitrary variety X its sections on an open subset $U \subset X$ are $\mathscr{D}_X(U) = \cup_{n\geq 0}\mathscr{D}_X^n(U)$, where $\mathscr{D}_X^0(U) = \mathscr{O}_X(U)$, and for $n \geq 1$,

$$\mathscr{D}_X^n(U) := \{\theta \in \mathrm{End}(\mathscr{O}_X(U)) \mid [\theta, f] \in \mathscr{D}_X^{n-1}(U), \text{ for all } f \in \mathscr{O}_X(U)\}.$$

Given a line bundle $\mathscr{L}$ on X, we obtain a sheaf of twisted differential operators whose sections on the open set $U \subset X$ are $\mathscr{D}_{\mathscr{L}}(U) := \cup_{n\geq 0}\mathscr{D}_{\mathscr{L}}^n(U)$, where again, $\mathscr{D}_{\mathscr{L}}^0 = \mathscr{O}_X^0$, but instead we have for $n \geq 1$,

$$\mathscr{D}_{\mathscr{L}}^n(U) := \{\theta \in \mathrm{End}(\mathscr{L}(U)) \mid [\theta, f] \in \mathscr{D}_{\mathscr{L}}^{n-1}(U), \text{ for all } f \in \mathscr{O}_X(U)\}.$$

To see that this is indeed a TDO on X, it is immediately clear that there is an embedding $\iota : \mathscr{O}_X \hookrightarrow \mathscr{D}_{\mathscr{L}}$ and a filtration $\mathscr{D}_{\mathscr{L}}^\bullet$ on $\mathscr{D}_{\mathscr{L}}$ such that $\iota(\mathscr{O}_X) = \mathscr{D}_{\mathscr{L}}^0$, since we have defined $\mathscr{D}_{\mathscr{L}}^0 = \mathscr{O}_X$. The isomorphisms $\mathscr{D}_{\mathscr{L}}^1/\mathscr{D}_{\mathscr{L}}^0 \xrightarrow{\sim} \Theta_X$ and $\mathrm{Sym}(\mathscr{D}_{\mathscr{L}}^1/\mathscr{D}_{\mathscr{L}}^0) \xrightarrow{\sim} \mathrm{gr}(\mathscr{D}_{\mathscr{L}})$ are clear since $\mathscr{L}$ is a line bundle. When $X = G/B$, any line bundle $\mathscr{L}$ corresponds uniquely to an integral weight $\lambda \in \mathfrak{h}^*$, and so in this case we shall denote by $\mathscr{D}_\lambda$ the TDO of differential endomorphisms of the line bundle corresponding to the weight $\lambda - \rho$, where ρ denotes the Weyl vector (the half sum of positive roots). This ρ-shift occurs due to the action of the Weyl group being given by the dot-action $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Example 5.2.1. Let $G = \mathrm{SL}_2(\mathbb{C})$, and $X = G/B$. This example is detailed in [Rom20] (p. 4-5). We fix an integral weight $\lambda \in \mathfrak{h}^*$, and noting that $\mathfrak{h}^*$ may be identified with $\mathbb{C}$, we can view λ to be an integer. Furthermore, under this identification, the only positive root is 2, so $\rho = 1$. To describe the sheaf $\mathscr{D}_\lambda$, it shall suffice to describe the gluing of $\mathscr{D}_\lambda$ on an open affine cover, since $\mathscr{D}_\lambda$ is quasi-coherent. As usual, let $U_0 = X \setminus (0:1)$, and $U_\infty = X \setminus (1:0)$. Since both U_0 and U_∞ are isomorphic to $\mathbb{C}$, it is easy to compute $\mathscr{D}_\lambda(U_0)$ and $\mathscr{D}_\lambda(U_\infty)$; they are both isomorphic to the usual rings of differential operators on the polynomial ring $\mathbb{C}[t]$. The twist in $\mathscr{D}_\lambda$ arises in how we glue together the rings on these two open sets. Recall that the sheaf $\mathscr{O}(\lambda - 1)$ has sections

$$\mathscr{O}(\lambda-1)(U_0) = \mathbb{C}[z],$$

$$\mathscr{O}(\lambda-1)(U_\infty) = \mathbb{C}[w],$$

where we identify $w = z^{-1}$, and so on the intersection,

$$\mathscr{O}(\lambda-1)(U_0 \cap U_\infty) = \mathbb{C}[z, z^{-1}].$$

The global sections may be regarded as

$$\mathscr{O}(\lambda-1)(X) = \{f(z) \in \mathbb{C}[z] \mid \deg f \leq \lambda - 1\},$$

and then we have the commutative diagram

$$\begin{array}{ccccc} & & \mathscr{O}(\lambda-1)(U_0) & & \\ & \nearrow & & \searrow & \\ \mathscr{O}(\lambda-1)(X) & & & & \mathscr{O}(\lambda-1)(U_0 \cap U_\infty) \\ & \searrow & & \nearrow & \\ & & \mathscr{O}(\lambda-1)(U_\infty) & & \end{array}$$

where the top two arrows are just the natural inclusions, the arrow $\mathscr{O}(\lambda-1)(X) \to \mathscr{O}(\lambda-1)(U_\infty)$ is given by $1 \mapsto z^{-\lambda+1} = w^{\lambda-1}$, and the arrow $\mathscr{O}(\lambda-1)(U_\infty) \to \mathscr{O}(\lambda-1)(U_0 \cap U_\infty)$ is given by $1 \mapsto z^{\lambda-1}$. Now the differential endomorphisms on $\mathscr{O}(\lambda-1)(U_0)$, $\mathscr{O}(\lambda-1)(U_\infty)$ and $\mathscr{O}(\lambda-1)(U_0 \cap U_\infty)$ are the obvious ones, so we just need to find the differential endomorphisms on the global sections. The key point is that the differential endomorphisms commute with the arrows in the above diagram, so a global section of $\mathscr{D}_\lambda$ will correspond to differential endomorphisms on $\mathscr{O}(\lambda-1)(U_0)$ and $\mathscr{O}(\lambda-1)(U_\infty)$ which coincide on their restriction to $\mathscr{O}(\lambda-1)(U_0 \cap U_\infty)$. In other words, we may consider a global section of $\mathscr{D}_\lambda$ to be a pair (θ, θ'), where $\theta \in \mathscr{D}_\lambda(U_0)$, $\theta' \in \mathscr{D}_\lambda(U_\infty)$, and if f corresponds to a global section in $\mathscr{O}(\lambda-1)$, then

$$\theta f = z^{\lambda-1} \cdot \theta'(z^{1-\lambda} f). \tag{5.1}$$

Since $\mathscr{D}_\lambda$ is a TDO, to describe the global sections of $\mathscr{D}_\lambda$ it suffices to find the pair (θ, θ'), in the case that $\theta = \partial_z$. Since f is a polynomial, by linearity of ∂_z, we may replace f by z^m in (5.1). Hence, we have

$$mz^{m-1} = \partial_z z^m = z^\lambda \cdot \theta' z^{m-\lambda},$$

and a quick computation reveals that $\theta' = \partial_z + \frac{\lambda-1}{z}$.

Next, we want to find a morphism from $\mathcal{U}(\mathfrak{g})$ to $\Gamma(X, \mathscr{D}_\lambda)$. There is a natural G-action on $\Gamma(X, \mathscr{O}(\lambda))$ given by $(g \cdot f)(g'B) = f(g^{-1}gB)$, for $g, g' \in G$, $f \in \Gamma(X, \mathscr{O}(\lambda-1))$. Differentiating this action gives an action of the Lie algebra $\mathfrak{g}$ on $\Gamma(X, \mathscr{O}(\lambda-1))$; explicitly, for $A \in \mathfrak{g}$, $g \in G$, and $f \in \Gamma(X, \mathscr{O}(\lambda-1))$,

$$(A \cdot f)(gB) = \frac{d}{dt}\bigg|_{t=0} f(\exp(At)^{-1} gB).$$

Now if p is another element of $\Gamma(X, \mathscr{O}(\lambda-1))$, then we have

$$\begin{aligned}
([A,p] \cdot f)(gB) &= A \cdot p \cdot f - p \cdot A \cdot f)(gB) \\
&= \frac{d}{dt}\bigg|_{t=0} p(\exp(At)^{-1} gB) f(\exp(At)^{-1} gB) - p(gB) \frac{d}{dt}\bigg|_{t=0} f(\exp(At)^{-1} gB) \\
&= \frac{d}{dt}\bigg|_{t=0} p(\exp(At)^{-1} gB) \cdot f(gB),
\end{aligned}$$

where the last line is just an application of the product rule. Hence, $[A,p] \cdot f \in \Gamma(X, \mathscr{O}(\lambda-1))$, and so it follows that any $A \in \mathfrak{g}$ corresponds to a derivation on $\mathscr{O}(\lambda-1)$. Thus, we have a Lie algebra morphism $\mathfrak{g} \to \operatorname{Der} \Gamma(X, \mathscr{O}(\lambda-1))$, which extends naturally to the desired algebra morphism from $\mathcal{U}(\mathfrak{g})$ to $\Gamma(X, \mathscr{D}_\lambda)$. It is clear then that any $\Gamma(X, \mathscr{D}_\lambda)$-module has the structure of a $\mathcal{U}(\mathfrak{g})$-module via the morphism $\mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathscr{D}_\lambda)$.

Now, let $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ be the (infinitesimal) central character associated to the weight λ. The following theorem due to Konstant is needed to give the desired equivalence of categories in the Beilinson-Bernstein localization theorem.

Theorem 5.2.2. *The morphism $\mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathscr{D}_\lambda)$ is surjective with kernel $\mathcal{U}(\mathfrak{g}) \cdot \ker \chi_\lambda$. Moreover, it sends $z \in \mathcal{Z}(\mathfrak{g})$ to $\chi_\lambda(z) \cdot \mathrm{id}$.*

The point of this theorem is that if M is a $\Gamma(X, \mathscr{D}_\lambda)$-module, and we give it the structure of a $\mathcal{U}(\mathfrak{g})$-module as described above, then the center $\mathcal{Z}(\mathfrak{g})$ will act on M via the infinitesimal character, and so M has the structure of a $\mathcal{U}(\mathfrak{g})$-module of infinitesimal character χ_λ. If we let $\mathcal{J}_\lambda$ denote the maximal ideal corresponding to $\lambda \in \mathfrak{h}^*$, and $I_{|\lambda|} = \pi^{-1}\mathcal{J}_\lambda$ the pullback along $\pi : \mathfrak{h}^* \to \mathfrak{h}^*/W$, then $I_{|\lambda|}$ is the maximal ideal in $\mathcal{Z}(\mathfrak{g})$ of polynomials which vanish at any element in the Weyl group orbit of λ. Next, let $\mathcal{U}_{|\lambda|} := \mathcal{U}(\mathfrak{g})/I_{|\lambda|}\mathcal{U}(\mathfrak{g})$. It is clear that the above theorem can now be restated to say that $\mathcal{U}_{|\lambda|} \to \Gamma(X, \mathscr{D}_\lambda)$ is an isomorphism. We can therefore view $\mathscr{D}_\lambda$ as a sheafified version of $\mathcal{U}_{|\lambda|}$, and by either localizing or taking global sections, we can go back and forth between the two.

Example 5.2.3. Returning to our $G = \mathrm{SL}_2$ example, to describe the morphism $\mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathscr{D}_\lambda)$, it suffices to find the elements of $\Gamma(X, \mathscr{D}_\lambda)$ corresponding to the generators $E, F, H \in \mathfrak{g}$. Recall that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, acts on the elements $(x : y) \in \mathbb{P}^1$ by

$$g \cdot (x : y) = (ax + by : cx + dy).$$

In particular, it acts on $z = (1 : z) \in U_0$ by

$$g \cdot z = \frac{c + dz}{a + bz},$$

and on $w = (w : 1) \in U_\infty$ by

$$g \cdot w = \frac{aw + b}{cw + d}.$$

There are naturally induced $\mathrm{SL}_2(\mathbb{C})$-actions on $\mathscr{O}(\lambda - 1)(U_0)$ and $\mathscr{O}(\lambda - 1)(U_\infty)$ given by

$$(g \cdot p)(z) = p(g^{-1} \cdot z) = p\left(\frac{-c + az}{d - bz}\right),$$

and

$$(g \cdot q)(w) = q(g^{-1} \cdot w) = q\left(\frac{dw - b}{-cw + a}\right),$$

where $p \in \mathscr{O}(\lambda - 1)(U_0)$, and $q \in \mathscr{O}(\lambda - 1)(U_\infty)$. In order to obtain a well-defined action on the global sections $\Gamma(X, \mathscr{O}(\lambda - 1))$, we need to twist these actions in the following way. If $p \in \mathscr{O}(\lambda - 1)(U_0)$ and $q \in \mathscr{O}(\lambda - 1)(U_\infty)$ correspond to the same global section, then $p(z) = z^{\lambda-1} q(z^{-1})$, and so we want our action to satisfy $(g \cdot p)(z) = z^{\lambda-1}(g \cdot q)(z^{-1})$. The equation

$$\begin{aligned}(d - bz)^{\lambda-1} p\left(\frac{-c + az}{d - bz}\right) &= (d - bz)^{\lambda-1}\left(\frac{-c + az}{d - bz}\right)^{\lambda-1} q\left(\frac{d - bz}{-c + az}\right)\\ &= z^{\lambda-1}(-cz^{-1} + a)^{\lambda-1} q\left(\frac{dz^{-1} - b}{-cz^{-1} + a}\right).\end{aligned}$$

implies that taking

$$(g \cdot p)(z) = (d - bz)^{\lambda-1} p\left(\frac{-c + az}{d - bz}\right),$$

and

$$(g \cdot q)(w) = (-cw + a)^{\lambda-1} q\left(\frac{dw - b}{-cw + a}\right)$$

will give a well-defined action on the global sections $\Gamma(X, \mathscr{O}(\lambda))$, as desired. Now we can see that if $p \in \mathscr{O}(\lambda)(U_0)$ has degree $\leq \lambda - 1$, then it describes a global sections, and E, F and H act on p by:

$$\begin{aligned}(E \cdot p)(z) &= \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot p(z)\\ &= \frac{d}{dt}\bigg|_{t=0} (1 - tz)^{\lambda-1} p\left(\frac{z}{1 - tz}\right)\\ &= z^2 p'(z) - (\lambda - 1) z p(z),\end{aligned}$$

$$\begin{aligned}(F \cdot p)(z) &= \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot p(z)\\ &= \frac{d}{dt}\bigg|_{t=0} p(-t + z)\\ &= -p'(z),\end{aligned}$$

and

$$(H \cdot p)(z) = \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot p(z)$$
$$= \frac{d}{dt}\bigg|_{t=0} e^{-\lambda t} p(e^{2t} z)$$
$$= 2zp'(z) + (\lambda - 1)p(z).$$

Thus, the morphism from $\mathcal{U}(\mathfrak{g})$ to $\mathscr{D}_\lambda$ is the one sending

$$E \mapsto z^2 \partial_z - (\lambda - 1)z,$$

$$F \mapsto -\partial_z,$$

and

$$H \mapsto 2z\partial_z - (\lambda - 1).$$

Now looking from the point of view of differential operators on the base affine space, we recall that $\widetilde{X} = \mathrm{SL}_2(\mathbb{C})/N \simeq \mathbb{C}^2 \setminus \{0\}$, so the global differential operators of $\widetilde{X}$ are precisely $\mathscr{D}(\widetilde{X}) = \mathbb{C}[t_1, t_2, \partial_1, \partial_2]$, with the usual commutativity relations. The H-action on $\mathscr{D}_{\widetilde{X}}$ can be written explicitly as $h \cdot t_i = ht_i$, and $h \cdot \partial_i = \frac{1}{h}\partial_i$, for $h \in H$, $i = 1, 2$, and where we identify H with $\mathbb{C}^*$. The algebra of H-invariant differential operators is then the one generated by $t_1\partial_1$, $t_2\partial_2$, $t_1\partial_2$ and $t_2\partial_1$. We see that this is just the same as the algebra $\widetilde{\mathcal{U}}$, which is generated by $E = -t_2\partial_1$, $F = -t_1\partial_2$, $H = -t_1\partial_1 + t_2\partial_2$ and $T = t_1\partial_1 + t_2\partial_2$. Here T generates the center of $\widetilde{\mathcal{U}}$, so selecting a twist λ amounts to choosing some $\lambda \in \mathfrak{h}^* = \mathbb{C}$ such that T acts by λ.

5.3 $\mathscr{D}$-affine varieties and the Beilinson-Bernstein localization theorem

The last component to understand the localization theorem is the notion of a $\mathscr{D}$-affine variety. Somewhat analogously to how an affine variety is one whose structure sheaf has many global sections, a $\mathscr{D}$-affine variety is one whose sheaf of differential operators has many global sections. More precisely,

Definition 5.3.1. A variety X is called *$\mathscr{D}$-affine* if every $\mathscr{D}_X$-module M is generated by its global sections, and $H^i(X, M) = 0$ for $i > 0$.

In particular, if X is $\mathscr{D}$-affine, then the category of $\mathscr{D}_X$-modules is equivalent to the category of $\mathscr{D}(X) = \Gamma(X, \mathscr{D}_X)$-modules. The mutually inverse functors which provide this equivalence of categories are the global sections functor Γ and the localization functor $\Delta = \mathscr{D}_X \otimes_{\mathscr{D}(X)} -$. We can generalize this definition to any sheaf of twisted differential operators.

Definition 5.3.2. Let $\mathscr{A}$ be a TDO on a variety X. We say that X is *$\mathscr{A}$-affine* if every $\mathscr{A}$-module M is generated by its global sections, and $H^i(X, M) = 0$ for $i > 0$.

Then the desired equivalence of categories follows from the following theorem.

Theorem 5.3.3 (Beilinson-Bernstein)**.** *Let $\lambda \in \mathfrak{h}^*$ be dominant and regular. The flag variety G/B is $\mathscr{D}_\lambda$-affine.*

Corollary 5.3.4. *Let $\lambda \in \mathfrak{h}^*$ be dominant and regular. The category of $\mathcal{U}_{|\lambda|}$-modules is equivalent to the category of $\mathscr{D}_\lambda$-modules. The mutually quasi-inverse functors giving this equivalence are the global sections functor and the localization functor.*

Corollary 5.3.5. *The global sections functor Γ is exact.*

These statements and their proofs are the subject of [BB81].

Chapter 6

Representations of a semisimple Lie algebra from global sections of D-modules

In this section, we construct modules over a semisimple Lie algebra by taking global sections of some interesting D-modules in the base affine space $\mathrm{SL}_2(\mathbb{C})/N$. Throughout, we fix the notation G an algebraic group with semisimple Lie algebra $\mathfrak{g}$, let $X = G/B$ be the flag variety and $\widetilde{X} = G/N$ the base affine space.

6.1 Differential operators on G/N

Since the base affine space G/N is quasi-affine, the following proposition found in [Gro61] (p. 94) will be rather useful.

Proposition 6.1.1. *A quasi-compact scheme X is quasi-affine if and only if every quasi-coherent $\mathscr{O}_X$-module is generated by its global sections.*

Many of the sheaves we are working with are quasi-coherent, and so this proposition allows us to describe them purely by describing their global sections. For example, when $G = \mathrm{SL}_2(\mathbb{C})$, we identify $\widetilde{X} = \mathrm{SL}_2(\mathbb{C})/N$ with $\mathbb{C}^2 \setminus \{0\}$ as usual, and the structure sheaf is the one generated by its global sections $\mathbb{C}[t_1, t_2]$. In particular, the sets $V_1 = \widetilde{X} \setminus V(t_1)$ and $V_2 = \widetilde{X} \setminus V(t_2)$ form an open affine cover of $\widetilde{X}$, and the structure sheaf has sections

$$\mathscr{O}_{\widetilde{X}}(V_1) = \mathbb{C}[t_1, t_1^{-1}, t_2],$$
$$\mathscr{O}_{\widetilde{X}}(V_2) = \mathbb{C}[t_1, t_2, t_2^{-1}],$$

and

$$\mathscr{O}_{\widetilde{X}}(V_1 \cap V_2) = \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}].$$

Given any global section $f(t_1, t_2)$, it is clear that

$$\mathrm{res}_{V_1}^{V_1 \cap V_2} \mathrm{res}_{\widetilde{X}}^{V_1} f = \mathrm{res}_{V_2}^{V_1 \cap V_2} \mathrm{res}_{\widetilde{X}}^{V_2} f.$$

Using Proposition 6.1.1, the sheaf of differential operators $\mathscr{D}_{\widetilde{X}}$ on $\widetilde{X}$ can easily be found to be the quasi-coherent sheaf whose global sections are the differential operators on $\mathbb{C}[t_1, t_2]$, viz. $\Gamma(\mathscr{D}_{\widetilde{X}}) = \mathbb{C}[t_1, t_2, \partial_1, \partial_2]$, where it is understood that $\partial_i t_i - t_i \partial_i = 1$, for $i = 1, 2$.

We now proceed to give a description of the H-invariant differential operators on the base affine space G/N, following [BGG].

Let G be an algebraic group with corresponding Lie algebra $\mathfrak{g}$, let $B \subset G$ be a Borel subgroup and $N = [B, B] \subset G$ the corresponding maximal unipotent subgroup with corresponding Lie algebras of $\mathfrak{b}$ and $\mathfrak{b}$ respectively. Also let $H \subset B$ be the maximal torus subgroup of B whose Lie algebra is $\mathfrak{h}$. Now G has a natural left (respectively right) action on itself by left (resp. right) multiplication. These actions each induce a left action on the regular functions on G, and to distinguish between these two actions, we shall denote the one arising from the left action on G by L_g and the one arising from the right action on G by R_g. Explicitly, these actions are given by

$$L_g f(g') = f(g^{-1} g')$$

and

$$R_g f(g') = f(g' g),$$

where g, $g' \in G$ and $f \in \mathscr{O}(G)$, where $\mathscr{O}(G)$ is the algebra of (global) differential operators on G. In other words, $L : g \mapsto L_g$ and $R : g \mapsto R_g$ define representations of G. By differentiating these representations, we obtain actions of the Lie algebra $\mathfrak{g}$ on G as follows

$$L_A f(g) = \frac{d}{dt}\bigg|_{t=0} f(\exp(tA)^{-1}g),$$

$$R_A f(g) = \frac{d}{dt}\bigg|_{t=0} f(g\exp(tA)),$$

where $A \in \mathfrak{g}$, $g \in G$, $f \in \mathscr{O}(G)$, and as with the group action, we denote the action arising from L by L_A and the action arising from R by R_A. It is immediate from the Leibniz rule that these $\mathfrak{g}$-actions define derivations on $\mathscr{O}(G)$. We can naturally extend these $\mathfrak{g}$-actions to $\mathcal{U}(\mathfrak{g})$-actions, which correspond with differential operators on G. Also, since left multiplication commutes with right multiplication, for any $g, g' \in G$, (resp. $A, A' \in \mathcal{U}(\mathfrak{g})$,) L_g and $R_{g'}$ (resp. L_A and $R_{A'}$) commute.

Now base affine space $\widetilde{X} = G/N$ consists of cosets gN, where $g \in G$. We see that the L-action of $\mathcal{U}(\mathfrak{g})$ on $\mathscr{O}(G)$ defines an action of $\mathcal{U}(\mathfrak{g})$ on $\mathscr{O}(\widetilde{X})$, for which we shall use the same notation. The R-action of $\mathcal{U}(\mathfrak{g})$ is not well-defined on $\mathscr{O}(\widetilde{X})$, however since H normalizes N, restricting this action to $\mathcal{U}(\mathfrak{h})$ does give a well-defined action on $\mathscr{O}(\widetilde{X})$. As above, the $\mathcal{U}(\mathfrak{g})$-action L and the $\mathcal{U}(\mathfrak{h})$-action R define differential operators on $\widetilde{X}$, and again, the differential operators associated to this $\mathcal{U}(\mathfrak{h})$-action commute with the differential operators associated to this $\mathcal{U}(\mathfrak{g})$-action. It is also clear that $\mathcal{U}(\mathfrak{h})$ commutes with itself, so $\mathcal{U}(\mathfrak{h})$ is in the center of the algebra generated by the action L of $\mathcal{U}(\mathfrak{g})$ and the action R of $\mathcal{U}(\mathfrak{h})$. Obtaining differential operators from the algebra $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$ in this way leads us towards the general approach in which we can study $\mathcal{M}od_\lambda(\widetilde{\mathcal{U}})$ by looking at certain types of D-modules on $\widetilde{X} = G/N$.

Now let $X = G/B$ be the flag variety. Since $H \simeq B/N$, we have that $\pi : \widetilde{X} \to X$ is an H-torsor. As such, H acts naturally on $\widetilde{X}$ via right multiplication, and this action is compatible with π in the sense that for any $h \in H$, and any $x \in \widetilde{X}$, we have $\pi(x \cdot h) = \pi(x)$. This H-action naturally induces an action of H on $\mathscr{O}_{\widetilde{X}}$ described above, and then there is a natural action on $\mathscr{D}_{\widetilde{X}}$, the algebra of differential operators on $\widetilde{X}$ so that whenever $\theta \in \mathscr{D}_{\widetilde{X}}$, $h \in H$ and $f \in \mathscr{O}_{\widetilde{X}}$, we have

$$(R_h\theta)(R_h f) = R_h(\theta(f)). \tag{6.1}$$

We can check that this indeed defines a left action, since if h, $h' \in H$,

$$\begin{aligned}(R_h R_{h'}\theta)(f) &= R_h((R_{h'}\theta)(R_{h^{-1}}f)) \\ &= R_h(R_{h'}(\theta(R_{h'^{-1}}R_{h^{-1}}f))) \\ &= R_{hh'}(\theta(R_{(hh')^{-1}}f)) \\ &= (R_{hh'} \cdot \theta)(f).\end{aligned}$$

Then considering the H-invariant differential operators on $\widetilde{X}$, we see that they are those satisfying

$$\theta(R_h f) = R_h(\theta(f)). \tag{6.2}$$

Now since the differential operators obtained above from the $\mathcal{U}(\mathfrak{g})$-action L and the $\mathcal{U}(\mathfrak{h})$-action R all commute with those corresponding to this $\mathcal{U}(\mathfrak{h})$-action, it can be seen that they satisfy (6.2). We may consider on X, the sheaf $\pi_*\mathscr{D}_{\widetilde{X}}$, and this contains a subsheaf $\widetilde{\mathscr{D}} := [\pi_*\mathscr{D}_{\widetilde{X}}]^H \subset \pi_*\mathscr{D}_{\widetilde{X}}$ of H-invariant differential operators, and the above discussion shows that we have an injective morphism $\widetilde{\mathcal{U}} \to \Gamma(\widetilde{\mathscr{D}})$. In fact, this morphism is an isomorphism [BG97] (p. 7).

Example 6.1.2. Let $G = \mathrm{SL}_2(\mathbb{C})$, so that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, take B to be the group of upper triangular matrices, and N to be the subgroup of B with 1's along the main diagonal. We have seen that there is an isomorphism of varieties $G/N \simeq \mathbb{C}^2 \setminus \{0\}$ via the map

$$\begin{pmatrix} x_1 & * \\ x_2 & * \end{pmatrix} N \mapsto (x_1, x_2),$$

and so we shall make this identification between $\widetilde{X}$ and $\mathbb{C}^2 \setminus \{0\}$. Hence, regular functions on $\widetilde{X}$ are just polynomials in x_1 and x_2. We can then describe the actions of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{h}$ via L and R as follows. Here, we shall simply denote L_A by $A\cdot$, for $A \in \mathfrak{sl}_2(\mathbb{C})$, and for the generator $H \in \mathfrak{h}$, we write $T\cdot$ for R_H. Let $f \in \mathscr{O}(\widetilde{X})$. For the action of E, we compute explicitly,

$$\begin{aligned} E \cdot f(x_1, x_2) &= \frac{d}{dt}\bigg|_{t=0} f(\exp(E)^{-1} \cdot (x_1, x_2)) \\ &= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 & * \\ x_2 & * \end{pmatrix}\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f(x_1 - tx_2, x_2) \\ &= -x_2\partial_1 \cdot f(x_1, x_2), \end{aligned}$$

and via similar routine checks, we have

$$F \cdot f(x_1, x_2) = -x_1\partial_2 \cdot f(x_1, x_2),$$

and

$$H \cdot f(x_1, x_2) = (-x_1\partial_1 + x_2\partial_2) \cdot f(x_1, x_2).$$

Additionally, we have

$$\begin{aligned} T \cdot f(x_1, x_2) &= \frac{d}{dt}\bigg|_{t=0} f((x_1, x_2) \cdot \exp(H)) \\ &= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_1 & * \\ x_2 & * \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f(e^t x_1, e^t x_2) \\ &= (x_1\partial_1 + x_2\partial_2) \cdot f(x_1, x_2). \end{aligned}$$

Next, we look to describe the sheaves $\pi_*\mathscr{D}_{\widetilde{X}}$ and $\widetilde{\mathscr{D}}$ on $X = G/B \simeq \mathbb{P}^1$. It suffices to describe them on the open affine patches $U_1 := \mathbb{P}^1 \setminus \{(0,1)\}$ and $U_2 := \mathbb{P}^1 \setminus \{(1,0)\}$ and to give the gluing conditions. We let $V_1 = \pi^{-1}U_1 = \mathbb{C}^2 \setminus V(x_1)$ and $V_2 = \pi^{-1}U_2 = \mathbb{C}^2 \setminus V(x_2)$ where $V(f(x_1, x_2))$ denotes the vanishing of the polynomial $f(x_1, x_2)$. It is clear then that

$$\pi_*\mathscr{D}_X(U_1) = \mathscr{D}_X(V_1) = \mathbb{C}[x_1, x_1^{-1}, x_2, \partial_1, \partial_2],$$

$$\pi_*\mathscr{D}_X(U_2) = \mathscr{D}_X(V_2) = \mathbb{C}[x_1, x_2, x_2^{-1}, \partial_1, \partial_2],$$

and since $\pi_*\mathscr{D}_X(\mathbb{P}^1) = \mathscr{D}_X(\mathbb{C}^2 \setminus \{0\}) = \mathbb{C}[x_1, x_2, \partial_1, \partial_2]$, the gluing conditions are the obvious ones. Now using (6.1), we can conclude that the H-action on $\pi_*\mathscr{D}_X$ can be described by

$$R_t x_i = t x_i,$$

and

$$R_t \partial_i = t^{-1}\partial_i,$$

for $i = 1, 2$, and where $t \in H$ can be regarded as an element of $\mathbb{C}^*$ under the identification of the groups $H \simeq \mathbb{C}^*$. We then obtain a clear description of $\widetilde{\mathscr{D}}$, as $\widetilde{\mathscr{D}}(U_1)$ is generated by the differential operators $x_1^{-1}x_2$, and $x_i\partial_j$, $i, j = 1, 2$, and $\widetilde{\mathscr{D}}(U_2)$ is generated by $x_1x_2^{-1}$ and the $x_i\partial_j$. The global sections at then those generated by the $x_i\partial_j$. Noting that

$$E = -x_2\partial_1,$$

$$F = -x_1\partial_2,$$

$$H + T = 2x_2\partial_2,$$

and

$$H - T = -2x_1\partial_1,$$

we may instead describe the global sections of $\widetilde{\mathscr{D}}$ as those elements generated by the differential operators E, F, H and T, i.e. precisely those differential operators arising from elements in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{h})$. We can see that $\widetilde{\mathscr{D}}$ acts on the structure sheaf of $\mathbb{P}^1$, since the natural projection map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ defined by $(x_1, x_2) \mapsto (x_1 : x_2)$ allows us to view $z = (1 : z) \in U_0 \subset \mathbb{P}^1$ as $\frac{x_2}{x_1}$, and similarly, $w = (w : 1)$ as $\frac{x_1}{x_2}$. Then E, F, H, and T act on a monomial z^n by

$$E \cdot z^n = nz^{n+1} = z^2 \partial_z z^n,$$

$$F \cdot z^n = -nz^{n-1} = -\partial_z z^n,$$

$$H \cdot z^n = 2nz^n = 2z\partial_z z^n,$$

and

$$T \cdot z^n = 0.$$

One can similarly find the actions on the monomial w^n, and this describes the action of $\widetilde{\mathscr{D}}$ on $\mathscr{O}_{\mathbb{P}^1} = \mathscr{O}(0)$. In particular, we remark that since $T \cdot z^n = 0$, the center of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{h})$, and hence the center of $\mathcal{U}(\mathfrak{g})$ acts by 0. In fact, this description shows that the $\widetilde{\mathscr{D}}$-modules on which T acts by 0 are precisely the D-modules on $\mathbb{P}^1$ (with no twist). If we instead consider $\widetilde{\mathscr{D}}$-modules on which T acts by a fixed positive integer λ, then we would get D-modules twisted by λ on $\mathbb{P}^1$. To see this, we actually observe a rather nice way to view the twisted sheaves $\mathscr{O}(\lambda)$. Note that if T acts on a submodule $M \subset \mathscr{O}_{\mathbb{C}^2 \setminus \{0\}}$ by a constant integer λ, then any $f \in M$ is made up of monomials of the form $x_1^m x_2^n$ such that $m + n = \lambda$. We consider the "twisted" projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ which is defined on $V_1 = \mathbb{C}^2 \setminus V(x_1)$ by

$$(x_1, x_2) \mapsto (x_1^\lambda : x_2) = (1 : x_1^{-\lambda} x_2) = (1 : z x_1^{1-\lambda}),$$

and on $V_2 = \mathbb{C}^2 \setminus V(x_2)$ by

$$(x_1, x_2) \mapsto (x_1 : x_2^\lambda) = (x_1 x_2^{-\lambda} : 1) = (w x_2^{1-\lambda} : 1),$$

where $z = x_1^{-1} x_2$ and $w = x_1 x_2^{-1}$. In particular, on U_0, we can view

$$z^n = \frac{x_2^n}{x_1^{n-(\lambda-1)}}$$

and in U_∞,

$$w^n = \frac{x_1^n}{x_2^{n-(\lambda-1)}}.$$

We can then compute

$$E \cdot z^n = (n - (\lambda - 1)) z^{n+1} = (z^2 \partial_z - (\lambda - 1) z) z^n,$$

$$F \cdot z^n = -nz^{n-1} = -\partial_z z^n,$$

$$H \cdot z^n = (2n - (\lambda - 1)) z^n = (2z\partial_z - (\lambda - 1)) z^n,$$

and

$$T \cdot z^n = (n - (n - \lambda)) z^n = \lambda z^n,$$

and we can compute the actions similarly for w. Observe that the actions of E, F and H are identical to those we computed for the TDO $\mathscr{D}_\lambda$ in Section 5.2. Thus, we see explicitly in the case of $\mathfrak{sl}_2(\mathbb{C})$, that selecting a particular monodromy - i.e. a particular constant by which T should act - is the same as selecting the twist of differential operators, which is the same as selecting the infinitesimal character of the $\mathcal{U}(\mathfrak{g})$-module. This idea holds in the more general setting, and its utility should become evident in the next example, where twisted sheaves of differential operators over the flag variety will be much more cumbersome to work with.

Example 6.1.3. Let $G = \mathrm{SL}_3(\mathbb{C})$, so that $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, let B be the Borel subgroup consisting of upper triangular matrices, $N \subset B$ the maximal unipotent subgroup, $H \simeq B/N$ the Cartan subgroup. Let $\widetilde{X} = G/N$ be the base affine space. We have an open embedding of $\widetilde{X}$ into an affine subvariety $Y \subset \mathbb{C}^6$ defined by mapping the coset

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N \mapsto (x_1, x_2, x_3, c_1, c_2, c_3),$$

where $x_1 = x_{11}$, $x_2 = x_{21}$, $x_3 = x_{31}$, and c_1, c_2 and c_3 are the minors

$$c_1 = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix},$$

$$c_2 = -\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix},$$

$$c_3 = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

The obvious fact that

$$\det \begin{pmatrix} x_{11} & x_{11} & x_{12} \\ x_{21} & x_{21} & x_{22} \\ x_{31} & x_{31} & x_{32} \end{pmatrix} = 0$$

implies that $Y \subset V(x_1c_1 + x_2c_2 + x_3c_3)$, and since $\widetilde{X}$ is a 5-dimensional variety, we in fact have $Y = V(x_1c_1 + x_2c_2 + x_3c_3)$. Linear independence conditions of the columns of cosets in $\widetilde{X}$ allows us to obtain an explicit isomorphism

$$\widetilde{X} \simeq Y \setminus (V(x_1, x_2, x_3) \cup V(c_1, c_2, c_3)).$$

The regular functions on $\widetilde{X}$ are then precisely the regular functions on Y, which are elements of

$$\frac{\mathbb{C}[x_1, x_2, x_3, c_1, c_2, c_3]}{(x_1c_1 + x_2c_2 + x_3c_3)}.$$

Differential operators on Y (and hence $\widetilde{X}$) are then those on $\mathbb{C}^6$ which stabilize the ideal $(x_1c_1 + x_2c_2 + x_3c_3)$. Since Y is a smooth variety, the ring of differential operators on Y is generated by the derivations on Y - which are the derivations on $\mathbb{C}^6$ which preserve the ideal $(x_1c_1 + x_2c_2 + x_3c_3)$. Some quick derivative computations and staring will then convince oneself that the ring of differential operators on Y is generated by all those of the form

$$x_i\partial_{x_j} - c_j\partial_{c_i}, \qquad c_i\partial_{x_j} - c_j\partial_{x_i}, \qquad x_i\partial_{c_j} - x_j\partial_{c_i},$$

$$x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}, \qquad c_1\partial_{c_1} + c_2\partial_{c_2} + c_3\partial_{c_3}.$$

By the above discussion, the differential operators arising from the left $\mathcal{U}(\mathfrak{g})$-action and from the right $\mathcal{U}(\mathfrak{h})$-action can be seen to be differential operators on $\widetilde{X}$, and moreover these are all the H-invariant ones. We shall compute these differential operators in the sequel. For $1 \leq i, j \leq 3$, let E_{ij} be the 3×3 matrix with a 1 in the (i, j) position and 0's everywhere else. Let $H_1 = E_{11} - E_{22}$ and $H_2 = E_{22} - E_{33}$. A basis for $\mathfrak{sl}_3(\mathbb{C})$ is given by H_1, H_2, and the E_{ij}, with $i \neq j$, The subalgebra $\mathfrak{n} = \mathfrak{Lie}(N)$ (respectively $\mathfrak{n}^-$) has a basis given by the E_{ij} with $i < j$ (respectively $i > j$), and $\mathfrak{h}$ has a basis given by H_1 and H_2. We can find the differential operator associated to e.g. E_{12} via the explicit computation

$$\begin{aligned} E_{12} \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) &= \frac{d}{dt}\bigg|_{t=0} f(\exp(E_{12})^{-1} \cdot (x_1, x_2, x_3, c_1, c_2, c_3)) \\ &= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} - tx_{21} & x_{12} - tx_{22} & x_{13} - tx_{23} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f(x_1 - tx_2, x_2, x_3, c_1, c_2 + tc_1, c_3) \\ &= (-x_2\partial_{x_1} + c_1\partial_{c_2}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3). \end{aligned}$$

In fact, via a similar computation, we can find that for $i \neq j$,

$$E_{ij} \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) = (-x_j\partial_{x_i} + c_i\partial_{c_j}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3).$$

We can also find the corresponding differential operator for H_1 as

$$
\begin{aligned}
H_1 \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) &= \frac{d}{dt}\bigg|_{t=0} f(\exp(H_1)^{-1} \cdot (x_1, x_2, x_3, c_1, c_2, c_3)) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} e^{-t}x_{11} & e^{-t}x_{12} & e^{-t}x_{13} \\ e^t x_{21} & e^t x_{22} & e^t x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f(e^{-t}x_1, e^t x_2, x_3, e^t c_1, e^{-t} c_2, c_3) \\
&= (-x_1\partial_{x_1} + x_2\partial_{x_2} + c_1\partial_{c_1} - c_2\partial_{c_2}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3),
\end{aligned}
$$

and similarly, we can find

$$
H_2 \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) = (-x_2\partial_{x_2} + x_3\partial_{x_3} + c_2\partial_{c_2} - c_3\partial_{c_3}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3).
$$

Finally, we can compute the differential operators corresponding to the right $\mathfrak{h}$-action. We shall denote by T_i the differential operator corresponding to the right action of H. We have as follows:

$$
\begin{aligned}
T_1 \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) &= \frac{d}{dt}\bigg|_{t=0} f((x_1, x_2, x_3, c_1, c_2, c_3) \cdot \exp(H_1)) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N \cdot \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} e^t x_{11} & e^{-t}x_{12} & x_{13} \\ e^t x_{21} & e^{-t}x_{22} & x_{23} \\ e^t x_{31} & e^{-t}x_{32} & x_{33} \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f(e^t x_1, e^t x_2, e^t x_3, c_1, c_2, c_3) \\
&= (x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3),
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 \cdot f(x_1, x_2, x_3, c_1, c_2, c_3) &= \frac{d}{dt}\bigg|_{t=0} f((x_1, x_2, x_3, c_1, c_2, c_3) \cdot \exp(H_2)) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} N \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{pmatrix} x_{11} & e^t x_{12} & e^{-t}x_{13} \\ x_{21} & e^t x_{22} & e^{-t}x_{23} \\ x_{31} & e^t x_{32} & e^{-t}x_{33} \end{pmatrix} N\right) \\
&= \frac{d}{dt}\bigg|_{t=0} f(x_1, x_2, x_3, e^t c_1, e^t c_2, e^t c_3) \\
&= (c_1\partial_{c_1} + c_2\partial_{c_2} + c_3\partial_{c_3}) \cdot f(x_1, x_2, x_3, c_1, c_2, c_3).
\end{aligned}
$$

The H-invariance of these operators can be seen since they all commute with T_1 and T_2. A regular central character of an $\mathfrak{sl}_3$-module M then amounts to a fixed pair of complex numbers $(\chi(T_1), \chi(T_2))$ which form a regular weight, and so that for any $v \in M$, we have $T_1 \cdot v = \chi(T_1)v$, and $T_2 \cdot v = \chi(T_2)v$. Then rather than dealing with twisted D-modules on the flag variety, we can simply consider D-modules on $\widetilde{X}$ such that T_1 and T_2 act by fixed constants.

Example 6.1.4. The more general case of $G = \mathrm{SL}_n(\mathbb{C})$ is quite similar to the case of $\mathrm{SL}_3(\mathbb{C})$, although the notation cannot be made to be so nice. We recall the structure of the base affine space as an affine variety is given in Section 2.6. It seems straightforward albeit tedious to compute the differential operators corresponding to the left G-action, if we just follow a similar procedure as in the case of $\mathfrak{sl}_3(\mathbb{C})$. It is not be too difficult, however, to describe explicitly the differential operators corresponding to the right H-action. For $1 \leq i \leq n-1$, let $H_i = E_{ii} - E_{(i+1)(i+1)}$, where E_{ij} denotes the matrix with 1 in the (i,j)-position and 0's elsewhere. In $\mathbb{C}^{2^n-2}$, we denote coordinates by x_{ij}, where the corresponding minor is $i \times i$, $1 \leq i \leq n-1$, and $1 \leq j \leq \binom{n}{i}$. Then the differential operator corresponding to T_i is

$$\sum_j x_{ij}\partial_{ij},$$

where the sum is taken over all $1 \leq j \leq \binom{n}{i}$.

6.2 Finite Dimensional Representations

The following proposition will allow us to characterize the finite dimensional $\mathfrak{g}$-modules.

Proposition 6.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Any finite dimensional irreducible $\mathfrak{g}$-module V is a highest weight module of dominant integral weight.*

Proof. We can use the weight space decomposition of $\mathfrak{g}$ to see that there exists a nonzero vector $v \in V$ and a weight $\lambda \in \mathfrak{h}^*$ for which $\mathfrak{n}^+ v = 0$ and $h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}$, so v is the maximal weight vector in V (of weight λ). Since V is irreducible, it is generated by v, so V is a highest weight module of weight λ. All that's left to show is that λ is dominant and integral; in fact, if λ is not both dominant and integral, then the irreducible highest weight module $L(\lambda)$ is not finite dimensional. If this result holds for $\mathfrak{sl}_2(\mathbb{C})$, then we can show that it will also hold in the general case, as we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha).$$

Given any $\alpha \in \Phi^+$ we can then find an $\mathfrak{sl}_2$-triple $(x_\alpha, y_\alpha, h_\alpha)$, where $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$. We denote the Lie algebra generated by x_α, y_α and h_α by ${}_2(\alpha)$. Then a $\mathfrak{g}$-module V is finite dimensional if and only if it is finite dimensional as an $\mathfrak{sl}_2(\alpha)$-module for every $\alpha \in \Phi^+$. Since $\mathfrak{sl}_2(\alpha) \simeq \mathfrak{sl}_2(\mathbb{C})$, all that remains to see is that a choice of weight $\lambda \in \mathfrak{h}^*$ is precisely a choice of constants $\lambda(h_\alpha) \in \mathbb{C}$, and moreover λ is dominant and integral if and only if every $\lambda(h_\alpha) \in \mathbb{N}$. But this just follows immediately from the definition of a dominant and integral weight. This completes the proof modulo the result for $\mathfrak{sl}_2(\mathbb{C})$ which we shall now show. Let $v \in L(\lambda)$ be a vector of highest weight λ. For any $n \geq 0$, we have

$$\begin{aligned}
E \cdot F^{n+1} \cdot v &= [E,F] \cdot F^n \cdot v + F \cdot E \cdot F^n \cdot v \\
&= H \cdot F^n \cdot v + F \cdot E \cdot F^n \cdot v \\
&= (\lambda - 2n)F^n \cdot v + F \cdot E \cdot F^n \cdot v \\
&= F^{n+1} \cdot E \cdot v + \sum_{i=0}^{n} F^{n-i} \cdot (\lambda - 2i)F^i \cdot v \\
&= \sum_{i=0}^{n} (\lambda - 2i)F^n \cdot v \\
&= \left((n+1)\lambda - 2\frac{n(n+1)}{2}\right) F^n \cdot v \\
&= (n+1)(\lambda - n)F^n \cdot v.
\end{aligned}$$

By induction on n, we see that if $\lambda \notin \mathbb{N}$, then $F^n \cdot v \neq 0$ for all $n \in \mathbb{N}$, in which case $L(\lambda)$ is infinite dimensional. ☐

With this proposition under our belt, along with the discussion of differential operators in the base affine space, we have the following result.

Theorem 6.2.2. *The finite dimensional irreducible $\mathfrak{g}$-modules exactly correspond to the monodromic global sections of the $\mathscr{D}_{\widetilde{X}}$-module $\mathscr{O}_{\widetilde{X}}$ on which $\mathfrak{h}^*$ acts by a dominant integral character.*

We will not prove this theorem in its full generality, but we shall look at some concrete examples which will give some intuition for why it holds, at least in the case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

Example 6.2.3. First, if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have seen that the global sections of the structure sheaf are $\Gamma(\mathscr{O}_{\widetilde{X}}) = \mathbb{C}[x_1, x_2]$, and the global sections of the ring of differential operators on $\widetilde{X}$ are $\Gamma(\mathscr{D}_{\widetilde{X}}) = \mathbb{C}[x_1, x_2, \partial_1, \partial_2]$. We wish to show that the global sections of $\mathscr{O}_{\widetilde{X}}$, viewed as a $\mathscr{D}_{\widetilde{X}}$-module, give us all the irreducible finite dimensional modules. We can identify the H-invariant operators of $\Gamma(\mathscr{D}_{\widetilde{X}})$ with $\widetilde{\mathcal{U}}$, where $E = -x_2\partial_1$, $F = -x_1\partial_2$, $H = -x_1\partial_1 + x_2\partial_2$, and $T = x_1\partial_1 + x_2\partial_2$. Since the elements of $\Gamma(\mathscr{O}_{\widetilde{X}}) = \mathbb{C}[x_1, x_2]$ are linear combinations of monomials in x_1 and x_2, it suffices to check the $\widetilde{\mathcal{U}}$ action on an arbitrary monomial $x_1^m x_2^n$, $m, n \geq 0$. We can compute

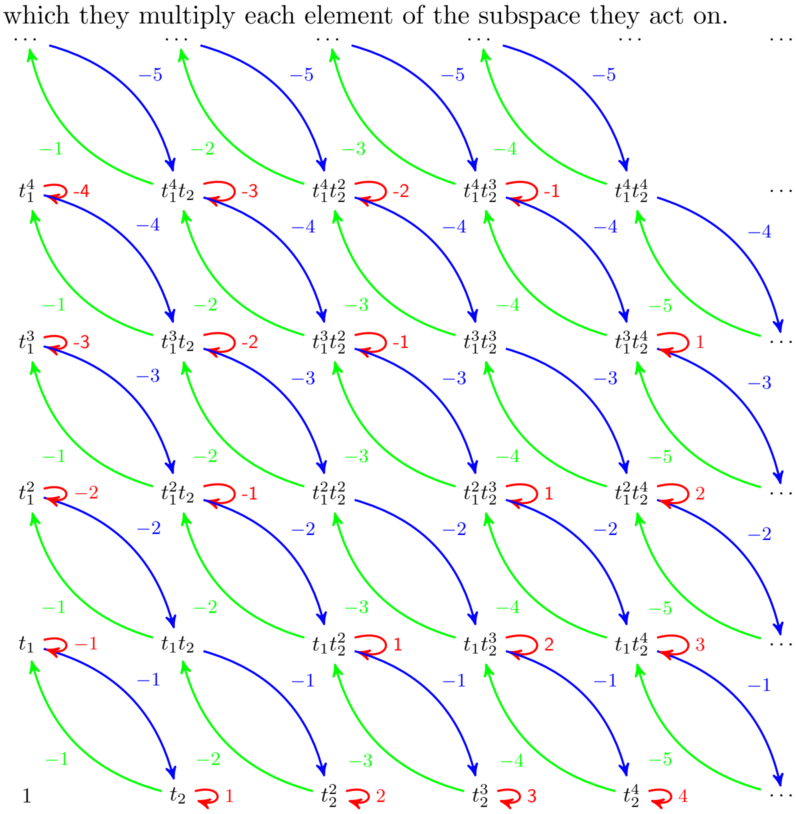
$$E \cdot x_1^m x_2^n = -m x_1^{m-1} x_2^{n+1}, \tag{6.3}$$

$$F \cdot x_1^m x_2^n = -n x_1^{m+1} x_2^{n-1}, \tag{6.4}$$

$$H \cdot x_1^m x_2^n = (n-m) x_1^m x_2^n, \tag{6.5}$$

$$T \cdot x_1^m x_2^n = (n+m) x_1^m x_2^n. \tag{6.6}$$

We observe that $m+n$ is invariant under the action of $\widetilde{\mathcal{U}}$. For $\lambda \geq 0$, the submodule $L(\lambda)$ generated by x_2^λ is then the finite dimensional irreducible submodule of $\Gamma(\mathscr{O}_X)$ with infinitesimal character λ. Indeed, by applying F enough times to x_2^λ and multiplying by some constant, we can obtain any monomial $x_1^m x_2^n$ where $m+n = \lambda$, $m, n \in \mathbb{N}$. On the other hand, given any such monomial, applying E enough times and multiplying by some constant gives x_2^λ. We can see that T acts by λ, so $L(\lambda)$ is irreducible with infinitesimal character λ, and since $m, n \geq 0$, it is clear that its dimension is finite. This module can be described by the picture below, where the blue arrows denote the action of E, the green arrows denote the action of F and the red arrows denote the action of H. Accompanied with the arrows is the constant by which they multiply each element of the subspace they act on.



Example 6.2.4. Next we look at the case where $\mathfrak{g} = \mathfrak{sl}_3$. We found that the global sections of the structure sheaf of $\widetilde{X}$ are

$$\Gamma(\mathscr{O}_{\widetilde{X}}) = \frac{\mathbb{C}[x_1, x_2, x_3, c_1, c_2, c_3]}{(x_1c_1 + x_2c_2 + x_3c_3)},$$

and the H-invariant differential operators on $\mathscr{O}_{\widetilde{X}}$ are the E_{ij}, $i \neq j$, H_1, H_2 and T_1, T_2, as in Example 6.1.3. Hence, the elements of $\Gamma(\mathscr{O}_{\widetilde{X}})$ are linear combinations of monomials of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}c_1^{b_1}c_2^{b_2}c_3^{b_3}$, where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{N}$, modulo the ideal $(x_1c_1 + x_2c_2 + x_3c_3)$. Moreover, since T_1 acts by $x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$ and T_2 acts by $c_1\partial_{c_1} + c_2\partial_{c_2} + c_3\partial_{c_3}$, all the H-invariant differential operators acting on the above monomial preserve the values of both $a_1 + a_2 + a_3$ and $b_1 + b_2 + b_3$. Now choosing a weight $\lambda \in \mathfrak{h}^*$ amounts to choosing constants $\lambda(T_1)$ and $\lambda(T_2)$ so that the T_i act by $\lambda(T_i)$. Additionally, λ is dominant and integral (therefore corresponds to a finite dimensional irreducible $\mathfrak{sl}_3$-module) if and only if $\lambda(T_1), \lambda(T_2) \in \mathbb{N}$. Assuming $\lambda \in \mathfrak{h}^*$ is dominant and integral, the finite dimensional module of highest weight λ can be seen to be the one consisting of monomials $x_1^{a_1}x_2^{a_2}x_3^{a_3}c_1^{b_1}c_2^{b_2}c_3^{b_3}$, where $a_1+a_2+a_3 = \lambda(T_1)$ and $b_1 + b_2 + b_3 = \lambda(T_2)$. By considering the action of the E_{ij} with $i < j$, we can see that a vector of maximal weight is $x_3^{\lambda(T_1)}c_1^{\lambda(T_2)}$.

In the more general case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the notation becomes more cumbersome to deal with, but essentially the same ideas used in the $\mathfrak{sl}_3(\mathbb{C})$ example will show that the structure sheaf on $\widetilde{X}$ will give all the irreducible finite dimensional $\mathfrak{sl}_n(\mathbb{C})$-modules.

6.3 Verma modules and dual Verma modules

We can most easily obtain the picture by considering those modules of integral weight. In this case, we consider the structure sheaf on the open B-orbit of $\widetilde{X}$ and push it forward via either the shriek or the star push-forward functor and this will give the Verma modules and the dual Verma modules of integral weight. More precisely, let U be the open B-orbit of $\widetilde{X}$ inside $\widetilde{X}$. Since $\widetilde{X}$ is quasi-affine, so too is U, and so we may utilize Proposition 6.1.1 when talking of quasi-coherent sheaves on U. Naturally, we have the open embedding $j : U \hookrightarrow \widetilde{X}$. The structure sheaf $\mathscr{O}_U$ is naturally a $\mathscr{D}_U$-module, and we may take the D-module pushforwards $j_*\mathscr{O}_U$ and $j_!\mathscr{O}_U$ to obtain our $\mathscr{D}_{\widetilde{X}}$-modules.

Theorem 6.3.1. *The Verma (respectively dual Verma) modules of integral weight exactly correspond to the H-monodromic global sections of the $\mathscr{D}_{\widetilde{X}}$-module $j_!\mathscr{O}_{\widetilde{X}}$ (respectively $j_*\mathscr{O}_{\widetilde{X}}$) on which $\mathfrak{h}^*$ acts by an integral character.*

Of particular note in this theorem is the duality between the !-pushforward and the $*$-pushforward which is exhibited when taking global sections. This duality is discussed in [BB93], in particular in section 3.6.

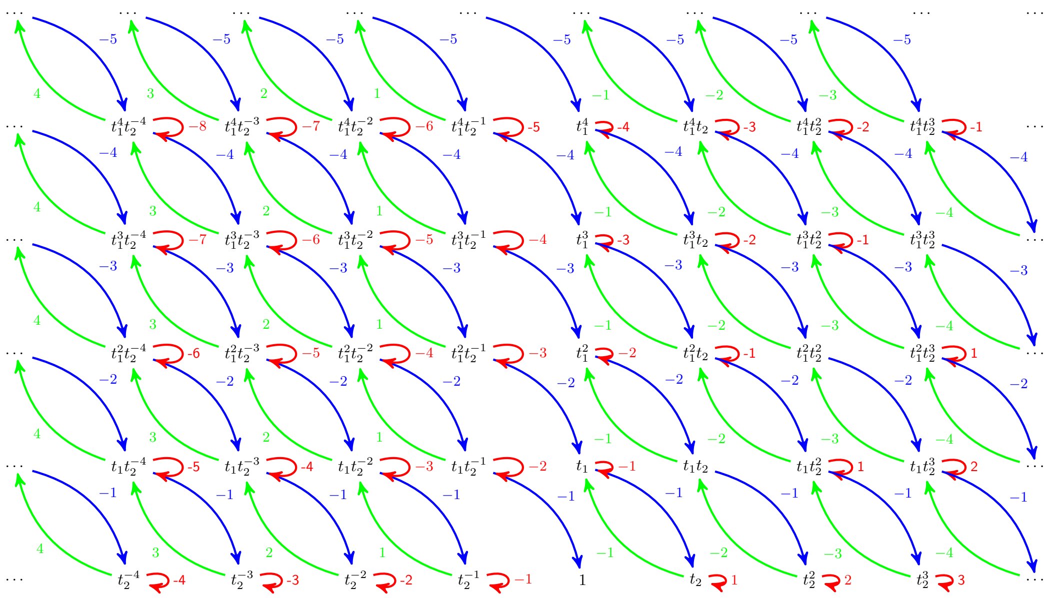
Example 6.3.2 ((Dual) Verma modules of integral weight for $\mathfrak{sl}_2(\mathbb{C})$)**.** If $\mathfrak{g} = \mathfrak{sl}_2$, then the B-orbits of $\widetilde{X} = \mathbb{C}^2 \setminus \{0\}$ are the sets

$$\ell = \{(x_1, 0) \,|\, x_1 \in \mathbb{C}^*\}$$

and

$$U = \{(x_1, x_2), \,|\, x_1 \in \mathbb{C},\ x_2 \in \mathbb{C}^*\}.$$

The structure sheaf on U has global sections $\Gamma(\mathscr{O}_U) = \mathbb{C}[x_1, x_2, x_2^{-1}]$, and the sheaf of differential operators has global sections $\Gamma(\mathscr{D}_U) = \mathbb{C}[x_1, x_2, x_2^{-1}, \partial_1, \partial_2]$, with usual commutativity relations. We wish to compute the D-module pushforwards j_* and $j_!$ of $\mathscr{O}_U$ along $j : U \hookrightarrow \widetilde{X}$. Since j is an open embedding, $j_*\mathscr{O}_U$ is easily seen to just be the restriction of the D-module structure of $\mathscr{O}_U$ to $\mathscr{D}_{\widetilde{X}}$. The $\widetilde{\mathcal{U}}$-submodule of $\Gamma(j_*\mathscr{O}_U)$ on which T acts by the integer λ consists of linear combinations of monomials of the form $x_1^m x_2^n$, where $m+n = \lambda$, $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. The $\widetilde{\mathcal{U}}$-module structure is given by the equations (6.3)-(6.6), and it is straightforward to see that this is the dual Verma module $M(\lambda)^*$. The below diagram shows the $\mathcal{U}(\mathfrak{g})$-structure of $j_*\mathscr{O}_U$, and we can see quite clearly how the dual Verma modules are obtained. Again, the blue arrows denote the E-action, the green arrows denote the F-action, and the red arrows denote the H-action.



The explicit computation of the shriek pushforward is a little more involved; recall that $j_! = \mathbb{D}_{\widetilde{X}} j_* \mathbb{D}_U$, where $\mathbb{D}$ denotes the duality functor. Since we are computing the pushforward of the structure sheaf, $\mathbb{D}_U$ just sends the left $\mathscr{D}_U$-module $\mathscr{O}_U$ to $\mathscr{O}_U$ as a right $\mathscr{D}_U$-module, and as above, j_* simply restricts the module structure to $\mathscr{D}_X$. We just need to find what $\mathbb{D}_{\widetilde{X}} = \mathcal{E}xt^2(-, \mathscr{D}_{\widetilde{X}})$ does when operating on the right $\mathscr{D}_{\widetilde{X}}$-module $\mathscr{O}_U$ We may take the free (hence projective) resolution of $j_*\mathbb{D}_U\mathscr{O}_U \simeq \mathscr{D}_U/\langle \partial_1, \partial_2\rangle \mathscr{D}_U$

$$0 \leftarrow j_*\mathbb{D}_U\mathscr{O}_U \xleftarrow{\varepsilon} \mathscr{D}_{\widetilde{X}} \xleftarrow{d_0} \mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}} \xleftarrow{d_1} \mathscr{D}_x \xleftarrow{d_2} 0,$$

where the maps are defined by

$$\varepsilon : 1 \mapsto x_2^{-1},$$

$$d_0 : (\theta_1, \theta_2) \mapsto \partial_1\theta_1 - x_2\partial_2\theta_2,$$

and

$$d_1 : 1 \mapsto (x_2\partial_2, \partial_1).$$

We can see that the image of d_0 is the kernel of ε, the image of d_1 is the kernel of d_0, d_1 is injective, and ε is surjective, so indeed this is a (projective) resolution. Applying the $\mathcal{H}om(-, \mathscr{D}_{\widetilde{X}})$ functor to the complex

$$0 \leftarrow \mathscr{D}_{\widetilde{X}} \xleftarrow{d_0} \mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}} \xleftarrow{d_1} \mathscr{D}_{\widetilde{X}} \xleftarrow{d_2} 0$$

gives a complex

$$0 \to \mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_0^*} \mathcal{H}om(\mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_1^*} \mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_2^*} 0,$$

where d_i^* sends a morphism f to a morphism $f \circ d_i$, for $i = 0, 1, 2$. Either by noting that $\mathscr{O}_U$ is holonomic or by explicit computation, it can be seen that the only nonzero cohomology from this complex is $\ker d_2^* / \operatorname{im} d_1^*$. Since $\mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \simeq \mathscr{D}_{\widetilde{X}}$ via the morphism $\mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \ni f \mapsto f(1)$, it is clear that $\ker d_2^* = \mathscr{D}_{\widetilde{X}}$, and so it suffices to compute $\operatorname{im} d_1^*$. Suppose $f \in \mathcal{H}om(\mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}})$, and let $\theta_1 = f(1,0)$ and $\theta_2 = f(0,1)$. Then $d_1^* f(1) = \theta_1 x_2\partial_2 + \theta_2\partial_1$. It follows that the image of d_1^* is the left ideal generated by ∂_1 and $x_2\partial_2$. Hence, we have

$$j_!\mathscr{O}_U = \mathcal{E}xt^2(j_*\mathbb{D}_U, \mathscr{D}_{\widetilde{X}}) = \frac{\mathscr{D}_{\widetilde{X}}}{\mathscr{D}_{\widetilde{X}}\langle \partial_1, x_2\partial_2\rangle}.$$

Global sections of this module consist of linear combinations of monomials of the form $x_1^m x_2^n$, and $x_1^m\partial_2^n$ where $m, n \geq 0$. The action of $\widetilde{\mathcal{U}}$ on $x_1^m x_2^n \in \Gamma(j_!\mathscr{O}_U)$, where $n \geq 1$ is given by equations (6.3)-(6.6), and the action of $\widetilde{\mathcal{U}}$ on $x_1^m\partial_2^n \in \Gamma(j_!\mathscr{O}_U)$ where $n \geq 0$ is given by

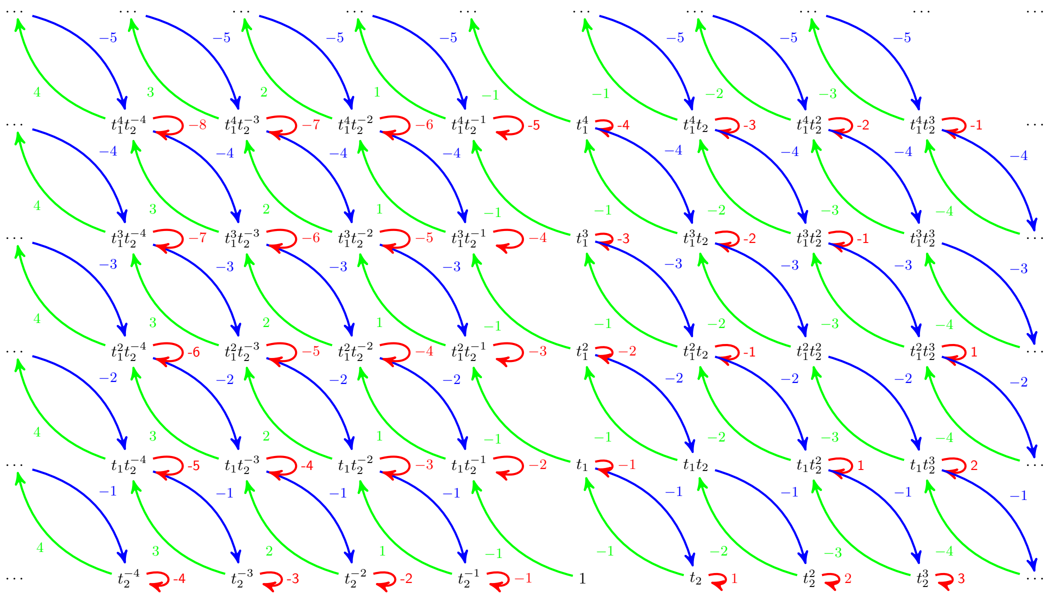
$$E \cdot x_1^m\partial_2^n = m(n-1)x_1^{m-1}\partial_2^{n-1}, \tag{6.7}$$

$$F \cdot x_1^m \partial_2^n = -x_1^{m+1} \partial_2^{n+1}, \tag{6.8}$$

$$H \cdot x_1^m \partial_2^n = -(m+n) x_1^m \partial_2^n, \tag{6.9}$$

$$T \cdot x_1^m \partial_2^n = (m-n) x_1^m \partial_2^n. \tag{6.10}$$

Now let $\lambda \in \mathbb{Z}$. We can see that for $\lambda \geq 0$, the submodule generated by x_2^λ has elements being linear combinations of $x_1^m x_2^n$ where $m+n=\lambda$, and $x_1^m \partial_2^l$ where $m-l=\lambda$, $m,n,l \in \mathbb{N}$. Also, for $\lambda < 0$, the submodule generated by $\partial_2^{-\lambda}$ has elements being linear combinations of $x_1^m \partial_2^n$, where $m-n=\lambda$. In either case, we denote this submodule by $M(\lambda)$, and see that T acts by λ on $M(\lambda)$, and in fact, $M(\lambda)$ is precisely the Verma module of highest weight λ. We can describe these Verma modules, again, using a picture in the same format as with the dual Verma modules.



Before proceeding with the next example, we should remark that if $\lambda \in \mathfrak{h}^*$ is not dominant and integral, then the Verma module $M(\lambda)$ is irreducible and coincides with the dual Verma module $M(\lambda)^*$. This statement is actually equivalent to Theorem 6.2.2, since Verma modules have a unique maximal submodule and unique irreducible quotient, and the finite dimensional irreducible modules are precisely obtained as quotients of Verma modules.

Example 6.3.3 ((Dual) Verma modules of non-integral weight for $\mathfrak{sl}_2(\mathbb{C})$)**.** We saw in the previous example that all the Verma (resp. dual Verma) modules with integral weight λ are obtained from the shriek (resp. star) pushforward of the structure sheaf on $U \subset \widetilde{X}$, the open B-orbit. By considering D-modules which are isomorphic to $\mathscr{O}_U$ as $\mathscr{O}_U$-modules, but with differing $\mathscr{D}_U$-module structure, we shall obtain the Verma and dual Verma modules of arbitrary highest weight.

Fix $\mu \in \mathbb{C} \setminus \mathbb{Z} \subset \mathbb{C} = \mathfrak{h}^*$, and let $\mathscr{O}_U(\mu)$ denote the the $\mathscr{D}_U$ module which is generated by the symbol x_2^μ, and is isomorphic to $\mathscr{O}_U$ as an $\mathscr{O}_U$-module. In other words, the global sections of $\mathscr{O}_U(\mu)$ are linear combinations of monomials of the form $x_1^m x_2^{n+\mu}$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and the $\mathscr{D}_U$-module structure is the one you'd expect, i.e.

$$\partial_1 \cdot x_1^m x_2^{n+\mu} = m x_1^{m-1} x_2^{n+\mu},$$

and

$$\partial_2 \cdot x_1^m x_2^{n+\mu} = (n+\mu) x_1^m x_2^{n-1+\mu}.$$

In the same fashion as the previous example, we can take the star pushforward $j_* \mathscr{O}_U(\mu)$, which is just the restriction of the $\mathscr{D}_U$-module structure to $\mathscr{D}_X$. In this case, the $\widetilde{\mathcal{U}}$-module structure on the global sections $\Gamma(\mathscr{O}_U(\mu))$ is given by

$$E \cdot x_1^m x_2^n x_2^\mu = -m x_1^{m-1} x_2^{n+1} x_2^\mu,$$
$$F \cdot x_1^m x_2^n x_2^\mu = -(n+\mu) x_1^{m+1} x_2^{n-1} x_2^\mu,$$
$$H \cdot x_1^m x_2^n x_2^\mu = (n-m+\mu) x_1^m x_2^n x_2^\mu,$$

$$T \cdot x_1^m x_2^n x_2^\mu = (m+n+\mu) x_1^m x_2^n x_2^\mu.$$

Now as usual, we choose a constant λ, where $\lambda = l + \mu$, $l \in \mathbb{Z}$, and consider the submodule of $j_*\mathscr{O}_U(\mu)$ consisting of all the elements on which T acts by λ. Then this submodule consists of all the linear combinations of monomials of the form $x_1^m x_2^n x_2^\mu$, where $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and $m+n+\mu = \lambda$. The above equations also show that this submodule is $M(\lambda)$, the Verma (hence dual Verma) module of highest weight λ.

It is curious to note that if we had allowed μ in the previous example to be an integer, then we would not get all the Verma modules of integral weight from the star pushforward, and we needed to take the shriek pushforward, but we saw that if μ is not an integer, then the star pushforward gives us everything. This leads us to wonder what we get if we take the shriek pushforward of $\mathscr{O}_U(\mu)$ when $\mu \notin \mathbb{Z}$. In this case we may regard $j_*\mathbb{D}_U\mathscr{O}_U(\mu)$ as the right $\mathscr{D}_{\widetilde{X}}$-module

$$j_*\mathbb{D}_U\mathscr{O}_U(\mu) = \frac{\mathscr{D}_U}{\langle \partial_1, \partial_2 - \mu x_2^{-1}\rangle \mathscr{D}_U}.$$

We again have a free resolution of $\mathscr{D}_X$-modules

$$0 \leftarrow j_*\mathbb{D}_U\mathscr{O}_U \xleftarrow{\varepsilon} \mathscr{D}_{\widetilde{X}} \xleftarrow{d_0} \mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}} \xleftarrow{d_1} \mathscr{D}_x \xleftarrow{d_2} 0,$$

however here the maps are given by

$$\varepsilon : 1 \mapsto 1,$$
$$d_0 : (\theta_1, \theta_2) \mapsto \partial_1\theta_1 - (\partial_2 x_2 - \mu)\theta_2,$$
$$d_1 : 1 \mapsto (\partial_2 x_2 - \mu, \partial_1),$$

so in the complex

$$0 \to \mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_0^*} \mathcal{H}om(\mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_1^*} \mathcal{H}om(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \xrightarrow{d_2^*} 0,$$

the image of d_1^* is $\mathscr{D}_{\widetilde{X}}\langle \partial_1, \partial_2 x_2 - \mu\rangle$. Hence,

$$j_!\mathscr{O}_U = \frac{\mathscr{D}_{\widetilde{X}}}{\mathscr{D}_{\widetilde{X}}\langle \partial_1, \partial_2 x_2 - \mu\rangle},$$

but this is just isomorphic to the $\mathscr{D}_{\widetilde{X}}$-module $\mathscr{O}_U(\lambda)$ via the isomorphism $1 \mapsto x_2^\mu$.

Example 6.3.4 ((Dual) Verma modules of integral weight for $\mathfrak{sl}_3(\mathbb{C})$)**.** Here we want to take the open B-orbit U on $\widetilde{X} = G/N$, where $G = \mathrm{SL}_3(\mathbb{C})$. To find what this is explicitly, we consider the product

$$\begin{pmatrix} \beta_1 & \alpha_1 & \alpha_2 \\ 0 & \beta_2 & \alpha_3 \\ 0 & 0 & \beta_3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} \beta_1 x_{11} + \alpha_1 x_{21} + \alpha_2 x_{31} & \dots & \dots \\ \beta_2 x_{21} + \alpha_3 x_{31} & \beta_2 x_{22} + \alpha_3 x_{22} & \dots \\ \beta_3 x_{31} & \beta_3 x_{32} & \dots \end{pmatrix},$$

where the $\alpha_i \in \mathbb{C}$ and the $\beta_i \in \mathbb{C}^*$ with $\beta_1\beta_2\beta_3 = 1$. The point is that as long as x_{31} and $c_1 = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}$ are nonzero, then any matrix of the form

$$\begin{pmatrix} \beta_1 & \alpha_1 & \alpha_2 \\ 0 & \beta_2 & \alpha_3 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

will map x_3 and c_1 to nonzero elements, and moreover given any two tuples $(x_1, x_2, x_3, c_1, c_2, c_3), (x_1', x_2', x_3', c_1', c_2', c_3') \in \widetilde{X}$, with $x_1, x_2, c_2, c_3, x_1', x_2', c_2', c_3' \in \mathbb{C}$ and $x_3, c_1, x_3', c_1' \in \mathbb{C}^*$, there exists a matrix of the above form which sends $(x_1, x_2, x_3, c_1, c_2, c_3)$ to $(x_1', x_2', x_3', c_1', c_2', c_3')$. This shows that the open B-orbit of $\widetilde{X}$ is

$$U = \{(x_1, x_2, x_3, c_1, c_2, c_3) \in \widetilde{X} \mid x_3, c_1 \neq 0\}.$$

The structure sheaf of U can be seen to then be

$$\mathscr{O}_U = \frac{\mathbb{C}[x_1, x_2, x_3, x_3^{-1}, c_1, c_1^{-1}, c_2, c_3]}{(x_1c_1 + x_2c_2 + x_3c_3)} \simeq \mathbb{C}[x_1, x_2, x_3, x_3^{-1}, c_1, c_1^{-1}, c_2].$$

Since $j : U \hookrightarrow \widetilde{X}$ is an open embedding, the star pushforward is the restriction of the module structure to $\mathscr{D}_{\widetilde{X}}$. The description of the differential operators on $\widetilde{X}$ corresponding to the generators of $\widetilde{\mathcal{U}}$ in Example 6.1.3 shows that a weight $\lambda \in \mathfrak{h}^*$ is described by a pair $(\lambda(T_1), \lambda(T_2))$ of complex numbers, and that the actions of T_1 and T_2 on a monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3}c_1^{b_1}c_2^{b_2}c_3^{b_3}$ are multiplication by $(a_1 + a_2 + a_3)$ and by $(b_1 + b_2 + b_3)$ respectively. One can check by applying the appropriate combination of E_{ij}'s that the H-monodromic submodule of $j_*\mathscr{O}_U$ where T_1 and T_2 act by $\lambda(T_1)$ and $\lambda(T_2)$ respectively is isomorphic to the dual Verma module $M(\lambda)^*$. This pretty much boils down to noting that we saw that the module generated by $x_3^{\lambda(T_1)}c_1^{\lambda(T)2)}$ is finite dimensional, and the quotient of the submodule generated by $x_1^{\lambda(T_1)+1}x_3^{-1}c_1^{-1}c_3^{\lambda(T_2)+1}$ by this finite dimensional submodule is isomorphic to an irreducible Verma module. To explicitly compute the !-pushforward in the $\mathfrak{sl}_3$ case seems like a more challenging task, especially since the global sections of the differential operators on $\widetilde{X}$ are not as nice as in the $\mathfrak{sl}_2$ example. Instead, we appeal to the intuitive sense that after taking H-monodromic global sections of the !-pushforward, they should be dual in the category $\mathcal{O}$ to the H-monodromic global sections of the $*$-pushforward, and this appears sensible enough when noting that $j_! = \mathbb{D}j_*\mathbb{D}$ and that the global sections functor on the flag variety is exact.

The previous example lays out the general idea that can be applied in the case of $\mathfrak{sl}_n(\mathbb{C})$. We have seen how we can describe the structure of the base affine space in this case, and the open B-orbit U will be the one consisting of cosets of $n \times n$ matrices whose representatives' $r \times r$ minors made up of the leftmost r columns and bottom-most r rows are all nonzero. The structure sheaf of U is then straightforward to compute, and the $*$-pushforward to $\widetilde{X}$ is just the restriction of the D-module structure to $\mathscr{D}_{\widetilde{X}}$, and just considering how the E_{ij} will act on monomials in $\mathscr{O}_U$ should reveal that the $*$-pushforward gives the dual Verma modules of integral weight, and then the !-pushforward will give the Verma modules of integral weight.

6.4 Principal series representations

In the previous section, we pushed forward D-modules on the open B-orbit of G/N. Some of the motivation for considering B-orbits arises from the particularly nice decomposition of G in general, called the Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} NwB,$$

which gives rise to a decomposition of the flag variety

$$G/B = \bigsqcup_{w \in W} NwB/B.$$

The B-orbits on G/N can then be viewed as the inverse image of N-orbits on the flag variety G/B. The purpose of this view is motivated by the idea of certain nice orbits on the flag variety called admissible orbits [BB93] (section 3.4). We shall not discuss admissible orbits in any detail, as they require some rather sophisticated machinery, however it is worthwhile to mention that if $K \subset G$ is a subgroup, and Q is a K-orbit, then if Q is an admissible orbit, the embeddings $Q \hookrightarrow X$ and $\pi^{-1}(Q) \hookrightarrow \widetilde{X}$ are affine, and we say that the K-action on $(X, \widetilde{X})$ is admissible if there are finitely many K-orbits on X and they are all admissible.

The action of the maximal unipotent subgroup $N \subset G$ is an example of an admissible action. Another example of an admissible action is the action of a maximal compact subgroup $K \subset G$. We look at this situation in the case that $G = \mathrm{SL}_2(\mathbb{C})$.

In this case, any maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ is conjugate to the subgroup K of matrices of the form

$$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1}, \end{pmatrix}$$

where $\beta \in \mathbb{C}^*$. It can be readily seen that the K-orbits of $G/B = \mathbb{P}^1$ are $\{0\}$, $\{\infty\}$, and $V = \mathbb{P}^1 \setminus \{0, \infty\}$. Taking the inverse image of these orbits along $\pi : \widetilde{X} \to X$ we get

$$\pi^{-1}(0) = \{(x_1, 0) \,|\, x_1 \in \mathbb{C}^*\},$$

$$\pi^{-1}(\infty) = \{(0, x_2) \,|\, x_2 \in \mathbb{C}^*\},$$

and

$$\pi^{-1}(V) = \{(x_1, x_2) \,|\, x_1, x_2 \in \mathbb{C}^*\}.$$

The structure sheaf on $\pi^{-1}(V)$ is $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$, and since $j : \pi^{-1}(V) \hookrightarrow \widetilde{X}$ is an open embedding, the $*$-pushforward will be the restriction of the module structure to $\mathscr{D}_{\widetilde{X}}$. The equations (6.3) - (6.6) give the $\widetilde{\mathcal{U}}$-action on an arbitrary monomial $x_1^m x_2^n$, where here $m, n \in \mathbb{Z}$. As in the previous section, we consider the submodule consisting of linear combinations of monomials of the form $x_1^m x_2^n$ where $m + n = \lambda$, and this module can be seen to be the module associated to the principal series representation of $\mathrm{SL}_2(\mathbb{R})$ corresponding to the parameter λ.

We can also take the !-pushforward of $\mathscr{O}_{\pi^{-1}(V)}$ along the embedding j. We have the free (therefore projective) resolution

$$0 \leftarrow j_* \mathbb{D}_{\pi^{-1}(V)} \mathscr{O}_{\pi^{-1}(V)} \xleftarrow{\varepsilon} \mathscr{D}_{\widetilde{X}} \xleftarrow{d_0} \mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}} \xleftarrow{d_1} \mathscr{D}_{\widetilde{X}} \xleftarrow{d_2} 0,$$

where

$$\varepsilon : 1 \mapsto x_1^{-1} x_2^{-1},$$

$$d_0 : (\theta_1, \theta_2) \mapsto x_1 \partial_1 \theta_1 - x_2 \partial_2 \theta_2,$$

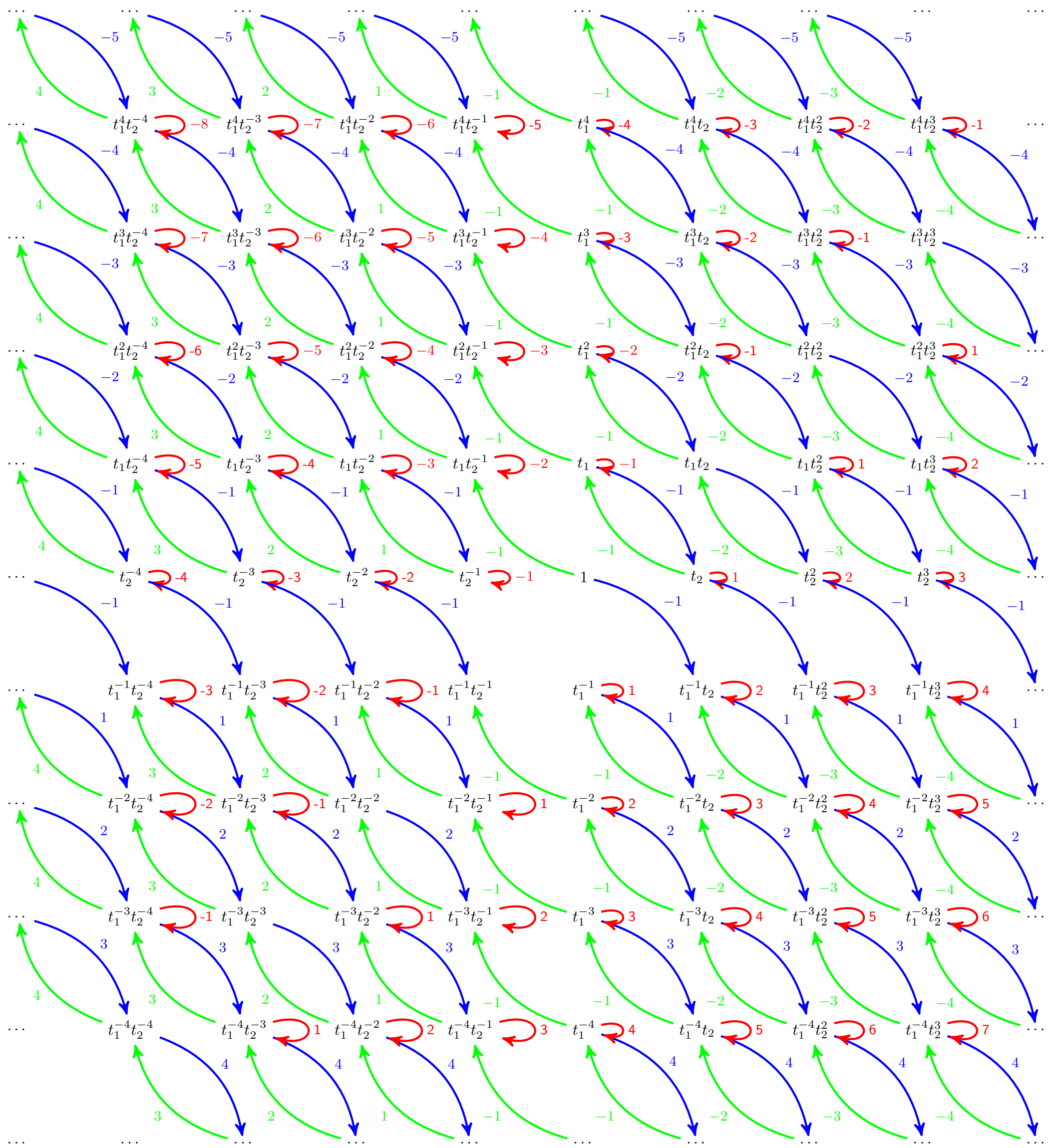
and

$$d_1 : 1 \mapsto (x_2 \partial_2, x_1 \partial_1).$$

Hence, the image of $d_1^* : \mathrm{Hom}(\mathscr{D}_{\widetilde{X}} \oplus \mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \to \mathrm{Hom}(\mathscr{D}_{\widetilde{X}}, \mathscr{D}_{\widetilde{X}}) \simeq \mathscr{D}_{\widetilde{X}}$ is the left ideal $\mathscr{D}_{\widetilde{X}} \langle x_1 \partial_1, x_2, \partial_2 \rangle$, and so

$$j_! \mathscr{O}_{\pi^{-1}(V)} = \frac{\mathscr{D}_{\widetilde{X}}}{\mathscr{D}_{\widetilde{X}} \langle x_1 \partial_1, x_2 \partial_2 \rangle}.$$

This module actually turns out to be isomorphic to $j_* \mathscr{O}_{\pi^{-1}(V)}$. The picture below, in a similar nature to the previous examples, can be seen to describe the $\mathcal{U}(\mathfrak{g})$-structure of this module.


$t_1^4t_2^{-4}$ -8
$t_1^4t_2^{-3}$ -7
$t_1^4t_2^{-2}$ -6
$t_1^4t_2^{-1}$ -5
t_1^4 -4
$t_1^4t_2$ -3
$t_1^4t_2^2$ -2
$t_1^4t_2^3$ -1
$t_1^3t_2^{-4}$ -7
$t_1^3t_2^{-3}$ -6
$t_1^3t_2^{-2}$ -5
$t_1^3t_2^{-1}$ -4
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$t_1^2t_2^{-4}$ -6
$t_1^2t_2^{-3}$ -5
$t_1^2t_2^{-2}$ -4
$t_1^2t_2^{-1}$ -3
t_1^2 -2
$t_1^2t_2$ -1
$t_1^2t_2^2$
$t_1^2t_2^3$ 1
$t_1t_2^{-4}$ -5
$t_1t_2^{-3}$ -4
$t_1t_2^{-2}$ -3
$t_1t_2^{-1}$ -2
t_1 -1
t_1t_2
$t_1t_2^2$ 1
$t_1t_2^3$ 2
t_2^{-4} -4
t_2^{-3} -3
t_2^{-2} -2
t_2^{-1} -1
1
t_2 1
t_2^2 2
t_2^3 3
$t_1^{-1}t_2^{-4}$ -3
$t_1^{-1}t_2^{-3}$ -2
$t_1^{-1}t_2^{-2}$ -1
$t_1^{-1}t_2^{-1}$
t_1^{-1} 1
$t_1^{-1}t_2$ 2
$t_1^{-1}t_2^2$ 3
$t_1^{-1}t_2^3$ 4
$t_1^{-2}t_2^{-4}$ -2
$t_1^{-2}t_2^{-3}$ -1
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t_1^{-2} 2
$t_1^{-2}t_2$ 3
$t_1^{-2}t_2^2$ 4
$t_1^{-2}t_2^3$ 5
$t_1^{-3}t_2^{-4}$ -1
$t_1^{-3}t_2^{-3}$
$t_1^{-3}t_2^{-2}$ 1
$t_1^{-3}t_2^{-1}$ 2
t_1^{-3} 3
$t_1^{-3}t_2$ 4
$t_1^{-3}t_2^2$ 5
$t_1^{-3}t_2^3$ 6
$t_1^{-4}t_2^{-4}$
$t_1^{-4}t_2^{-3}$ 1
$t_1^{-4}t_2^{-2}$ 2
$t_1^{-4}t_2^{-1}$ 3
t_1^{-4} 4
$t_1^{-4}t_2$ 5
$t_1^{-4}t_2^2$ 6
$t_1^{-4}t_2^3$ 7

Chapter 7

Jantzen filtrations

7.1 The Jantzen filtration on Verma modules

A very nice (and more general) account of Jantzen filtrations is given in [IK11], which is the reference we follow for the most part. The goal here is to define a filtration of a Verma module satisfying certain nice properties, which finds use in describing the multiplicities of irreducibles occurring in a given Verma module. It is not much more difficult to define, more generally, a filtration for any k-vector space which has these nice properties when applied to Verma modules, and so we shall do so. To do this, we need to work in a deformed setting (which we shall make more precise soon enough), however for the moment, we shall mention some facts in the undeformed setting. Let $v_\lambda \in M(\lambda)$ be a maximal vector, and $v_\lambda^* \in M(\lambda)^\vee$ be the element which maps v_λ to 1 and maps any vector not in the same weight space as v_λ to 0. Recall τ is the anti-involution fixing $\mathfrak{h}$ and sending $x_\alpha \in \mathfrak{g}_\alpha$ to $y_\alpha \in \mathfrak{g}_{-\alpha}$ for each $\alpha \in \Delta$. We can check that v_λ^* is a maximal vector in $M(\lambda)^\vee$ as for any $v \in M(\lambda)$, and any $x \in \mathfrak{n}$,

$$(x \cdot v_\lambda^*)(v) = v_\lambda^*(\tau(x) \cdot v) = 0,$$

since $\tau(x) \in \mathfrak{n}^-$, and so $\tau(x) \cdot v$ cannot be in the same weight space as the maximal vector v_λ. There is a canonical homomorphism $\varphi : M(\lambda) \to M(\lambda)^\vee$ defined by sending the generator

$$v_\lambda \mapsto v_\lambda^*.$$

The image of the map φ is the unique simple submodule $L(\lambda)$ of $M(\lambda)^\vee$. Indeed, let π be the projection defined by the composition

$$\pi : \mathcal{U}(\mathfrak{g}) \to M(\lambda) \to N(\lambda);$$

we shall show that $\pi^{-1}(N(\lambda))v_\lambda^* = 0$ and thence the claim will follow. For any $\alpha \in \Delta$ where $(x_\alpha, y_\alpha, h_\alpha)$ is the corresponding $\mathfrak{sl}_2$-triple, we first have

$$(x_\alpha \cdot v_\lambda^*)(v) = v_\lambda^*(y_\alpha \cdot v) = 0,$$

since for any $v \in M(\lambda)$, the weight of $y_\alpha \cdot v$ is lower than the weight of v. We also have the equality

$$(y_\alpha \cdot v_\lambda^*)(v) = v_\lambda^*(x_\alpha \cdot v),$$

for any $v \in M(\lambda)$. Now if $v \in N(\lambda)$, the unique maximal submodule of $M(\lambda)$, then for any $x \in \mathcal{U}(\mathfrak{g})$, we have $v_\lambda^*(x \cdot v) = 0$, and so $(x \cdot v_\lambda^*)(v) = v_\lambda^*(\tau(x) \cdot v) = 0$. On the other hand, if $v \notin N(\lambda)$, we have $v + N(\lambda) \in M(\lambda)/N(\lambda) \simeq L(\lambda)$, and then there exists some $x \in \mathcal{U}(\mathfrak{g})$ for which $\tau(x) \cdot v = v_\lambda$, and so $(x \cdot v_\lambda^*)(v) = 1$. Suppose that this $x \in \pi^{-1}(N(\lambda))$. Then $v_\lambda = \tau(x) \cdot v$, and so $N(\lambda) \ni x \cdot v_\lambda = x \cdot \tau(x) \cdot v$, contradicting that $v \notin N(\lambda)$, since $x \cdot \tau(x) \cdot v$ lies in the same weight space as v. It follows that if $x \in \pi^{-1}(N(\lambda))$, then $x \cdot v_\lambda^* = 0$.

Definition 7.1.1. There is a symmetric bilinear form $\langle -, - \rangle$ on $M(\lambda)$, called the *Shapovalov form*, defined by

$$\langle v, w \rangle = \varphi(v)(w).$$

Bilinearity of this form is clear. Symmetry can be seen by writing $v = x \cdot v_\lambda$ and $w = x' \cdot v_\lambda$, for $x, x' \in \mathcal{U}(\mathfrak{g})$, and then

$$\begin{aligned} \varphi(v)(w) &= \varphi(x \cdot v_\lambda)(x' \cdot v_\lambda) \\ &= (x \cdot v_\lambda^*)(x' \cdot v_\lambda) \\ &= v_\lambda^*(\tau(x) \cdot x' \cdot v_\lambda) \\ &= v_\lambda^*(\tau(x') \cdot x \cdot v_\lambda) \\ &= \varphi(w)(v), \end{aligned}$$

where the second-last line follows from the fact that v_λ^* vanishes everywhere except for the 1-dimensional weight space of v_λ and τ fixes $\mathfrak{h}$. The Shapovalov form is also contravariant in the sense that for any $x \in \mathfrak{g}$, $\langle x \cdot v, w \rangle = \langle v, \tau(x) \cdot w \rangle$, which immediately follows from the definition of the $\mathfrak{g}$-module structure on $M(\lambda)^\vee$. Since the Verma module $M(\lambda)$ is a highest weight module generated by the element v_λ, any contravariant bilinear form on $M(\lambda)$ is uniquely determined by the value of $\langle v_\lambda, v_\lambda \rangle$. As such, the Shapovalov form is the unique contravariant bilinear form on $M(\lambda)$ such that $\langle v_\lambda, v_\lambda \rangle = 1$.

Now given a field k, the Jantzen filtration for a k-vector space can be defined as follows: choose a PID $\mathcal{R}$ and a prime element $t \in \mathcal{R}$ so that the quotient ring $\mathcal{R}/t\mathcal{R}$ coincides with the field k. We have the canonical map $\phi : \mathcal{R} \to k$ which allows us to view k as a right (or left) $\mathcal{R}$-module. Then we can turn any left $\mathcal{R}$-module $\widetilde{M}$, into a k-vector space via $\widetilde{M} \mapsto k \otimes_{\mathcal{R}} \widetilde{M}$. Additionally, if $f : \widetilde{M} \to \widetilde{N}$ is a morphism of $\mathcal{R}$-modules, then $\mathrm{id}_k \otimes f : k \otimes_{\mathcal{R}} \widetilde{M} \to k \otimes_{\mathcal{R}} \widetilde{N}$ defines a morphism of k-algebras. It is easy to see that this describes a functor F_ϕ, induced by ϕ, from the category of left $\mathcal{R}$-modules to the category of k-vector spaces.

Now let M be a k-vector space. We can lift M to a free $\mathcal{R}$-module $\widetilde{M}$ with a non-degenerate symmetric bilinear form $\langle -, - \rangle_{\widetilde{M}} \to \mathcal{R}$, so that $F_\phi \widetilde{M} = M$. One can check that for $v_1, v_2 \in M$, where $v_1 = F_\phi w_1$ and $v_2 = F_\phi w_2$ for $w_1, w_2 \in \widetilde{M}$, we can define a symmetric bilinear form on M by

$$\langle v_1, v_2 \rangle_M := \phi \langle w_1, w_2 \rangle_{\widetilde{M}}.$$

With this setup, we can construct the following filtration on the k-vector space M :

Definition 7.1.2. For $n \in \mathbb{N}$, put

$$\widetilde{M}(n) := \{ w \in \widetilde{M} \mid \langle w, w' \rangle \in t^n \mathcal{R}, \text{ for every } w' \in \widetilde{M} \}.$$

Let $\iota_n : \widetilde{M}(n) \to \widetilde{M}$ be the inclusion, so

$$F_\phi \iota_n = \mathrm{id}_k \otimes \iota_n : k \otimes_{\mathcal{R}} \widetilde{M}(n) \to k \otimes_{\mathcal{R}} \widetilde{M}.$$

Now let $M(n) := \operatorname{im} F_\phi \iota_n$. The *Jantzen filtration* of M is defined to be the decreasing filtration

$$M = M(0) \supset M(1) \supset M(2) \supset \dots$$

We now follow this definition to construct the Jantzen filtration on the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$. Here, we have $k = \mathbb{C}$, and we shall put $\mathcal{R} = \mathbb{C}[t]$, for some indeterminate t which will be the prime element seen in the definition of the Jantzen filtration, i.e. $\phi : \mathbb{C}[t] \to \mathbb{C}$ will be the quotient by t map. Before going further, it might be nice to note that we have a new Lie algebra $\mathfrak{g}_t := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$, with Borel subalgebra $\mathfrak{b}_t = \mathfrak{b} \otimes_{\mathbb{C}} \mathbb{C}[t]$ and Cartan subalgebra $\mathfrak{h}_t = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t]$.

Definition 7.1.3. A *deformed Verma module* (over the Lie algebra $\mathfrak{g}$) is the module

$$M(\lambda + \rho t) := \mathcal{U}(\mathfrak{g}_t) \otimes_{\mathcal{U}(\mathfrak{b}_t)} \mathbb{C}[t]_{\lambda + \rho t},$$

where ρ is the Weyl vector and $\mathbb{C}[t]_{\lambda+\rho t}$ is the $\mathfrak{b}_t$-module looking like $\mathbb{C}[t]$ as a $\mathbb{C}$-vector space and where the $\mathfrak{b}_t$-action factors through $\mathfrak{b}_t \to \mathfrak{h}_t \xrightarrow{\lambda + \rho t} \mathbb{C}[t]$.

We see then that the functor F_ϕ acting on the deformed Verma module will be that which sets $t = 0$. All that's left is to see that $M(\lambda + \rho t)$ has a non-degenerate symmetric bilinear form. In fact, Jantzen, in 1979 proved that $M(\lambda + \rho t)$ has a unique symmetric contravariant bilinear form (valued in $\mathbb{C}[t]$), and using this fact, along with the discussion above, allows us to construct a Jantzen filtration for any Verma module.

The fact that the contravariant bilinear form on $M(\lambda + \rho t)$ is unique, along with the fact that there is a canonical map from a Verma module to a dual Verma module suggests that if we can define a deformed version of a dual Verma module, then the canonical map from the deformed Verma module to the deformed dual Verma module will correspond to the desired contravariant form on $M(\lambda + \rho t)$. Indeed, the anti-involution τ on $\mathcal{U}(\mathfrak{g})$ can be naturally extended to an anti-involution of $\mathcal{U}(\mathfrak{g}_t)$, and so we can consider the $\mathcal{U}(\mathfrak{g}_t)$-module $M(\lambda + \rho t)^*$ under the action

$$(x(t) \cdot f(t))(v(t)) = f(t)(\tau(x(t)) \cdot v(t)),$$

where $x(t) \in \mathfrak{g}_t$, $v(t) \in M(\lambda + \rho t)$, $f(t) \in M(\lambda + \rho t)^*$, and where we write them all as functions of t to emphasize that they are all elements of deformed objects. We can decompose $M(\lambda + \rho t)$ into weight spaces

$$M(\lambda + \rho t) = \bigoplus_{\alpha(t) \in \mathfrak{h}_t^*} M(\lambda + \rho t)_{\alpha(t)},$$

and after checking that $(M(\lambda + \rho t)_{\alpha(t)})^* = (M(\lambda + \rho t)^*)_{\alpha(t)}$, we arrive at the following:

Definition 7.1.4. The deformed dual Verma module with maximal weight $\lambda + \rho t$ is the $\mathfrak{g}_t$-module

$$M(\lambda + \rho t)^\vee := \bigoplus_{\alpha(t) \in \mathfrak{h}_t} M(\lambda + \rho t)^*_{\alpha(t)}.$$

Now with this, the canonical morphism $M(\lambda + \rho t) \to M(\lambda + \rho t)^\vee$ will induce the desired $\mathbb{C}[t]$-valued symmetric contravariant bilinear form on $M(\lambda+\rho t)$ which is used to obtain the Jantzen filtration on $M(\lambda)$. Before proceeding to a theorem due to Jantzen which provides some nice properties of this filtration, we need the following definition.

Definition 7.1.5. Let M be a $\mathfrak{g}$-module in category $\mathcal{O}$, and Λ the additive group of integral weights of the root system associated to $\mathfrak{g}$. We make Λ into a multiplicative group by associating to every $\lambda \in \Lambda$ the symbol $e(\lambda)$ so that $e(\lambda)e(\mu) = e(\lambda + \mu)$, where μ is another element of Λ. The *formal character* of M is then the element in group ring $\mathbb{Z}\Lambda$

$$\operatorname{ch} M := \sum_{\lambda \in \Lambda} \dim M_\lambda e(\lambda),$$

where M_λ denotes the $\mathfrak{h}$-eigenspace associated to the weight λ.

We also recall that to each simple root $\alpha \in \Delta$, we have in the Weyl group the simple reflection associated to that root, which is denoted by s_α.

Theorem 7.1.6. *Let $\lambda \in \mathfrak{h}^*$ and $M(\lambda)$ denote the Verma module of highest weight λ and $N(\lambda) \subset M(\lambda)$ the unique maximal submodule. The Jantzen filtration*

$$M(\lambda) = M(\lambda)(0) \supset M(\lambda)(1) \supset M(\lambda)(2) \supset \dots$$

on $M(\lambda)$ satisfies the following:

a) $M(\lambda)(i) = 0$ for large enough i;

b) $M(\lambda)(1) = N(\lambda)$;

c) Every nonzero quotient $M(\lambda)(i)/M(\lambda)(i+1)$ has a unique non-degenerate contravariant form; and

d) The Jantzen character sum formula

$$\sum_{i>0} \operatorname{ch} M(\lambda)(i) = \sum_{\substack{\alpha>0, \\ s_\alpha \cdot \lambda < \lambda}} \operatorname{ch} M(s_\alpha \cdot \lambda)$$

is satisfied.

Remark 7.1.7. *In the process of obtaining the Jantzen filtration of a Verma module, at some point we set t to 0. This suggests that we don't actually need to adjoin $\mathbb{C}[t]$ to the gain the deformity, rather we could just adjoin $\mathbb{C}[t]/t^n$ for sufficiently large n, and we shall call such modules* finitely deformed (dual) Verma modules. *Indeed, this is the setting in which we will find ourselves when dealing in the geometric Jantzen filtration.*

Remark 7.1.8. *This section has mostly been about the Jantzen filtration on Verma modules, though we did give a general definition for the Jantzen filtration on any k-vector space. It follows that there is a similarly defined Jantzen filtration on dual Verma modules, which we shall not explicitly write about* *here, but we will see it arise naturally in the geometric context, and it will look pretty much like what one would expect when comparing it to the Jantzen filtration of a Verma module.*

7.2 The monodromy filtration of an arbitrary object in an abelian category

Before going to the geometric construction of the Jantzen filtration, we shall need some preliminaries, and these will hopefully elude the reader to a deeper and possibly more general idea of the Jantzen filtration. The main proposition we need is due to Deligne.

Proposition 7.2.1. *Let s be a nilpotent endomorphism of an object V in an abelian category $\mathcal{A}$. Then there exists a unique finite filtration $\mu_\bullet$ on V such that $s\mu_i \subset \mu_{i-2}$ and for $k \in \mathbb{N}$, s^k induces an isomorphism $\mathrm{gr}^\mu_k \simeq \mathrm{gr}^\mu_{-k}$, where $\mathrm{gr}^\mu_i = \mu_i/\mu_{i-1}$.*

Existence is proved in [Del80] using induction, and uniqueness can be proved in a similar manner. In fact, by just following the steps of this proof, we can get an explicit description of the monodromy filtration in terms of the image and kernel of powers of the nilpotent endomorphism s. Let

$$\mathfrak{F}_p V = \begin{cases} \ker s^{k+1} & \text{if} \quad p \geq 0 \\ 0 \quad \text{if} \quad p < 0 \end{cases}$$

be the kernel filtration of V, and

$$\mathfrak{G}_q V = \begin{cases} \operatorname{im} s^{k} & \text{if} \quad q > 0 \\ V \quad \text{if} \quad q \leq 0 \end{cases}$$

be the image filtration of V. The monodromy filtration $\mu_\bullet$ of V is then given by

$$\mu_r = \sum_{p-q=r} \mathfrak{F}_p V \cap \mathfrak{G}_q V,$$

for each $r \in \mathbb{Z}$. There are two related filtrations $J_{!\bullet}$ and $J_{*\bullet}$ introduced in [BB93], which carry the name of Jantzen filtrations. These are defined as

$$J_{!i} = \ker s \cap \mathfrak{G}_{-i}$$

and

$$J_{*i} = \frac{\mathfrak{F}_i + \operatorname{im} s}{\operatorname{im} s},$$

and it can be seen that $J_{!\bullet}$ is the restriction of $\mu_\bullet$ to $\ker s$ and $J_{*\bullet}$ is the restriction of $\mu_\bullet$ to $\operatorname{coker} s$.

7.3 The monodromy and Jantzen filtrations in the geometric setting

We wish to view the filtrations defined in the previous section in a geometric setting - in particular to D-modules on the base affine space. We follow closely [BB93], and for more details one should refer to the construction of nearby cycles as in e.g. [Bei87].

In general, given a smooth variety Y and a regular function $f : Y \to \mathbb{A}^1$, we consider the open embedding $j : U := f^{-1}(\mathbb{A}^1 \setminus \{0\}) \hookrightarrow Y$. Given $n \in \mathbb{N}$, let $I^{(n)}$ be the $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}}$-module which is isomorphic to $\mathscr{O}_{\mathbb{A}^1\setminus\{0\}} \otimes \mathbb{C}[s]/s^n$ as an $\mathscr{O}_{\mathbb{A}^1\setminus\{0\}}$-module, but is generated by the symbol x^s, which gives it the $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}}$-module structure defined by $x\partial x^s = sx^s$. As well as being a $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}}$-module, $I^{(n)}$ also has the obvious structure of a $\mathbb{C}[s]/s^n$-module, and hence we may view it as a $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}} \otimes \mathbb{C}[s]/s^n$-module. It can be readily seen that $I^{(n)}/s^n = I^{(n-1)}$, for $n \geq 1$. We can pull back this $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}}$-module $I^{(n)}$ along f (restricted to U) to obtain a $\mathscr{D}_U$-module $f^*I^{(n)}$.

Now given any $\mathscr{D}_U$-module M_U, we can consider a deformed version which is $f^sM_U^{(n)} := f^*I^{(n)} \otimes_{\mathscr{O}_U} M_U$, which has the structure of a $\mathscr{D}_U \otimes \mathbb{C}[s]/s^n$-module. We note that $f^sM_U^{(1)} = M_U$. We can take the $*$-pushforward along the open embedding $j : U \hookrightarrow Y$ to obtain the $\mathscr{D}_Y \otimes \mathbb{C}[s]/s^n$-module $j_*f^sM_U^{(n)}$ and assuming M_U is holonomic, we can take the !-pushforward to obtain another $\mathscr{D}_Y \otimes \mathbb{C}[s]/s^n$-module $j_!f^sM_U^{(n)}$. Since $f^sM_U^{(1)} = M_U$, we remark that $j_!f^sM_U^{(1)} = j_!M_U$ and $j_*f^sM_U^{(1)} = j_*M_U$. For any $a \in \mathbb{N}$, we may compose the multiplication by s^a map with the canonical morphism $j_!f^sM_U^{(n)} \to j_*f^sM_U^{(n)}$ to obtain a morphism $s^a(n) : j_!f^sM_U^{(n)} \to j_*f^sM_U^{(n)}$ which coincides with the multiplication by s^a on the open set U. We shall take for granted that there exists a large enough $N \in \mathbb{N}$, such that for every $n > N$, $\operatorname{coker} s^a(n) = \operatorname{coker} s^a(n-1)$. With this fact, we may define a functor π^a_f from the category of holonomic

$\mathscr{D}_U$-modules to the category of holonomic $\mathscr{D}_Y$-modules, sending M_U to $\operatorname{coker} s^a(N)$. Here, we are most interested in the case where $a = 1$, which is called the maximal extension functor $\Xi_f = \pi_f^1$.

We have the following exact sequences of holonomic $\mathscr{D}_Y$-modules (for details see [Bei87])

$$0 \to j_! f^s M_U^{(a)} \to \pi_f^{a+b} M_U \to \pi_f^b M_U \to 0$$

and

$$0 \to \pi_f^b M_U \to \pi_f^{a+b} M_U \to j_* f^s M_U^{(a)} \to 0.$$

Moreover, multiplication by s^a defines a map from $\pi_f^{a+b} M_U \to \pi_f^{a+b} M_U$ with image isomorphic to $\pi_f^b M_U$.

When $a = 1$, and $b = 0$, we can be a little more specific, and see that we have exact sequences

$$0 \to j_! M_U \to \Xi_f M_U \to \pi_f^0 M_U \to 0, \tag{7.1}$$

and

$$0 \to \pi_f^0 M_U \to \Xi_f M_U \to j_* M_U \to 0. \tag{7.2}$$

Moreover, $\ker(s : \Xi_f M_U \to \Xi_f M_U) = j_! M_U$ and $\operatorname{coker}(s : \Xi_f M_U \to \Xi_f M_U) = j_* M_U$. Now since s is a nilpotent endomorphism of $\Xi_f M_U$, we can take the monodromy filtration $\mu_\bullet$ of $\Xi_f M_U$, and we obtain the so-called Jantzen filtrations $J_{!\bullet}$ and $J_{*\bullet}$ by restricting to the kernel and cokernel of $\Xi_f M_U$, which are respectively $j_! M_U$ and $j_* M_U$.

Now, using Theorem 6.3.1, if we take $Y = \mathscr{D}_{\widetilde{X}}$ to be the base affine space, $U \subset \mathscr{D}_{\widetilde{X}}$ to be the open B-orbit, $f : (x_1, x_2) \mapsto x_2$, and $M_U = \mathscr{O}_U$ to be the structure sheaf on U, then these geometric Jantzen filtrations yield filtrations on $j_! \mathscr{O}_U$ and on $j_* \mathscr{O}_U$. When we take H-monodromic global sections of these filtrations, we claim that we obtain the Jantzen filtrations of Verma modules and dual Verma modules as in Definition 7.1.2, and in the sequel we shall sketch the main ideas behind why this is for the Jantzen filtration of Verma modules.

The first part is to see that the global sections of $j_! f^s \mathscr{O}_U$ and $j_* f^s \mathscr{O}_U$ give finitely deformed Verma modules and dual Verma modules respectively. What is essentially going on is that we are adjoining this $\mathbb{C}[s]/s^n$ to the $\mathscr{D}_{\widetilde{X}}$-module $j_! \mathscr{O}_U$, and so when we take global sections, we still have adjoined this $\mathbb{C}[s]/s^n$, and upon selecting a monodromy, this $\mathbb{C}[s]/s^n$ still remains, and so it follows from Remark 7.1.7 that these give the finitely deformed Verma and dual Verma modules. An interesting thing to not here is that these finitely deformed Verma and dual Verma modules are examples of $\mathfrak{g}$-modules with generalized infinitesimal character. In particular, the quotients $j_! f^s \mathscr{O}_U^{(n)} / s j_! f^s \mathscr{O}_U^{(n)}$ and $j_* f^s \mathscr{O}_U^{(n)} / s j_* f^s \mathscr{O}_U^{(n)}$ are respectively the (undeformed) Verma and dual Verma modules.

Now we have the canonical morphism $j_! f^s \mathscr{O}_U^{(n)} \to j_* f^s \mathscr{O}_U^{(n)}$, and restricting to H-monodromic global sections, yields the canonical map from finitely deformed Verma modules to finitely deformed dual Verma modules. Recognizing that $\Xi_f \mathscr{O}_U$ is the quotient of $j_* f^s \mathscr{O}_U^{(n)}$ by the image of $s^1(n)$ for sufficiently large n, the injection $j_! \mathscr{O}_U \to \Xi_f \mathscr{O}_U$ from the short exact sequence in (7.1) can be seen to be the restriction to $j_! \mathscr{O}_U$ of the canonical morphism $j_! f^s \mathscr{O}_U^{(n)} \to j_* f^s \mathscr{O}_U^{(n)}$ composed with the quotient by the image of $s^1(n)$. Now we have an obvious filtration on $j_* f^s \mathscr{O}_U^{(n)}$ by powers of s, and this induces a filtration on $\Xi_f \mathscr{O}_U$, which upon pulling back along $j_! \mathscr{O}_U \to \Xi_f \mathscr{O}_U$ yields a filtration of $j_! \mathscr{O}_U$. Looking on the H-monodromic global sections, what we described lines up with the description of the Jantzen filtration for Verma modules in Section 6.1.

On the other hand, the geometric description of the Jantzen filtration on $j_! \mathscr{O}_U$ is the restriction of the monodromy filtration to the kernel of s. But this filtration $J_{!\bullet}$ is precisely the intersection of $\ker s$ with a shifted and flipped version of the image filtration of $\Xi_f \mathscr{O}_U$. So in fact, we see that this geometric Jantzen filtration matches the filtration we described above, and hence we have a geometric description for the Jantzen filtration of a Verma module.

7.4 A computation of the Jantzen filtration for Verma modules via geometric methods in the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Here we concretely illustrate the above sketch for the geometric Jantzen filtration in our typical example, where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Here, we take $Y = \mathbb{C}^2 \setminus \{0\}$ the base affine space, and let $f : Y \to \mathbb{C}$ defined by $f : (x_1, x_2) \mapsto x_2$. Now put $U = f^{-1}(\mathbb{C}^*)$, which is the open B-orbit of Y, so that we have the open embedding $j : U \hookrightarrow Y$. For $n > 0$, let $I^{(n)}$ be the left free rank 1 $\mathscr{O}_{\mathbb{C}^*} \otimes \mathbb{C}[s]/s^n$-module generated by the symbol x^s, with the $\mathscr{D}_{\mathbb{C}^*}$-module structure satisfying the relation $\partial_x \cdot x^s = sx^{-1} \cdot x^s$. For a holonomic $\mathscr{D}_U$-module M_U, define

$$f^s M_U^{(n)} := f^* I^{(n)} \otimes_{\mathscr{O}_U} M_U.$$

Taking $M_U = \mathscr{O}_U$, we have $f^s \mathscr{O}_U^{(n)} = f^* I^{(n)} \otimes_{\mathscr{O}_U} \mathscr{O}_U = f^* I^{(n)}$. The global sections of this are elements of $\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^n$.

Here, for an arbitrary variety X, we shall write $\mathscr{A}_X := \mathscr{D}_X \otimes \mathbb{C}[s]/s^n$, and remark that we may view $f^s \mathscr{O}_U^{(n)}$ as an $\mathscr{A}_U$-module, and its pushforward to Y will be an $\mathscr{A}_Y$-module. Since j is an open embedding, the pushforward $j_* f^s \mathscr{O}_U^{(n)}$ is just the restriction of the module structure of $f^s \mathscr{O}_U^{(n)}$ to $\mathscr{A}_Y$. Alternatively, we may regard this as the left $\mathscr{A}_Y$-module

$$\frac{\mathscr{A}_U}{\mathscr{A}_U \langle \partial_1, x_2\partial_2 - s\rangle}$$

Now $j_! f^s \mathscr{O}_U^{(n)} = \mathbb{D}_Y j_* \mathbb{D}_U f^s \mathscr{O}_U^{(n)}$, where $\mathbb{D}_Y$ and $\mathbb{D}_U$ denote the dualizing functors (on Y and U respectively). We know that applying the dualizing functor on U gives the right $\mathscr{A}_U$-module

$$\mathbb{D}_U f^s \mathscr{O}_U^{(n)} = \frac{\mathscr{A}_U}{\langle \partial_1, \partial_2 - x_2^{-1}s\rangle \mathscr{A}_U},$$

and applying j_* restricts the module structure to $\mathscr{A}_Y$. Then, to compute $\mathbb{D}_Y j_* \mathbb{D}_U f^s \mathscr{O}_U^{(n)}$, we find a projective resolution, e.g.

$$0 \leftarrow \frac{\mathscr{A}_U}{\langle \partial_1, \partial_2 - x_2^{-1}s\rangle \mathscr{A}_U} \xleftarrow{\varepsilon} \mathscr{A}_Y \xleftarrow{d_0} \mathscr{A}_Y \oplus \mathscr{A}_Y \xleftarrow{d_1} \mathscr{A}_y \xleftarrow{d_2} 0,$$

where

$$\begin{aligned} \varepsilon &: 1 \mapsto x_2^{-1}, \\ d_0 &: (\theta_1, \theta_2) \mapsto \partial_1\theta_1 - (x_2\partial_2 - s)\theta_2, \\ d_1 &: 1 \mapsto (x_2\partial_2 - s, \partial_1). \end{aligned}$$

Applying the $\mathcal{H}om(-, \mathscr{A}_Y)$-functor to the complex $0 \leftarrow \mathscr{A}_Y \xleftarrow{d_0} \mathscr{A}_Y \oplus \mathscr{A}_Y \xleftarrow{d_1} \mathscr{A}_Y \xleftarrow{d_2} 0$ and computing the cohomology gives

$$j_! f^s \mathscr{O}_U^{(n)} = \frac{\mathscr{A}_Y}{\mathscr{A}_Y \langle \partial_1, x_2\partial_2 - s\rangle}.$$

We remark that every global section of $j_! f^s \mathscr{O}_U^{(n)}$ may be written as a linear combinations of elements either in the form $s^k x_1^l x_2^m$, or $s^k x_1^l \partial_2^m$ where $0 \le s < n$, $l \ge 0$, $m \ge 0$. We shall denote the generator of $j_! f^s \mathscr{O}_U^{(n)}$ by x_2^s.

Recall that $\widetilde{\mathcal{U}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathbb{C}[h^*]$ is the extended universal enveloping algebra, and for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we identify the generators of $\widetilde{\mathcal{U}}$ with derivations

$$E = -x_2\partial_1, \quad F = -x_1\partial_2, \quad H = -x_1\partial_1 + x_2\partial_2, \quad T = x_1\partial_1 + x_2\partial_2.$$

We can look at the $\widetilde{\mathcal{U}}$-module structure of the global sections of $j_* f^s \mathscr{O}_U^{(n)}$. These are elements of $\mathbb{C}[x_1, x_2, x_2^{-1}]x_2^s$ and has $\widetilde{\mathcal{U}}$-action given by

$$\begin{aligned} E \cdot s^k x_1^l x_2^m x_2^s &= -l s^k x_1^{l-1} x_2^{m+1} x_2^s, \\ F \cdot s^k x_1^l x_2^m x_2^s &= (s - m) s^k x_1^{l+1} x_2^{m-1} x_2^s, \\ H \cdot s^k x_1^l x_2^m x_2^s &= (s - l + m) s^k x_1^l x_2^m x_2^s, \\ T \cdot s^k x_1^l x_2^m x_2^s &= (s + l + m) s^k x_1^l x_2^m x_2^s. \end{aligned}$$

We can also consider the $\widetilde{\mathcal{U}}$-module structure of global sections of $j_! f^s \mathscr{O}_U^{(n)}$. If $m \ge 1$, then the action on $s^k x_1^l x_2^m x_2^s$ is the same as above, and the action on $s^k x_1^l \partial_2^m x_2^s$ is given by

$$E \cdot s^k x_1^l \partial_2^m x_2^s = (m - l - 1 - s) s^k x_1^{l-1} \partial_2^{m-1} x_2^s,$$

$$F \cdot s^k x_1^l \partial_2^m x_2^s = -s^k x_1^{l+1} \partial_2^{m+1} x_2^s,$$
$$H \cdot s^k x_1^l \partial_2^m x_2^s = (s-l-m) s^k x_1^l \partial_2^m x_2^s,$$
$$T \cdot s^k x_1^l \partial_2^m x_2^s = (s+l-m) s^k x_1^l \partial_2^m x_2^s.$$

If $m = 0$, then the actions of H and T are the same as above and we have

$$E \cdot s^k x_1^l x_2^s = -l s^k x_1^{l-1} x_2 x_2^s,$$
$$F \cdot s^k x_1^l x_2^s = -s^k x_1^{l+1} \partial_2 x_2^s.$$

We remark here that in both cases - $j_* f^s \mathscr{O}_U^{(n)}$ and $j_! f^s \mathscr{O}_U^{(n)}$ - selecting a monodromy amounts to choosing an integer λ so that $l+m=\lambda$, and so T acts by $s+\lambda$. In this case, $T-\lambda$ acts nilpotently, so the H-monodromic global sections with monodromy λ form a module of generalized infinitesimal character λ.

The canonical morphism $j_! f^s \mathscr{O}_U^{(n)} \to j_* f^s \mathscr{O}_U^{(n)}$ is that which just sends $x_2^s \mapsto x_2^s$. In particular, this sends $\partial_2 x_2^s \mapsto x_2^{-1} s x_2^s$, and more generally,

$$x_1^l \partial_2^m x_2^s \mapsto s(s-1)(s-2)\dots(s-m+1) x_1^l x_2^{-m} x_2^s, \quad \text{and} \tag{7.3}$$
$$x_1^l x_2^m x_2^s \mapsto x_1^l x_2^m x_2^s \tag{7.4}$$

The key point is that on the global sections (or e.g. the sections on $Y \setminus V(x_1)$), any time we find a x_2^{-m} term where $m > 0$ in the image of the canonical map, it is always accompanied by a factor of s. Thus, the global sections of the image of the canonical map may be viewed as elements from

$$(\mathbb{C}[x_1, x_2] \otimes \mathbb{C}[s]/s^n) x_2^s + (\mathbb{C}[x_1, x_2^{-1}] \otimes \mathbb{C}[s]/s^n) s x_2^s,$$

that is, all the elements in $j_* f^s \mathscr{O}_U^{(n)}$ except for linear combinations of elements of the form $x_1^l x_2^{-m} x_2^s$, with $m \geq 0$, $l \geq 1$. Although we only really need to work with the global sections here, it is worthwhile to look explicitly on the affine open subset $Y \setminus V(x_2) = U$ to verify that $s^a(n)$ indeed coincides with multiplication by s^a when restricted to U. The sections on U are respectively

$$j_! f \mathscr{O}_U^{(n)}(U) = (\mathbb{C}[x_1, x_2, x_2^{-1}, \partial_2] \otimes \mathbb{C}[s]/s^n) x_2^s,$$

and

$$j_* f \mathscr{O}_U^{(n)}(U) = (\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^n) x_2^s,$$

so it is clear that the canonical map is surjective when we restrict to U.

The morphism $s^a(n) : j_! f^s \mathscr{O}_U^{(n)} \to j_* f^s \mathscr{O}_U^{(n)}$ is defined by composing the multiplication map on $j_! f^s \mathscr{O}_U^{(n)}$ by s^a with the canonical map $j_! f^s \mathscr{O}_U^{(n)} \to j_* f^s \mathscr{O}_U^{(n)}$. The image of this morphism has global sections

$$(\mathbb{C}[x_1, x_2] \otimes \mathbb{C}[s]/s^n) s^a x_2^s + (\mathbb{C}[x_1, x_2^{-1} \otimes \mathbb{C}[s]/s^n) s^{a+1} x_2^s,$$

and on U we can check that $s^a(n)$ coincides with multiplication by s^a. Provided $n > a$, the cokernel of $s^a(n)$ is the same for all $n > a$, and so

$$\pi_f^a(\mathscr{O}_U) := \operatorname{coker} s^a(n), \quad n > a$$

is well defined and has global sections

$$\begin{aligned} \frac{(\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^n) x_2^s}{\Gamma(\operatorname{im} s^a(n))} &= \frac{(\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^n) x_2^s}{(\mathbb{C}[x_1, x_2] \otimes \mathbb{C}[s]/s^n) s^a x_2^s + (\mathbb{C}[x_1, x_2^{-1}] \otimes \mathbb{C}[s]/s^n) s^{a+1} x_2^s} \\ &= \frac{(\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^{a+1}) x_2^s}{\mathbb{C}[x_1, x_2] s^a x_2^s}. \end{aligned} \tag{7.5}$$

In particular, the elements of coker $s^a(n)$ are linear combinations of elements of the form $s^k x_1^m x_2^l x_2^s$, where $m, k \in \mathbb{N}$, $l \in \mathbb{Z}$, and for $m \geq 0$ we identify $s^a x_1^l x_2^m x_2^s$ with 0 and for $m < 0$ we identify $s^{a+1} x_1^l x_2^m x_2^s$ with 0. The action of $\mathscr{D}_Y$ is the natural one, but with the exception that $x_2 \cdot s^a x_1^m x_2^{-1} x_2^s = 0$. Since $s^a(n)$

restricted to U coincides with multiplication by s^a, the sections of $\pi_f^a(\mathscr{O}_U)$ on U are simply those where s^a is identified with 0, i.e.

$$\pi_f^a(\mathscr{O}_U)(U) = (\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^a)x_2^s.$$

Perhaps it is interesting to view the action of the extended enveloping algebra $\widetilde{\mathcal{U}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathbb{C}[\mathfrak{h}^*]$ on $\Gamma(\pi_f^a(\mathscr{O}_U))$. For $k, m \in \mathbb{N}$, $l \in \mathbb{Z}$, we have

$$\begin{aligned} E \cdot s^k x_1^m x_2^l x_2^s &= -ms^k x_1^{m-1} x_2^{l+1} x_2^s, \\ F \cdot s^k x_1^m x_2^l x_2^s &= -(l+s)s^k x_1^{m+1} x_2^{l-1} x_2^s, \\ H \cdot s^k x_1^m x_2^l x_2^s &= (l-m+s)s^k x_1^m x_2^l x_2^s, \\ T \cdot s^k x_1^m x_2^l x_2^s &= (l+m+s)s^k x_1^m x_2^l x_2^s. \end{aligned}$$

One should note that if we have either the case that $k = a$, $l < 0$ or $k = a - 1$, $l \geq 0$, then the actions look like

$$\begin{aligned} F \cdot s^k x_1^m x_2^l x_2^s &= -ls^k x_1^{m+1} x_2^{l-1} x_2^s, \\ H \cdot s^k x_1^m x_2^l x_2^s &= (l-m)s^k x_1^m x_2^l x_2^s, \\ T \cdot s^k x_1^m x_2^l x_2^s &= (l+m)s^k x_1^m x_2^l x_2^s. \end{aligned}$$

Additionally, if $k = a$ and $l = -1$, then

$$E \cdot s^a x_1^m x_2^{-1} x_2^s = 0.$$

It can be seen that the Casimir operator $\Omega := H^2 + 2EF + 2FE$ acts by

$$\Omega \cdot s^k x_1^m x_2^l x_2^s = [(l+m+1)^2 - 1 + 2s(l+m+1) + s^2]s^k x_1^m x_2^l x_2^s$$

Following our characterizations of $j_! f^s \mathscr{O}_U^{(n)}$ and $\pi_f^a \mathscr{O}_U$, the morphisms in the short exact sequence

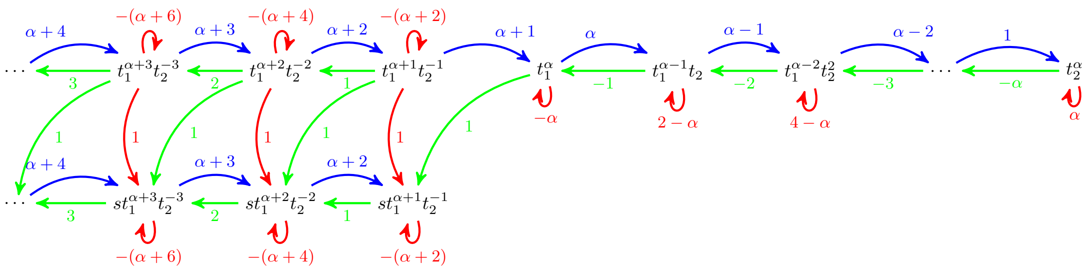
$$0 \to j_! f^s \mathscr{O}_U^{(a)} \to \pi_f^{a+b} \mathscr{O}_U \to \pi_f^b \mathscr{O}_U \to 0$$

are respectively $x^s \mapsto s^b x^s$ and the quotient map killing $x_2^m s^b$ if $m \geq 0$ and $x_2^m s^{b+1}$ if $m < 0$. Similarly, the morphisms in the short exact sequence

$$0 \to \pi_f^b \mathscr{O}_U \to \pi_f^{a+b} \mathscr{O}_U \to j_* f^s \mathscr{O}_U^{(a)} \to 0$$

are respectively $x_2^s \mapsto s^a x_2^s$ and the quotient map killing s^a.

Looking specifically at the maximal extensions functor $\Xi_f := \pi_f^1$, the global sections of the image of the map given by multiplication by s is isomorphic to $\mathbb{C}[x_1, x_2^{-1}]x_2^{-1}$, with $x_1 \cdot x_1^{-1} = 0$. Hence, the global sections of the cokernel of this map are simply elements in $\mathbb{C}[x_1, x_2, x_2^{-1}] = \Gamma(j_* \mathscr{O}_U)$. The global sections of the kernel of the map given by multiplication by s are precisely those elements in the image of the injection $j_! f^s \mathscr{O}_U^{(1)} \to \pi_f^1 \mathscr{O}_U$ defined by $x_2^s \mapsto x_2^s$. Indeed, an explicit description of this map is given by equations (7.3) and (7.4) so the image of this map consists of $\mathbb{C}$-linear combinations of elements in $\mathbb{C}[x_1, x_2]x_2^s$ and $\mathbb{C}[x_1, x_2^{-1}]sx_2^{-1}x_2^s$. On the other hand, in $\pi_f^1 \mathscr{O}_U$, elements in $\mathbb{C}[x_1, x_2]sx_2^s$ and $\mathbb{C}[x_1, x_2^{-1}]s^2 x_2^{-1} x_2^s$ are identified with 0, so $\ker s$ consists on $\mathbb{C}$-linear combinations of elements in $\mathbb{C}[x_1, x_2]x_2^s$ and $\mathbb{C}[x_1, x_2^{-1}]sx_2^{-1}x_2^s$. Hence, we have shown that the global sections of $\ker s$ in $\Xi_f \mathscr{O}_U$ coincide with the global sections of the map $j_! f^s \mathscr{O}_U^{(1)} \to \pi_f^1 \mathscr{O}_U$. Since this map is injective, we may identify $j_! f^s \mathscr{O}_U^{(1)}$ with its image under this map, and since $j_! \mathscr{O}_U = j_! f^s \mathscr{O}_U^{(1)}$, it follows that $j_! \mathscr{O}_U$ is the kernel of the morphism $s : \Xi_f \mathscr{O}_U \to \Xi_f \mathscr{O}_U$. The picture below, with a similar formatting to those diagrams in the previous chapter, shows what the monodromic global sections of $\Xi_f \mathscr{O}_U$ with monodromy α look like.



The morphism s on $\Xi_f \mathscr{O}_U$ is such that $s^2 \Xi_f \mathscr{O}_U = 0$, so following the construction given by Deligne in [Del80] (1.6.1), the monodromy filtration $\mu_\bullet$ on $\Xi_f \mathscr{O}_U$ is given by

$$0 \subset \operatorname{im} s \subset j_! \mathscr{O}_U \subset \Xi_f \mathscr{O}_U,$$

since $j_! \mathscr{O}_U = \ker s$, and $\operatorname{im} s \subset \ker s$, and $\operatorname{im} s \simeq \frac{\Xi_f \mathscr{O}_U}{\ker s}$ with isomorphism given by $s x_2^{-1} \mapsto x_2^{-1}$. Restricting to $\ker s = j_! \mathscr{O}_U$ gives the Jantzen filtration $J_{f!\bullet}$,

$$0 \subset \operatorname{im} s \subset \ker s,$$

and restricting to $\operatorname{coker} s = j_* \mathscr{O}_U$ gives the Jantzen filtration $J_{f*\bullet}$

$$0 \subset \frac{\ker s}{\operatorname{im} s} \subset \operatorname{coker} s.$$

To finish, we shall give explicit descriptions of the global sections of each of these submodules, and the $\widetilde{\mathcal{U}}$-action on each of them. First, the global sections of $\Xi_f \mathscr{O}_U$ are precisely the sections of π_f^a as described above for the case where $a = 1$. We may view the global sections of $\Xi_f \mathscr{O}_U$ as elements from

$$\frac{(\mathbb{C}[x_1, x_2, x_2^{-1}] \otimes \mathbb{C}[s]/s^2) x_2^s}{\mathbb{C}[x_1, x_2] s x_2^s}.$$

The $\widetilde{\mathcal{U}}$-action has already been described above.

The global sections of $\ker s = j_! \mathscr{O}_U = J_{f!0}$ can be viewed as elements of the $\mathbb{C}$-vector space $\mathbb{C}[x_1, x_2, x_2^{-1}]$, with the action of $\mathscr{D}_U$ being the usual one except that $x_2 \cdot x_2^{-1} = 0$, and $\partial_2 \cdot 1 = \lambda x_2^{-1}$. The $\widetilde{\mathcal{U}}$-module structure is given as follows. For, $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$E \cdot x_1^m x_2^n = -m x_1^{m-1} x_2^{n+1}, \quad n \neq -1,$$

$$E \cdot x_1^m x_2^{-1} = 0,$$

$$F \cdot x_1^m x_2^n = -n x_1^{m+1} x_2^{n-1}, \quad n \neq 0,$$

$$F \cdot x_1^m = -x_1^{m+1} x_2^{-1},$$

$$H \cdot x_1^m x_2^n = (n - m) x_1^m x_2^n,$$

$$T \cdot x_1^m x_2^n = (m + n) x_1^m x_2^n.$$

Also, the Casimir operator acts by

$$\Omega \cdot x_1^m x_2^n = ((n + m + 1)^2 - 1) x_1^m x_2^n.$$

The global sections of $\operatorname{im} s = J_{f!-1}$ can simply be seen as elements of $\mathbb{C}[x_1, x_2^{-1}] x_2^{-1}$. The $\widetilde{\mathcal{U}}$-action is the same as that described for $J_{f!0}$. Now for any $\alpha \in \mathbb{N}$, we consider the $\widetilde{\mathcal{U}}$-submodules of $J_{f!0}^\alpha \subset J_{f!0}$ and $J_{f!0}^\alpha \subset J_{f!-1}$ which are annihilated by $T - \alpha$. First note that $J_{f!0}^\alpha$ is generated by the single element t_2^α and $J_{f!-1}^\alpha$ is generated by the single element $x_1^{\alpha+1} x_2^{-1}$. Moreover, $J_{f!-1}^\alpha$ is a submodule of $J_{f!0}^\alpha$, so we may take a quotient, and we see that this quotient

$$\frac{J_{f!0}^\alpha}{J_{f!-1}^\alpha}$$

is finite dimensional.

Next, the global sections of coker $s = j_*\mathscr{O}_U = J_{f*1}$ can be seen to just be elements from $\mathbb{C}[x_1, x_2, x_2^{-1}]$, with the usual $\mathscr{D}_Y$-action (in particular, as opposed to the previous situation, here we have $\partial_2 \cdot 1 = 0$ and $x_2 \cdot x_2^{-1} = 1$), and the global sections of $\frac{\ker s}{\operatorname{im} s} = J_{f*0}$ are elements from $\mathbb{C}[x_1, x_2]$. We have already seen this situation before; the $\widetilde{\mathcal{U}}$-action on both of these modules is

$$E \cdot x_1^m x_2^n = -m x_1^{m-1} x_2^{n+1},$$

$$F \cdot x_1^m x_2^n = -n x_1^{m+1} x_2^{n-1},$$

$$H \cdot x_1^m x_2^n = (n-m) x_1^m x_2^n,$$

$$T \cdot x_1^m x_2^n = (m+n) x_1^m x_2^n,$$

for suitable n and m, and the Casimir operator acts as

$$\Omega \cdot x_1^m x_2^n = ((n+m+1)^2 - 1) x_1^m x_2^n.$$

Again, here we may fix $\alpha \in \mathbb{N}$ and consider the submodules $J_{f*1}^{\alpha} \subset J_{f*1}$ and $J_{f*0}^{\alpha} \subset J_{f*0}$ which are annihilated by $T - \alpha$. Once again, J_{f*0}^{α} is a submodule of J_{f*1}^{α}. Note that J_{f*0}^{α} is generated by the single element x_2^{α}, and J_{f*1}^{α} is generated by the single element $x_1^{\alpha+1} t_2^{-1}$. We also remark that J_{f*0}^{α} is finite dimensional.

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