ERRATUM TO "ON A PROBLEM OF LANG FOR MATRIX POLYNOMIALS, *BULL. LONDON MATH. SOC.*, 54 (2022), 1552–1567"

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In [6] the author initiates the study of a matrix analogue of the well known result conjectured by Lang [4, 8] in the 1960s and proved by Ihara, Serre and Tate, asserting the finiteness of so-called *torsion points* on curves, that is, points with all coordinates roots of unity. For the case of plane curves, Beukers and Smyth [1, Section 4.1] give a uniform bound for the number of such points. The paper [6] contains two main results, one in the case of $n \times n$ matrix specialisations commuting with the coefficients of the involved polynomials (Theorem 1.4) and one for arbitrary 2×2 matrix specialisations (Theorem 1.7).

Unfortunately the proof of Theorem 1.7 contains a gap in the application of [3, Theorem 1], that is, the author missed to consider also the case $s = 0$ in Case (I) on page 1564. As a result of this gap, [6, Theorem 1.7] does not hold true, and consequently Corollary 1.9 does not as well.

We regret that we have to retract the statement of [6, Theorem 1.7], and we correct in this note the statement of [6, Corollary 1.9] and give a corollary which resembles the one in [6, Theorem 1.7] for the polynomials $f(X) = X$ and $g(X) = X^d + C$, where C is a fixed 2×2 matrix over $\mathbb{C}$.

Before we state the result we introduce some notation: we denote by $\mathsf{M}_2(\mathbb{C})$ the set of all 2×2 complex matrices, and $\mathrm{GL}_2(\mathbb{C})$ its subset of invertible matrices. We call a matrix $A \in \mathrm{GL}_2(\mathbb{C})$ *torsion* if $A^n = I$ for some $n \geq 1$, where I represents the identity matrix. For a matrix A, $\mathrm{Tr}(A)$ denotes the trace of A and $\mathrm{Spec}(A)$ denotes the set of its eigenvalues.

We also recall that a conjugacy class $\mathcal{A}$ containing an element $A \in \mathsf{M}_2(\mathbb{C})$ is the set of all matrices of the form UAU^{-1}, $U \in \mathrm{GL}_2(\mathbb{C})$.

Theorem 1. *Let $C \in \mathsf{M}_2(\mathbb{C})$.*

(i) *Assume $C \neq \mu \cdot I$ for some $\mu \in \mathbb{C}$. If $\mathrm{Tr}(C)$ is not the sum of at most two roots of unity, then there are only finitely many pairs of conjugacy classes $(\mathcal{A}, \mathcal{B}) \subset \mathsf{M}_2(\mathbb{C})^2$ that contain the torsion matrix solutions to the equation*

$$X + Y = C.$$

Conversely, if there are only finitely many pairs of conjugacy classes $(\mathcal{A}, \mathcal{B}) \subset \mathsf{M}_2(\mathbb{C})^2$ that contain the torsion matrix solutions to the equation

$$X + Y = C,$$

then $\mathrm{Tr}(C)$ is not the sum of at most two roots of unity.

(ii) *If $C = \mu \cdot I$ for some $\mu \in \mathbb{C}^*$, then there are only finitely many conjugacy classes $\mathcal{A}$ that contain torsion matrices $A \in \mathrm{GL}_2(\mathbb{C})$ such that $\mu \cdot I - A$ is also torsion.*

Proof. We can write $C = VDV^{-1}$, where $V \in \mathrm{GL}_2(\mathbb{C})$ and

$$D = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}, \tag{1}$$

where μ_1, μ_2, μ are the eigenvalues of C.

We look for torsion solutions $(A, B) \in \mathrm{GL}_2(\mathbb{C})^2$ to the equation $X+Y = C$. Since A, B are torsion if and only if $V^{-1}AV$, $V^{-1}BV$ are as well, the problem is equivalent to looking at torsion solutions to the equation

$$X + Y = D, \tag{2}$$

where D has one of the forms in (1).

We first remark that if D is diagonal in (1), and $\mu_1 = \mu_2$, that is, $C = \mu_1 \cdot I$, then we look for torsion matrices A having eigenvalues λ_1, λ_2 that are roots of unity, such that $\mu_1 - \lambda_1$ and $\mu_1 - \lambda_2$ are also roots of unity. However, there are finitely many such roots of unity λ_1 and λ_2 if and only if $\mu_1 \neq 0$. Indeed, let λ_1 be a root of unity such that $\eta_1 = \mu_1 - \lambda_1$ is also a root of unity, that is, $\lambda_1 + \eta_1 = \mu_1$. If $\mu_1 \neq 0$, there are at most two solutions (λ_1, η_1) in roots of unity (and same discussion applies to λ_2) corresponding to the intersection of the unit circles in $\mathbb{C}$ centred at 0 and μ_1, proving (ii).

Therefore, from now on, if D is diagonal as above, we assume that $\mu_1 \neq \mu_2$, and we proceed with proving (i).

($\Longrightarrow$) We assume first that $\mathrm{Tr}(C)$ is not the sum of at most two roots of unity, and we want to show that, up to conjugacy, there are finitely many torsion solutions to (2). Let (A, B) be such a solution, and let $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$. Taking the trace of (2), we obtain the following equation in roots of unity:

$$\lambda_1 + \lambda_2 + \eta_1 + \eta_2 = \mathrm{Tr}(C). \tag{3}$$

If $\mathrm{Tr}(C)$ is not an algebraic number, (3) has no solution $(\lambda_1, \lambda_2, \eta_1, \eta_2)$ in roots of unity, therefore from now on we assume $\mathrm{Tr}(C)$ to be algebraic over $\mathbb{Q}$. Moreover, if $\mathrm{Tr}(C) = 0$, then it is the sum of two roots of unity, say 1 and -1, which contradicts our assumption. Therefore, we also have that $\mathrm{Tr}(C) \neq 0$.

We can apply now [2, Theorem 1] (see also [5, 7]) to conclude that there are finitely many non-degenerate solutions $(\lambda_1, \lambda_2, \eta_1, \eta_2)$ in roots of unity to (3), that is, for which there is no vanishing subsum. Therefore such solutions lead to finitely many torsion matrices A up to conjugacy and the same for B.

We consider now the possible vanishing subsums in (3), which up to symmetry, are as follows:

(i) $\lambda_1 = \mathrm{Tr}(C)$ and $\lambda_2 + \eta_1 + \eta_2 = 0$ (same discussion applies if λ_1 is replaced by λ_2; if λ_1 is replaced by any of η_1 or η_2, same discussion applies with A interchanged with B).

(ii) $\lambda_1 + \eta_1 = \mathrm{Tr}(C)$ and $\lambda_2 + \eta_2 = 0$ (same discussion applies if $\lambda_1 + \eta_1$ is replaced by any combination $\lambda_i + \eta_j$ with $i, j \in \{1, 2\}$).

(iii) $\lambda_1 + \lambda_2 = \mathrm{Tr}(C)$ and $\eta_1 + \eta_2 = 0$ (if $\lambda_1 + \lambda_2$ is replaced by $\eta_1 + \eta_2$, same discussion applies with A interchanged with B).

However, in all these cases we note that $\mathrm{Tr}(C)$ is the sum of at most two roots of unity, which contradicts our assumption. This concludes the proof of this implication.

($\Longleftarrow$) We assume now that the torsion solutions to (2) are contained in finitely many pairs of conjugacy classes. We want to show that $\mathrm{Tr}(C)$ is not the sum of at most two roots of unity. It is enough to construct examples when $\mathrm{Tr}(C)$ is a root of unity or the sum of two roots of unity that lead to infinitely many nonsimilar torsion matrices A, B that satisfy (2).

Example1: Assume $\mathrm{Tr}(C)$ is a root of unity. Let $\lambda_1, \lambda_2, \eta_1, \eta_2$ be roots of unity that satisfy

$$\lambda_1 = \mathrm{Tr}(C) \qquad \text{and} \qquad \lambda_2 + \eta_1 + \eta_2 = 0$$

(that is, we are in case (i) above).

Therefore λ_1 is uniquely defined, and the second equation above implies that

$$(-\eta_1/\lambda_2, -\eta_2/\lambda_2) \in \{e^{\pm\pi i/3}, e^{\mp\pi i/3}\}.$$

However, if one varies λ_2 over all roots of unity, one can construct matrices A with $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and B with $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$ satisfying the above system. Indeed, let $\lambda_2 \neq \lambda_1$ be any root of unity, $\eta_1 = -e^{\pi i/3}\lambda_2$ and $\eta_2 = -e^{-\pi i/3}\lambda_2$.

If D in (1) is diagonal, the matrix

$$A = \begin{pmatrix} \lambda_1 + \lambda_2 - d & d(\lambda_1 + \lambda_2 - d) - \lambda_1\lambda_2 \\ 1 & d \end{pmatrix}, \tag{4}$$

where

$$d = \frac{\lambda_2^2 - \lambda_1\lambda_2 - \mu_1\mu_2 + (\lambda_1 + \lambda_2)\mu_2}{\mu_2 - \mu_1},$$

satisfies $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and $B = C - A$ satisfies $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$. As λ_2 varies over all roots of unity, we obtain infinitely many matrices A and B which are not similar.

Similarly, if D in (1) has the second Jordan form with eigenvalue μ, then one can construct

$$A = \begin{pmatrix} 2\mu & 0 \\ \lambda_2^2 + \mu^2 - \mu\lambda_2 & \lambda_2 \end{pmatrix}. \tag{5}$$

For such A one has $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and $B = C - A$ satisfies $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$.

Example 2: Assume $\mathrm{Tr}(C)$ is a sum of two roots of unity. If $\mathrm{Tr}(C) = 0$ (which also falls in this case), then (3) becomes

$$\lambda_1 + \lambda_2 + \eta_1 + \eta_2 = 0,$$

which clearly has infinitely many torsion solutions $(\lambda_1, \lambda_2, \eta_1, \eta_2)$, and thus one can construct infinitely many non-similar torsion matrices A and B.

We give now a construction when $\mathrm{Tr}(C) \neq 0$. Let $\lambda_1, \lambda_2, \eta_1, \eta_2$ be roots of unity that satisfy

$$\lambda_1 + \eta_1 = \mathrm{Tr}(C) \qquad \text{and} \qquad \lambda_2 + \eta_2 = 0$$

(that is, we are in case (ii) above).

From the first equation $\lambda_1 + \eta_1 = \mathrm{Tr}(C)$, since $\mathrm{Tr}(C) \neq 0$, we have at most two solutions (λ_1, η_1) in roots of unity satisfying this equation. Let us fix one such torsion pair (λ_1, η_1) such that $\lambda_1 + \eta_1 = \mathrm{Tr}(C)$.

However, as above, if one varies λ_2 over all roots of unity, one can construct matrices A with $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and B with $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$ satisfying the above system. Indeed, let $\lambda_2 \neq \lambda_1$ be any root of unity.

If D in (1) is diagonal, let A be defined by (4), where

$$d = \frac{-\mathrm{Tr}(C)\lambda_2 - \mu_1\mu_2 + (\lambda_1 + \lambda_2)\mu_2}{\mu_2 - \mu_1}.$$

Then one has $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and $B = C - A$ satisfies $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$. As λ_2 varies over all roots of unity, we obtain again infinitely many matrices A and B which are not similar.

Similarly, if D in (1) has the second Jordan form with eigenvalue μ, then one can construct

$$A = \begin{pmatrix} \lambda_1 & 0 \\ -\mu^2 + \mu(\lambda_1 - \lambda_2) & \lambda_2 \end{pmatrix}.$$

For such A one has $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$ and $B = C - A$ satisfies $\mathrm{Spec}(B) = \{\eta_1, \eta_2\}$, and thus one obtains infinitely many such non-similar matrices.

Similar construction can be made to create examples for the case (iii) above. This concludes the proof. $\square$

We also note that the strategy of the proof can also be used to study torsion solutions of the equation (3) in higher dimension $n \geq 3$ as well, however one would have to consider all possible vanishing subsums to a linear equation in $2n$ roots of unity, which becomes quickly very complicated.

We conclude this note with the following consequence.

Corollary 2. *Let $C \in \mathsf{M}_2(\mathbb{C})$ be such that $\mathrm{Tr}(C)$ is not the sum of at most two roots of unity, and let $f(Z) = Z^d + C \in \mathsf{M}_2(\mathbb{C})[Z]$ be a polynomial of degree $d \geq 1$. Then, up to conjugacy, there are only finitely many torsion matrices U such that $f(U)$ is also torsion.*

Proof. If $d = 1$, then this is exactly one of the implications of the statement of Theorem 1 with A replaced by $-U$ therein. Let $d \geq 2$. Then we look for torsion specialisations $U, B \in \mathrm{GL}_2(\mathbb{C})$ such that

$$-U^d + B = C.$$

By Theorem 1, there are finitely many pairs of conjugacy classes $(\mathcal{A}, \mathcal{B})$ such that any torsion solution (A, B) to $X + Y = C$ belongs to one of these pairs, and thus, there are finitely many U, up to conjugacy, as well. This concludes the proof. $\square$

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