Partitions of networks that are robust to vertex permutation dynamics

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Abstract

Minimum disconnecting cuts of connected graphs provide fundamental information about the connectivity structure of the graph. Spectral methods are well-known as stable and efficient means of finding good solutions to the balanced minimum cut problem. In this paper we generalise the standard balanced bisection problem for static graphs to a new “dynamic balanced bisection problem”, in which the bisecting cut should be minimal when the graph is subjected to a general sequence of vertex permutations. We extend the standard spectral method for partitioning static graphs, based on eigenvectors of the Laplacian matrix of the graph, by constructing a new dynamic Laplacian matrix, with eigenvectors that generate good solutions to the dynamic minimum cut problem. We formulate and prove a dynamic Cheeger inequality for graphs, and demonstrate the effectiveness of the dynamic Laplacian matrix for both structured and unstructured graphs.

1 Introduction

Many spatio-temporal systems arising from physical processes can be modeled as dynamics on graphs. The motivation for our work is understanding the complex combination of dynamics and graph structure in terms of graph connectivity. The strength of graph connectivity concerns the number of edges that need to be removed in order to disconnect the graph, and is a fundamental characteriser of graph structure. Efficient algorithms for graph partitioning and the detection of community structures have led to applications in image segmentation [27, 19], parallel computing [21], social graph analysis [26], dynamical systems [16], image and video synthesis [22], nonlinear fluid flow [17], and route planning [9] (see [5] for a recent review on several numerical algorithms and applications).

Persistently highly interconnected subregions on a dynamic graph can highlight important physical properties of the underlying process, such as the stability of subprocesses and community structures over time. While it is possible to consider changes in graph structures simply by treating the changing graph as a series of static ones and evaluating
the connective structures in each step (see [8, 11] for applications in image processing and social graphs, respectively), considering a series of static connectivity problems can not be used to find subregions of dynamic graphs that are persistently highly interconnected as the graph evolves.

In this paper we focus on minimum-cut balanced partitions of the graph. Such partitions bisect a graph so that the number of edges cut is minimised, while satisfying lower bounds on the number of vertices in each of the two partition elements. It is well-known that the balanced graph partitioning problem is NP-complete (see [18]), however, the importance of the problem has generated an extensive collection of heuristic algorithms that can produce good solutions [14, 5]. A very popular class of graph partitioning methods are known as spectral partitioning methods. This approach was initiated by Fiedler [12] (also [3]), and has been developed by several authors (e.g. [11, 24]). In the present paper, we focus on the Laplacian matrix-based spectral method and follow the constructions of [7].

This paper is organised as follows. The basic machinery of the Laplacian matrix-based spectral partitioning method is described in section 2. In section 3, the construction of a new dynamic Laplacian matrix is outlined. In section 4, we state the new dynamic Cheeger inequality; the proof is deferred to the appendix. In section 5, a numerical algorithm for dynamic spectral partitioning is detailed, and numerical experiments using this algorithm are conducted on the 54-node graph Ellingham-Horton graph and a randomly generated graph.

2 Spectral partition for static graphs

In this section we briefly revisit the Laplacian-based spectral graph partitioning approach. We denote a graph by \( G = G(V, E) \), where \( V \) is the vertex set and \( E \) is the set of (undirected) edges. We assume that there are no self-loops or multiple edges, although the method we describe could be extended to cover these cases. We define a disconnection \( G' = G'(V, E') \) of \( G \) by partitioning \( V = V_1 \cup V_2 \) into two disjoint vertex sets \( V_1, V_2 \) and forming the reduced edge set \( E' = E \setminus \{ [i,j] \in E : i \in V_1, j \in V_2 \} \), where \([i,j] \in E \) is an undirected edge. The balanced graph bisection problem for a connected graph \( G(V, E) \) asks for a disconnection \( G'(V, E') \), where the set of removed edges \( E \setminus E' \) is minimised, while maintaining a relatively large number of vertices (counting multiplicity of degree) in both \( V_1 \) and \( V_2 \). We define the partition boundary \( C(V_1, V_2) \) between the partitions \( V_1 \) and \( V_2 \) as the set of edges removed to disconnect \( G \); that is, \( C(V_1, V_2) = \{ [i,j] \in E : i \in V_1, j \in V_2 \} \). The total degree of the vertex set \( V \) is denoted by \( D(V) := \sum_{i \in V} d(i) \), where \( d(i) \) is the degree of the vertex \( i \). A standard quantity to minimise is the Cheeger Constant \([4, 7, 19]\)

\[
h = \min_{V_1, V_2 \text{ partition } V} h(V_1, V_2),
\]

where

\[
h(V_1, V_2) = \frac{|C(V_1, V_2)|}{\min\{D(V_1), D(V_2)\}}.
\]

A partition \( \{V_1, V_2\} \) that achieves the minimum in (2.1) has high internal connectivity within each component corresponding to vertices \( V_i, i = 1, 2 \), and low connectivity
between the two components. Moreover, neither component is small in terms of total degree.

**Example 2.1.** In Figure 1, setting \( V_1 = \{3, 5\}, \ V_2 = \{1, 2, 4\}, \) we have \(|C(V_1, V_2)| = \left|\{[2, 3], [4, 5]\}\right| = 2, \ D(V_1) = 1 + 1 = 2, \ D(V_2) = 1 + 3 + 2 = 6. \ Thus, \( h(V_1, V_2) = \frac{2}{\min\{2, 6\}} = 1. \)

![Figure 1: Graph with 5 vertices, colored nodes = \( V_1 \), colored edges = \( E \setminus E' \) and \( h = 1. \)](image)

Following the construction of [7], we introduce a “normalised” version of the Laplacian matrix. Let \( N \) be a \(|V| \times |V|\) diagonal matrix with entries \( N_{ii} = \sqrt{d(i)} \), and consider the symmetric Laplacian matrix given by

\[
L_{ij} = \begin{cases} 
    d(i) & \text{if } i = j \\
    -1 & \text{if } [i, j] \in E, i \neq j \\
    0 & \text{otherwise}
\end{cases} \quad (2.3)
\]

The normalised Laplacian matrix is defined by \( \mathcal{L} = N^{-1}LN^{-1} \); i.e. \( \mathcal{L} \) is the symmetric matrix

\[
\mathcal{L}_{ij} = \begin{cases} 
    1 & \text{if } i = j \\
    -\frac{1}{\sqrt{d(i)d(j)}} & \text{if } [i, j] \in E, i \neq j \\
    0 & \text{otherwise}
\end{cases} \quad (2.4)
\]

Standard results concerning \( \mathcal{L} \) are: (i) the eigenvalues \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \) of \( \mathcal{L} \) are nonnegative and real, and if \( G \) is connected, \( \lambda_1 \) is of unit multiplicity and \( \lambda_2 > 0 \) [13, 3].

The eigenvector corresponding to \( \lambda_2 \) is commonly used to construct a balanced bisection \( V_1, V_2 \) of \( G \) with a small number of edges connecting \( V_1 \) to \( V_2 \); further details are provided in Section 5. One has the celebrated Cheeger inequality, which yields an upper bound for \( h \) in terms of \( \lambda_2 \); see [7], for example.

**Theorem 2.2.** Let \( G \) be a connected graph, and \( \lambda_2 \) the smallest nonzero eigenvalue of \( \mathcal{L} \). Then \( h \leq \sqrt{2\lambda_2} \).

## 3 Dynamics on Networks

We now consider the situation where the vertices of \( G \) are subjected to dynamics. Examples of dynamics on graphs include transmission of diseases in populations [28], transmission of happiness in social graphs [15], and synchronisation of community structures [23].

Abstractly, we have a permutation \( \pi_v : V \to V \), which induces an action \( \pi_e : E \to \hat{E} \) on edges via \( \pi_e([i, j]) = [\pi_v(i), \pi_v(j)], [i, j] \in E \). In this way, the entire graph \( G \) is transformed by \( \pi : G \to \hat{G} \), where \( \pi((V, E)) = (\pi_v(V), \pi_e(E)) \). The transformation \( \pi \) is a graph isomorphism: clearly edges \( \pi_e([i, j]), \pi_e([j, k]) \) are adjacent in \( \hat{G} \) if edges \([i, j], [j, k] \) edges are adjacent in \( G \).
Example 3.1. For example, in Figure 2, we see the image of the graph of Figure 1 under the cyclic permutation $\pi_v(i) = i + 1 \pmod{5}$, $i = 1, 2, 3, 4, 5$.

![Figure 2](image)

Figure 2: The graph of Figure 2 under cyclic permutation.

### 3.1 A Cheeger constant for dynamic graphs

The quality of a balanced minimum-cut on dynamic graphs can drastically alter as time progresses. In the following we first consider evolution on a graph over a single discrete time step, and then extend this to dynamics over a finite number of time steps.

One can ask the very natural question: for a permutation $\pi_v$, how well does a fixed partition $\{V_1, V_2\}$ represent a minimal disconnection of both $G$ and $\pi(G)$, according to the edge sets $E$ and $\pi_e(E)$, respectively.

To describe the disconnection of the graph $\pi(G)$ induced by $\{V_1, V_2\}$, we denote the reduced set of edges $\pi_e(E') = \pi_e(E) \setminus \{(i, j) \in \pi_e(E) : i \in V_1, j \in V_2\}$. Let $C_\pi$ denote the set of edges removed to disconnect $\pi(G)$; that is

$$C_{\pi}(V_1, V_2) = \{(i, j) \in \pi_e(E) : i \in V_1, j \in V_2\};$$

in words, fix $V_1$ and $V_2$ and compare edges in $\pi(G)$. Equivalently,

$$|\{3.1\}| = |\{(\pi_v^{-1}i, \pi_v^{-1}j) \in E : i \in V_1, j \in V_2\}| = |\{(i, j) \in E : i \in \pi_v^{-1}(V_1), j \in \pi_v^{-1}(V_2)\}|;

that is, pull back the vertex sets $V_1, V_2$ with $\pi_v$ and compare edges in $G$. Thus,

$$|C_\pi(V_1, V_2)| = |C(\pi_v^{-1}(V_1), \pi_v^{-1}(V_2))|. \quad (3.2)$$

We now consider the computation of vertex degree in $\pi(G)$. For $V' \subset V$, define $D_\pi(V') = \sum_{i \in V'} d_\pi(i)$, where $d_\pi(i)$ is the degree of $i$ computed in the graph $\pi(G)$:

$$d_\pi(i) = |\{j \in V : [i, j] \in \pi_e(E)\}|. \quad (3.3)$$

One can also do this degree computation in the original graph $G$ by noticing that

$$|\{3.3\}| = |\{j \in V : [\pi_v^{-1}i, \pi_v^{-1}j] \in E\}| = d(\pi_v^{-1}i) \quad (3.4)$$

Thus,

$$D_\pi(V') = D(\pi_v^{-1}(V')). \quad (3.5)$$

Example 3.2. Returning to Figure 1 and Figure 2 with $V_1 = \{3, 5\}$, $V_2 = \{1, 2, 4\}$, we have: $|C_\pi(V_1, V_2)| = |\{3, 2, 3, 4, 5, 1\}| = 3$ and $|C(\pi_v^{-1}(V_1), \pi_v^{-1}(V_2))| = |\{2, 1, 2, 3, 4, 5\}| = 3$ (using $\pi_v^{-1}(V_1) = \{2, 4\}$, $\pi_v^{-1}(V_2) = \{5, 1, 3\}$). Also, $D_\pi(V_1) = 5 = D(\pi_v^{-1}(V_1))$ and $D_\pi(V_2) = 3 = D(\pi_v^{-1}(V_2))$. 

4
We now define a **dynamic balanced graph bisection problem**:

\[
h^d = \min_{V_1, V_2 \text{ partition } V} h^d(V_1, V_2),
\]

(3.6)

where

\[
h^d(V_1, V_2) = \frac{|C(V_1, V_2)| + |C_{\pi}(V_1, V_2)|}{\min\{D(V_1), D(V_2)\} + \min\{D_{\pi}(V_1), D_{\pi}(V_2)\}}.
\]

(3.7)

**Example 3.3.** Combining Examples 2.1 and 3.2 we compute

\[
h^d(V_1, V_2) = \frac{(2+3)}{\min\{2, 6\} + \min\{5, 3\}} = 1.
\]

If we instead choose \(V_1' = \{1, 4, 5\}, V_2' = \{2, 3\}\), we find \(h^d(V_1', V_2') = \frac{(2+2)}{\min\{1 + 2 + 1, 3 + 1\} + \min\{1 + 1 + 2, 1 + 3\}} = 1/2; \text{ in fact, this is the unique partition achieving this minimal value } h^d.\)

### 3.2 A spectral method for dynamic graphs

We now introduce a dynamic Laplacian matrix to provide good solutions to the dynamic balanced graph bisection problem (3.6). Define the square permutation matrix

\[
P_{ij} = \begin{cases} 
1, & \text{if } \pi_v(i) = j; \\
0, & \text{otherwise.}
\end{cases}
\]

(3.8)

Motivated by the properties (3.2) and (3.5), we define the **dynamic Laplacian matrix**

\[
L_d = \frac{L + P^{-1}LP}{2}.
\]

(3.9)

The first term in (3.9) acts on \(G\), while the second term transforms from \(G\) to \(\pi(G)\) using \(P\), then applies \(L\) to \(\pi(G)\), and finally pulls the result back to \(G\) with \(P^{-1}\). If one defines \(L'\) to be the Laplacian matrix for the graph \(\pi(G)\), then by (3.4),

\[
L'_{ij} = \begin{cases} 
d(\pi^{-1}_v(i)) & \text{if } i = j \\
-1 & \text{if } [\pi^{-1}_v(i), \pi^{-1}_v(j)] \in E, i \neq j \\
0 & \text{otherwise.}
\end{cases}
\]

\[
= L_{\pi^{-1}v(i)\pi^{-1}v(j)}.
\]

From the definition of \(P\), it is straightforward to show that \(L' = P^{-1}LP\). As in the static case, we apply a degree normalisation to \(L_d\) and define the **normalised** dynamic Laplacian matrix by

\[
L_d = N^{-1}L_dN^{-1} = \frac{L + L'}{2}
\]

(3.10)

where \(L' = N^{-1}L'N^{-1}\). We will show that the spectral properties of \(L_d\) determine partitions of \(G\) with minimum-cut properties that are robust to the permutation \(\pi\). We have the following important characterisation of the eigenvalues of \(L_d\).
3.3 Rayleigh characterisation of $\mathcal{L}_d$

Let $f$ be a vector in $\mathbb{R}^n$, $n = |V|$. Let $e^i$ denote the $i$th standard basis vector in $\mathbb{R}^n$, and $\sum_{i\sim j}$ denote the summation over the set of all vertices such that $i$ is adjacent to $j$ in $G$. Then

$$Lf = \sum_{i\sim j} (f_i - f_j) e^i. \tag{3.11}$$

We equip $\mathbb{R}^n$ with the standard inner-product $\langle ., . \rangle$, so that

$$\langle f, Lf \rangle = \sum_{i\sim j} f_i (f_i - f_j) = \sum_{i < j} f_i (f_i - f_j) + \sum_{i > j} f_i (f_i - f_j) = \sum_{i < j} (f_i - f_j)^2. \tag{3.12}$$

For the remainder of this paper, we write $\sum_{i\sim j}$ as $\sum_{i < j}$ unless otherwise stated. Note that $Pf_i = f_{\pi v_i}$; thus

$$\langle Pf, LPf \rangle = \sum_{i\sim j} (f_{\pi v_i} - f_{\pi v_j})^2. \tag{3.13}$$

By (3.10) and (3.12), we arrive at the Rayleigh quotient

$$R(f) = \frac{\langle Nf, \mathcal{L}_d Nf \rangle}{\langle Nf, Nf \rangle} = \frac{\langle f, \mathcal{L}_d f \rangle}{\langle Nf, Nf \rangle} = \frac{\sum_{i\sim j} (f_i - f_j)^2 + (f_{\pi v_i} - f_{\pi v_j})^2}{2 \sum_{i=1}^n d(i) f_i^2} =: R(f). \tag{3.13}$$

Using (3.13), one can establish fundamental properties of $\mathcal{L}_d$ related to the connectivity of $G$.

**Lemma 3.4.** Let $G(V, E)$ be a connected graph of degree $n = |V|$, $\pi_v : V \to V$ a permutation, and $P$ and $\mathcal{L}_d$ be defined by (3.8) and (3.10) respectively.

1. The eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of $\mathcal{L}_d$ are nonnegative and real,

2. $\lambda_1$ is of unit multiplicity and $\lambda_2 > 0$.

3. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector with each element equal to 1. Then

$$\lambda_2 = \min_{f \in \mathbb{R}^n : \langle Nf, N\mathbf{1} \rangle = 0} R(f) \tag{3.14}$$

and the minimum of (3.14) is attained when $Nf$ is the eigenvector of $\mathcal{L}_d$ corresponding to $\lambda_2$.

**Proof.**
1. Let \( \mathbf{0} \) be the zero vector in \( \mathbb{R}^n \). Since \( L \mathbf{1} = \mathbf{0} \) by (3.11), \( L_d \mathbf{N} = N^{-1} L_d \mathbf{1} = \mathbf{0} \) by (3.9). Thus 0 an eigenvalue of \( L_d \). The matrix \( L_d \) is positive-definite by (3.13), and symmetric using orthogonality of \( P \). Symmetry implies the eigenvalues are real and positive-semidefiniteness implies nonnegativity.

2. If \( \lambda_2 = 0 \) then all terms of the form \( (f_i - f_j) \) and \( (f_{\pi_i} - f_{\pi_j}) \) are zero. But by connectivity of the graph if for one particular pair \( f_i = f_j = c \), then by traversing the graph along connected edges one sees that in fact all \( f_i, i \in V \) must take the same value \( c \). Thus, the eigenvalue \( \lambda_1 = 0 \) is of unit multiplicity.

3. Let \( g = N f \), by the Max-min Theorem [25], the symmetry of \( L_d \) implies the following characterisation of the eigenvalues of \( L_d \):

\[
\lambda_{k+1} = \max_{X^k \subset \mathbb{R}^n} \left( \min_{g \in \mathbb{R}^n: \langle g, X^k \rangle = 0, g \neq 0} \frac{\langle g, L_d g \rangle}{\langle g, g \rangle} \right)
\]

where \( X^k \) is a \( k \) dimensional subspace of \( \mathbb{R}^n \). For the case of \( k = 1 \), the maximum of (3.15) is achieved when \( X^1 \) is the kernel of \( L_d \). Since the kernel of \( L_d \) coincides with the eigenvector \( N \mathbf{1} \),

\[
\lambda_2 = \min_{g \in \mathbb{R}^n: \langle g, N \mathbf{1} \rangle = 0, g \neq 0} \frac{\langle g, L_d g \rangle}{\langle g, g \rangle}
\]

The minimum is achieved when \( g = g_2 \), the eigenvector of \( L_d \) corresponding to \( \lambda_2 \). Substituting \( g = N f \) yields (3.14).

\[\square\]

### 3.4 Dynamics over \( n \) time steps

If one has \( n - 1 \) permutations \( \pi_1, \ldots, \pi_{n-1} \), which are applied in sequence to the graph, then one can naturally extend (3.6)–(3.7) to form a dynamic Cheeger constant \( h_d^1 \) over \( n \) time steps. Denote \( \pi^{(0)} = \mathbf{I} \), \( \pi^{(i)} = \pi_i \circ \cdots \circ \pi_2 \circ \pi_1 \), \( i = 1, \ldots, n - 1 \), and define

\[
h_d^1 := \min_{V_1, V_2} \frac{\sum_{i=0}^{n-1} |C_{\pi^{(i)}}(V_1, V_2)|}{\sum_{i=0}^{n-1} \min\{D_{\pi^{(i)}}(V_1), D_{\pi^{(i)}}(V_2)\}}
\]

To construct the \( n \)-time step dynamic Laplacian, denote by \( P_i \) the permutation matrix for \( \pi_i \) (according to (3.8)), and define

\[
L_d = \frac{\sum_{i=0}^{n-1} (P_i \cdots P_2 P_1)^{-1} L (P_i \cdots P_2 P_1)}{n}
\]

The normalised dynamic Laplacian can be found by \( L_d = N^{-1} L_d N^{-1} \). The results of Lemma 3.4 also hold in this \( n \)-step situation.
4 A Dynamic Cheeger inequality

In this section, we derive our main result concerning the relationship between the spectrum of $L_d$ and the dynamic balanced graph bisection problem (3.6). This highlights the important role the second smallest eigenvalue $\lambda_2$ of $L_d$ plays in determining the persistent community structures in $G$.

**Theorem 4.1 (Dynamic Cheeger inequality).** Let $G = G(V, E)$ be a simple connected graph, and $L_d$, $h_d$ be defined by (3.10), (3.6) respectively. If $\lambda_2$ is the second smallest eigenvalue of $L_d$, then

$$h_d \leq 2\sqrt{\lambda_2} \quad (4.1)$$

*Proof.* See Appendix. □

The Laplacian matrix for a given graph is constructed from the graph’s adjacency matrix, thus all information regarding the graph’s connectivity is encoded within the graph’s Laplacian. Since our new dynamic Laplacian matrix was constructed from both the Laplacian for $G$ and $\pi(G)$, it is possible that the complex interactions between dynamics and graph connectivity are contained within the dynamic Laplacian matrix. Indeed, Theorem 4.1 tell us that the subregions on a dynamic graph that are persistently highly interconnected are closely related to the second smallest eigenvalue of $L_d$. In fact, in the proof of Theorem 4.1, it is shown that the eigenvector corresponding to $\lambda_2$ indicates how the graph of interest should be partitioned. In particular, if the vertices of $G$ are ordered according to the magnitude of each component of the degree normalised eigenvector of $\lambda_2$, then there exists a threshold in which the partition elements yielded would have a dynamic Cheeger constant that satisfies the inequality (4.1).

5 Numerical method and experiments

We can use the new Laplacian matrix $L_d$ to construct bisections of $G$ that are robust to permutation in an identical way to which the standard Laplacian matrix $L$ is used to construct bisections of a static graph. One computes $g_2$, the eigenvector of $L_d$ corresponding to $\lambda_2$, and sets $f_2 = N^{-1}g_2$. For each $\gamma \in \{f_2,i\}_{i=1}^{n-1}$, one defines the sets $V_1^\gamma = \{i \in V : f_2,i \leq \gamma\}$ and $V_2^\gamma = \{i \in V : f_2,i > \gamma\}$. The sets $V_1^\gamma$, $V_2^\gamma$ partition $V$ and there are at most $n-1$ nontrivial partitions of this form. One simply evaluates $h_d(V_1^\gamma, V_2^\gamma)$ for these at most $n-1$ distinct partitions and selects the partition that minimises $h_d(V_1^\gamma, V_2^\gamma)$. This approach was described in [10]; see [19] for a modern treatment.

To produce partitions of more than two components, one can iteratively apply the above procedure to components already identified. Alternatively, further eigenvectors $g_3, g_4, \ldots$, can be used to create partitions into more than two components that are robust to the permutation dynamics, in an analogous way to existing algorithms in the static case; see [20, 6, 2, 27] for the use of multiple eigenvectors to partition static graphs.

5.1 Numerical experiments

To illustrate our method, we use two graphs with very different connective structures. Firstly, a graph with obvious static community structures; we then apply a vertex permu-
tation that disrupts these community structures. Secondly, a randomly generated graph, where there are no clear static community structures, nor dynamic community structures.

5.2 Example 1: A structured graph

Let \( G \) be the 3-regular Ellingham-Horton 54-graph; see Figure 3.

![Figure 3: Ellingham-Horton 54-graph. Shown is the result of the spectral bisection method described in Section 5 using \( L \). The resulting partition is shown as \( V_1 \) (colored vertices), \( V_2 \) (non-colored vertices) and the partition boundary (red edges). (a) \( G \): \(|C(V_1, V_2)| = 4, D(V_1) = 54, D(V_2) = 108. \) (b) \( \pi(G) \): \(|C_\pi(V_1, V_2)| = 30, D_\pi(V_1) = 54, D_\pi(V_2) = 108. \)](image)

5.2.1 The standard (static) spectral bisection method

We first attempt to solve the static balanced bisection problem using the second eigenvector \( g_2 \) of the Laplacian matrix \( L \) in an identical way to that described in Section 5. The vector \( f_2 = N^{-1}g_2 \) (shown in Figure 4a) orders the vertices and produces at most \( n - 1 \) distinct partitions of the form \( \{V_1, V_2\} \); we select the partition with the lowest value of \( h \) given by (2.2). The results are shown in Figure 3a with the vertices corresponding to \( V_1 \) colored green and those in \( V_2 \) uncolored. The edges that connect \( V_1 \) and \( V_2 \) are colored red. The corresponding numerical quantities are in the “\( L \)” column of Table 1. The degree counts of \( V_1 \) and \( V_2 \) are relatively unbalanced; this is because the graph consists of three main clusters of approximately equal degree sum, and it is natural to statically partition the graph by grouping two clusters together.

We now introduce a vertex permutation \( \pi_v \), which will disrupt the cluster structure. We apply the cyclic permutation \( \pi_v = (18, 36, 18 + 2, 36 + 2, 18 + 4, 36 + 4, 18 + 6, 36 + 6, \ldots, 18 + 16, 36 + 16) \). The vertex collections \( V_1 \) and \( V_2 \) in \( \pi(G) \) are shown in Figure 3b colored green and white, respectively. The edges in \( \pi_v(E) \) that connect \( V_1 \) and \( V_2 \) are colored red, and one now sees a large increase in the number of these edges. Thus, the partition \( V_1, V_2 \), which nicely captured the cluster structure of the static graph, is not robust under the permutation \( \pi_v \); in other words, \( V_1, V_2 \) do not capture community structures for both \( G \) and \( \pi(G) \). The relevant numerical quantities are listed in the “\( L \)” column of Table 1. One sees a large increase in \( h^d \) compared to the value of \( h \).
Figure 4: Plots of $f_2 = N g_2$, where $g_2$ is the second eigenvector of either $\mathcal{L}$ or $\mathcal{L}_d$ for the Ellingham-Horton 54-graph. (a) $f_2$ from the static Laplacian $\mathcal{L}$. (b) $f_2$ from the dynamic Laplacian $\mathcal{L}_d$.

Table 1: Results of spectral bisection using the second eigenvectors of $\mathcal{L}$ and $\mathcal{L}_d$ to minimise $h$ and $h^d$, respectively, for the Ellingham-Horton 54 graph. The partitions $V_1, V_2$ are obtained using the method described in Section 5.

<table>
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<th>Quantity</th>
<th>$\mathcal{L}$</th>
<th>$\mathcal{L}_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C(V_1, V_2)</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>C_\pi(V_1, V_2)</td>
<td>$</td>
</tr>
<tr>
<td>$D(V_1), D(V_2)$</td>
<td>54, 108</td>
<td>114, 48</td>
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<tr>
<td>$D_\pi(V_1), D_\pi(V_2)$</td>
<td>54, 108</td>
<td>114, 48</td>
</tr>
<tr>
<td>$h$</td>
<td>0.0741</td>
<td>0.0833</td>
</tr>
<tr>
<td>$h^d$</td>
<td>0.3148</td>
<td>0.0833</td>
</tr>
</tbody>
</table>

5.2.2 New dynamic spectral bisection method

We now seek to determine community structures that are robust under the permutation $\pi$. To do this, we form the matrix $\mathcal{L}_d$ and compute the second eigenvector $g_2$. The vector $f_2 = N^{-1} g_2$ (shown in Figure 4b) orders the vertices and produces at most $n - 1$ distinct partitions of the form $\{V_1, V_2\}$ as described earlier in Section 5; we select the partition with the lowest value of $h^d$ given by (3.7).

The results are shown in Figure 5a and Figure 5b with the vertices corresponding to $V_1$ colored green and those in $V_2$ uncolored. The edges that connect $V_1$ and $V_2$ are colored red. In contrast to the partition in Figure 3, there are relatively few red edges in both Figure 5a and Figure 5b. The corresponding numerical quantities are in the “$\mathcal{L}_d$” column of Table 1.

The value of $h$ produced via $\mathcal{L}_d$ is slightly larger than that produced by $\mathcal{L}$ (0.0833 vs. 0.0741), as the latter is tailored to minimising $h$, however, the value of $h^d$ produced by $\mathcal{L}_d$ is much lower than that via $\mathcal{L}$ (0.0833 vs. 0.3148). Note that the partition found by the degree normalised eigenvector $f_2$ in Figure 4a cannot be found as a partition from $f_2$ in Figure 4b because $f_2$ arising from the latter vector assigns extreme negative and positive
values to the two lower clusters in Figure 3. Thus, the static Laplacian $\mathcal{L}$ will not group together these two lower clusters and prefers to adjoin the upper central cluster to one of the lower two.

5.3 Example 2: Randomly generated graph.

We now illustrate our method on a large random graph. We randomly generated a connected graph $G$ on 1000 vertices, with average degree approximately eight, as follows. Create a 4000-vector $x$ filled with uniformly randomly distributed integers sampled from \{1, 2, ..., 1000\}. Create a second vector $y$ by sorting $x$ in ascending order. Produce 4000 edges of the form \([x_i, y_i], i = 1, \ldots, 4000\), and remove all self-loops and duplicate edges. We arrive at a graph with 3980 edges (and a total degree sum of 7960).

The permutation $\pi_v : V \to V$ is given by $\pi_v(i) = i + 300 \pmod{1000}, i = 1, \ldots, 1000$. We computed the second eigenvector of both $\mathcal{L}$ and $\mathcal{L}_d$ for this graph and from these two eigenvectors we created the corresponding two partitions that minimise $h(V_1, V_2)$ and $h^d(V_1, V_2)$, respectively. The numerical results are summarised in Table 2.

Table 2: Results of spectral bisection using the second eigenvectors of $\mathcal{L}$ and $\mathcal{L}_d$ to minimise $h$ and $h^d$, respectively, for a randomly generated graph of 1000 vertices. The partitions $V_1, V_2$ are obtained using the method described in Section 5.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$\mathcal{L}$</th>
<th>$\mathcal{L}_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C(V_1, V_2)</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>C_\pi(V_1, V_2)</td>
<td>$</td>
</tr>
<tr>
<td>$D(V_1), D(V_2)$</td>
<td>3990, 3970</td>
<td>3976, 3984</td>
</tr>
<tr>
<td>$D_\pi(V_1), D_\pi(V_2)$</td>
<td>4090, 3878</td>
<td>3978, 3982</td>
</tr>
<tr>
<td>$h$</td>
<td>0.2831</td>
<td>0.3496</td>
</tr>
<tr>
<td>$h^d$</td>
<td>0.4049</td>
<td>0.3518</td>
</tr>
</tbody>
</table>
Referring first to the solution obtained from $L$, Table \ref{table:results} shows that the number of edges cut to disconnect the graph and minimise $h$ is just over one-quarter of all edges (1124 edges), indicating that there is no strong clustering in the random graph. Moreover, the bisections are almost perfectly balanced in terms of total degree counts. When subjected to the permutation dynamics $\pi$, the number of edges cut in $\pi(G)$ almost doubles to 2054 edges. This is because there is no particular relation between the structure of $G$ and the permutation $\pi$, so the bisection induced on $\pi(G)$ is effectively random, and cuts about half of the total number of edges.

Considering the bisection obtained from $L_d$, we see that this bisection cuts slightly more edges (1390 edges) than the bisection from $L$ (1124 edges) on $G$, however, when the dynamics of $\pi$ is applied to the graph, the number of edges traversing $V_1$ and $V_2$ in $\pi(G)$ are almost the same (1408 edges). Thus, one pays a little extra to bisect the initial graph, but this reaps large benefits when the dynamics is applied.

6 Conclusion

We considered the situation where a graph is subjected to general vertex permutation dynamics. Our aim was to determine community structures that are robust to the dynamics. We proposed a natural extension of the Laplacian matrix-based spectral method of graph partitioning and introduced a new dynamic Laplacian matrix for the dynamic graph. We stated a new dynamic Cheeger inequality, the proof of which follows in the Appendix. Furthermore, we demonstrated that eigenvectors of our dynamic Laplacian matrix efficiently separate the graph into components that retain their community structure under the dynamics.

Appendix

Let $G(V, E)$ be a graph with $n$ vertices $V = \{1, 2, \ldots, n\}$ and $\pi_v : V \rightarrow V$ a vertex permutation. We define the sets $S_i = \{1, 2, 3, \ldots, i\}$ and $\bar{S}_i = \{i + 1, \ldots, n\}$, with the set abbreviations $C(S_i) = C(S_i, \bar{S}_i)$ and $D(S_i) = \min\{D(S_i), D(\bar{S}_i)\}$, where the partition boundary $C$ and total vertex degree $D$ are defined as in Section 2. The ordered cut value $\alpha$ is defined by

$$\alpha = \min_{1 \leq i \leq |V|} \frac{|C(S_i)| + |C(\pi_v^{-1}(S_i))|}{\bar{D}(S_i) + \bar{D}(\pi_v^{-1}(S_i))}. \quad (6.1)$$

Furthermore, on the graph $G$ let $d(i)$ denote the degree of the vertex $i$, and $\sum_{i \sim j}$ denote the summation over the set of all vertices such that $i$ is adjacent to $j$, $i < j$. The following Lemma forms the crucial link between the cardinality of the partition boundary $C(\pi_v^{-1}(S_i))$ and a vector $f \in \mathbb{R}^n$.

**Lemma 6.1.** Let $G = G(V, E)$ be a simple connected graph of degree $n$, $n = |V|$, and suppose that the transformation $\pi = (\pi_v, \pi_e)$ is an isomorphism of graphs between $G$ and $\pi(G)$. If $f \in \mathbb{R}^n$ satisfies $f_i \leq f_{i+1}$ for all $i = 1, \ldots, n - 1$, then

$$\sum_{i \sim j} |f_{\pi_v i} - f_{\pi_v j}| = \sum_{i=1}^{n-1} |f_i - f_{i+1}| \cdot |C(\pi_v^{-1}(S_i))| \quad (6.2)$$
Proof. We perform induction on the number of vertices of $G$. For $n = 2$, $V = \{1, 2\}$ and $E = [1, 2]$, the vertex permutation $\pi_v$ either fixes both vertices or interchanges them. In both cases of $\pi_v$, the LHS of (6.2) is

$$\sum_{i \sim j} |f_{\pi_v i} - f_{\pi_v j}| = |f_1 - f_2|$$

while on the RHS of (6.2), the partition boundary $C(\pi_v^{-1}(S_i))$ contains the single edge $[1, 2]$, so that

$$\sum_{i=1}^{n-1} |f_i - f_{i+1}|.|C(\pi_v^{-1}(S_i))| = |f_1 - f_2|.$$

Thus (6.2) holds for a graph with two vertices. We proceed to show that the statement (6.2) is still valid for any finite number of vertices, by adding an addition vertex $n + 1$ to $G$, and counting the increase in both sides of (6.2).

Let $K = \{n_1, n_2, \ldots, n_{d(n+1)} \}$ ($d(n+1)$ is the degree of vertex $n + 1$) be the subvertex set of the vertices $\{1, \ldots, n\}$ such that each vertex in $K$ is adjacent to $n + 1$ in $G$ and $\pi_v n_1 \leq \pi_v n_2 \leq \ldots \leq \pi_v n_{|K|}$, $|K| = d(n + 1)$. Suppose $\pi_v(n + 1) = m$ where $1 \leq \pi_v n_{j-1} < m \leq \pi_v n_j \leq n + 1$, for some $1 < j \leq |K|$ (in the following arguments the situations where $m > \pi_v n_{|K|}$ or $m \leq \pi_v n_1$ can be treated in a similar fashion).

a. LHS

With the addition of the vertex $n + 1$ and its associated edges to $G$, the LHS of (6.2) is increased by an amount of $|f_{\pi_v n_k} - f_{\pi_v(n+1)}|$ for each $n_k \in K$, so the total increase to the LHS of (6.2) is given by

$$\sum_{k=1}^{|K|} |f_{\pi_v n_k} - f_{\pi_v(n+1)}|.$$  \hspace{1cm} (6.3)

We expand the summands of (6.3) using the assumption that $f_i \leq f_{i+1}$ for each $i \in V$, and $\pi_v n_1 \leq \pi_v n_2 \leq \ldots \leq \pi_v n_{|K|}$. There are two cases to consider, when $k < j$ and when $k \geq j$. First the case when $k < j$, for $k = 1 < j$,

$$|f_{\pi_v n_1} - f_m| = |f_{\pi_v n_1} - f_{\pi_v n_1+1}| + |f_{\pi_v n_1+1} - f_{\pi_v n_1+2}| + \ldots + |f_{\pi_v n_{j-1}} - f_{\pi_v n_{j-2}}| + |f_{\pi_v n_{j-1}} - f_{\pi_v n_{j-1+1}}| + |f_{\pi_v n_{j-1+1}} - f_{\pi_v n_{j-1+2}}| + \ldots + |f_{m-1} - f_m|$$

and for $k = 2 < j$,

$$|f_{\pi_v n_2} - f_m| = |f_{\pi_v n_2} - f_{\pi_v n_{2+1}}| + |f_{\pi_v n_{2+1}} - f_{\pi_v n_{2+2}}| + \ldots + |f_{\pi_v n_{j-1}} - f_{\pi_v n_{j-2}}| + |f_{\pi_v n_{j-1}} - f_{\pi_v n_{j-1+1}}| + |f_{\pi_v n_{j-1+1}} - f_{\pi_v n_{j-1+2}}| + \ldots + |f_{m-1} - f_m|$$

...
repeating this expansion for each $n_k \in K$ such that $k < j$ and adding them together, then we have an expression for the summation of (6.3) over the indices from 1 to $j - 1$:

$$
\sum_{k=1}^{j-1} |f_{\pi_n n_k} - f_{\pi_n (n+1)}| 
= |f_{\pi_n n_1} - f_{\pi_n n_1+1}| + |f_{\pi_n n_1+1} - f_{\pi_n n_2}| + \ldots + |f_{\pi_n n_{j-1}} - f_{\pi_n n_j}| + 2(|f_{\pi_n n_2} - f_{\pi_n n_2+1}| + |f_{\pi_n n_2+1} - f_{\pi_n n_3}| + \ldots + |f_{\pi_n n_{j-2}} - f_{\pi_n n_{j-1}}|)
\vdots
\vdots
+(j - 1) (|f_{\pi_n n_{j-1}} - f_{\pi_n n_{j-1}+1}| + |f_{\pi_n n_{j-1}+1} - f_{\pi_n n_{j+1}}| + \ldots + |f_{\pi_n n_{j-1}+2} - f_{\pi_n n_{j+1}}|)  
$$

Next we expand the summand of (6.3) when $k \geq j$. For $k = j$

$$
|f_{\pi_n n_j} - f_{m}| = |f_m - f_{m+1}| + |f_{m+1} - f_{m+2}| + \ldots + |f_{\pi_n n_j} - f_{\pi_n n_j}|
$$

and for $k = j + 1$,

$$
|f_{\pi_n n_{j+1}} - f_{m}| = |f_m - f_{m+1}| + |f_{m+1} - f_{m+2}| + \ldots + |f_{\pi_n n_j} - f_{\pi_n n_j}| + |f_{\pi_n n_j} - f_{\pi_n n_{j+1}}| + |f_{\pi_n n_{j+1}} - f_{\pi_n n_{j+2}}| + \ldots + |f_{\pi_n n_{j+1}} - f_{\pi_n n_{j+2}}|
$$

Repeating this expansion for each $n_k \in K$ such that $k \geq j$ and adding them together, then we have an expression for the summation of (6.3) over the indices from $j$ to $|K|:

$$
\sum_{k=j}^{[K]} |f_{\pi_n n_k} - f_{\pi_n (n+1)}| 
= (|K| - j + 1)(|f_m - f_{m+1}| + |f_{m+1} - f_{m+2}| + \ldots + |f_{\pi_n n_j} - f_{\pi_n n_j}|) + (|K| - j)(|f_{\pi_n n_j} - f_{\pi_n n_{j+1}}| + |f_{\pi_n n_{j+1}} - f_{\pi_n n_{j+2}}| + \ldots + |f_{\pi_n n_{j+1}} - f_{\pi_n n_{j+2}}|) + \ldots + |f_{\pi_n n_{[K]-1}} - f_{\pi_n n_{[K]-1}+1}| + |f_{\pi_n n_{[K]-1}+1} - f_{\pi_n n_{[K]-1}+2}| + \ldots + |f_{\pi_n n_{[K]-1}} - f_{\pi_n n_{[K]}}|
$$

The total increase in the LHS of (6.2) is accounted for by the sum of (6.4) and (6.6).

b. RHS  

Next we consider the increase in the RHS of (6.2) due to the additional vertex $n + 1$ to $G$ (or $m$ to $\pi(G)$) and the associated edges. We note that

$$
C(\pi^{-1}_v(S_i)) = \{ [p, q] \in E : p \in \pi^{-1}_v(S_i), q \in \pi^{-1}_v(\tilde{S}_i) \}
= \{ [\pi_v p, \pi_v q] \in \pi_e(E) : \pi_v p \in S_i, \pi_v q \in \tilde{S}_i \}
= \{ [\pi_v p, \pi_v q] \in \pi_e(E) : \pi_v p \leq i < \pi_v q \}
$$

so if $i < m = \pi_v(n + 1)$, then the increase in $|C(\pi^{-1}_v(S_i))|$ comes from the new set of edges connecting any vertex with label $\pi_v p \leq i$ to the vertex $m$, i.e the set of edges $[\pi_v n_k, m] \in \pi_e(E)$ for all $\pi_v n_k \leq i$. Otherwise $i \geq m$, and the increase in $|C(\pi^{-1}_v(S_i))|$ comes from the new edges connecting any vertex with label $\pi_v q > i$ to the vertex $m$, i.e the set of edges $[\pi_v n_k, m] \in \pi_e(E)$ for all $\pi_v n_k > i$. So by splitting the summation over
the index of \( i < m \) and \( i \geq m \), we can again account for the total increase in the RHS of (6.2) by only considering the summation indices \( i = \pi_v n_k \) for each \( n_k \in K \).

First we sum over the indices \( i < m \). Recall that \( 1 \leq \pi_v n_{j-1} < m \leq \pi_v n_j \leq n + 1 \) for some \( 1 < j \leq |K| \), so for this case we need to consider all \( n_k \in K \) such that \( k < j \). For \( k = 1 < j \), since \([n_1, n + 1] \in E\) the edge \([\pi_v n_1, m] \) is in \( \pi_v(E) \), therefore the partition boundary \( C(\pi_v^{-1}(S_n)) \) contains this additional edge due to the vertex \( n + 1 \). Furthermore, this edge is also contained in \( C(\pi_v^{-1}(S_i)) \) for all \( \pi_v n_i \leq i < m \), thus this edge contributes to the total increase in the RHS of (6.2) over the index \( 1 \leq i < m \) by an amount of

\[
|f_{\pi_v n_1} - f_{\pi_v n_1+1}| + |f_{\pi_v n_1+1} - f_{\pi_v n_1+2}| + \ldots + |f_{\pi_v n_2} - f_{\pi_v n_2+1}| + |f_{\pi_v n_2+1} - f_{\pi_v n_2+2}| + \ldots + |f_{\pi_v n_3} - f_{\pi_v n_3}|
\]

Similarly, when \( k = 2 < j \), since \([n_2, n + 1] \in E\), the edge \([\pi_v n_2, m] \) is in \( \pi_v(E) \) is contained in \( C(\pi_v^{-1}(S_i)) \) for all \( \pi_v n_i \leq i < m \), and contributes to the total increase over the index \( 1 \leq i < m \) by

\[
|f_{\pi_v n_2} - f_{\pi_v n_2+1}| + |f_{\pi_v n_2+1} - f_{\pi_v n_2+2}| + \ldots + |f_{\pi_v n_3} - f_{\pi_v n_3}|
\]

Repeating this for all \( n_k \in K \) such that \( k < j \), then we see that the total increase in the RHS of (6.2) over the index \( 1 \leq i < m \) is given by

\[
|f_{\pi_v n_1} - f_{\pi_v n_1+1}| + |f_{\pi_v n_1+1} - f_{\pi_v n_1+2}| + \ldots + |f_{\pi_v n_2} - f_{\pi_v n_2+1}| + |f_{\pi_v n_2+1} - f_{\pi_v n_2+2}| + \ldots + |f_{\pi_v n_3} - f_{\pi_v n_3}|
\]

Next we calculate the total increase to the RHS of (6.2) over the indices \( i \geq m \). For this case, we need to consider all \( n_k \in K \) such that \( k \geq j \). For \( k = j \), the edge \([m, \pi_v n_j] \) is in \( \pi_v(E) \) is contained in \( C(\pi_v^{-1}(S_i)) \) for all \( m \leq i < \pi_v n_j \). Thus this edge contributes to the total increase of the RHS of (6.2) over the indices \( m \leq i \leq n \) by an amount of

\[
|f_m - f_{m+1}| + |f_{m+1} - f_{m+2}| + \ldots + |f_{\pi_v n_j} - f_{\pi_v n_j}|
\]

Similarly, when \( k = j + 1 \), the edge \([m, \pi_v n_{j+1}] \) is in \( \pi_v(E) \) is contained in \( C(\pi_v^{-1}(S_i)) \) for all \( m \leq i < \pi_v n_{j+1} \), and contributes to the total increase over the indices \( m \leq i \leq n \) by

\[
|f_m - f_{m+1}| + |f_{m+1} - f_{m+2}| + \ldots + |f_{\pi_v n_j} - f_{\pi_v n_j}|
\]

\[
+|f_{\pi_v n_j} - f_{\pi_v n_{j+1}}| + |f_{\pi_v n_{j+1} - f_{\pi_v n_{j+2}}| + \ldots + |f_{\pi_v n_j - f_{\pi_v n_j}}|
\]

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Repeating this for all \( n_k \in K \) such that \( k \geq j \), we see that the total increase in the RHS of (6.2) over the indices \( m \leq i \leq n + 1 \) is given by

\[
\begin{align*}
&(|K| - j + 1)(|f_{m+1} - f_m| + |f_{m+2} - f_{m+1}| + \ldots + |f_{\pi_v n_j} - f_{\pi_v n_j - 1}| \\
&+ (|K| - j)(|f_{\pi_v n_j} - f_{\pi_v n_j+1}| + |f_{\pi_v n_j+1} - f_{\pi_v n_j+2}| + \ldots + |f_{\pi_v n_j+1 - 1} - f_{\pi_v n_j+1}| \\
&\vdots \\
&+ f_{\pi_v n_j K_i - 1} + 1 - f_{\pi_v n_j K_i - 1} + 2 - f_{\pi_v n_j K_i - 1} + 3 - f_{\pi_v n_j K_i - 1} + \ldots + |f_{\pi_v n_j K_i - 1} - f_{\pi_v n_j K_i - 1}|. (6.8)
\end{align*}
\]

Comparing the expression (6.4) to (6.7), and (6.6) to (6.8), we see that with the addition of the vertex \( n + 1 \) to \( G \), the increase in both sides of (6.2) are equal.

If we set \( \pi_v \) to be the identity permutation in Lemma 6.1, then we obtain

**Corollary 6.2.** Let \( G(V, E) \) be a simple connected graph of degree \( n \), \( n = |V| \). If a vector \( f \in \mathbb{R}^n \) satisfies \( f_i \leq f_{i+1} \) for each \( i = 1, \ldots, n - 1 \), then

\[
\sum_{i<j} |f_i - f_j| = \sum_{i=1}^{n-1} |f_i - f_{i+1}| \cdot |C(S_i)| \quad (6.9)
\]

**Lemma 6.3.** Let \( \pi_v : V \to V \) be a permutation, \( N \) the diagonal matrix with entries \( N_{ii} = \sqrt{d(i)} \), and \( L_d \) and \( R(f) \) be defined as in (3.10) and (3.13) respectively. If \( f \in \mathbb{R}^n \), \( n = |V| \) then

\[
2\sqrt{R(f)} \geq \sum_{i,j} \frac{|f_i^2 - f_j^2| + |f_{\pi_v i}^2 - f_{\pi_v j}^2|}{\sum_{i=1}^{n}(d(i)f_i^2 + d(\pi_v^{-1} i)f_i^2)} \quad (6.10)
\]

**Proof.** For each \( i \in V \),

\[
\sum_{i<j} (f_i + f_j)^2 \leq \sum_{i<j} (f_i + f_j)^2 + (f_i - f_j)^2 = 2 \sum_{i<j} f_i^2 + f_j^2
\]

\[
= 2 \sum_{i=1}^{n} d(i)f_i^2. \quad (6.11)
\]

Similarly

\[
\sum_{i<j} (f_{\pi_v i} + f_{\pi_v j})^2 \leq 2 \sum_{i=1}^{n} d(i)f_{\pi_v i}^2. \quad (6.12)
\]

By adding (6.11) to (6.12) then rearranging, we arrive at the inequality

\[
\sum_{i<j} (f_i + f_j)^2 + (f_{\pi_v i} + f_{\pi_v j})^2 \geq 2 \sum_{i=1}^{n}(d(i)f_i^2 + d(i)f_{\pi_v i}^2) \leq 1. \quad (6.13)
\]
Therefore

\[
R(f) = \frac{\sum_{i \sim j} (f_i - f_j)^2 + (f_{\pi, i} - f_{\pi, j})^2}{2 \sum_{i=1}^{n} d(i)f_i^2} \\
\geq \frac{\sum_{i \sim j} (f_i - f_j)^2 + (f_{\pi, i} - f_{\pi, j})^2}{2 \sum_{i=1}^{n} (d(i)f_i^2 + d(i)f_{\pi, i}^2)} \\
\geq \frac{\sum_{i \sim j} (f_i - f_j)^2 + (f_{\pi, i} - f_{\pi, j})^2}{2 \sum_{i=1}^{n} (d(i)f_i^2 + d(i)f_{\pi, i}^2)} \times \text{LHS of (6.13)} \\
= \frac{(a^2 + b^2)(\hat{a}^2 + \hat{b}^2)}{4 \left(\sum_{i=1}^{n} (d(i)f_i^2 + d(i)f_{\pi, i}^2)\right)^2}
\]

where

\[
a = \left(\sum_{i \sim j} (f_i - f_j)^2\right)^{\frac{1}{2}} \quad b = \left(\sum_{i \sim j} (f_{\pi, i} - f_{\pi, j})^2\right)^{\frac{1}{2}} \\
\hat{a} = \left(\sum_{i \sim j} (f_i + f_j)^2\right)^{\frac{1}{2}} \quad \hat{b} = \left(\sum_{i \sim j} (f_{\pi, i} + f_{\pi, j})^2\right)^{\frac{1}{2}}.
\]

Observe that each of the term \(a, b, \hat{a}\) and \(\hat{b}\) are real, so that

\[(a^2 + b^2)(\hat{a}^2 + \hat{b}^2) \geq (a\hat{a} + \hat{b})^2.
\]

Furthermore, application of the Cauchy-Schwartz inequality on the expressions \(a\hat{a}\) and \(bb\) yields

\[a\hat{a} = \left(\sum_{i \sim j} (f_i - f_j)^2\right)^{\frac{1}{2}} \left(\sum_{i \sim j} (f_i + f_j)^2\right)^{\frac{1}{2}} \geq \sum_{i \sim j} |f_i^2 - f_j^2|
\]

and

\[bb = \left(\sum_{i \sim j} (f_{\pi, i} - f_{\pi, j})^2\right)^{\frac{1}{2}} \left(\sum_{i \sim j} (f_{\pi, i} + f_{\pi, j})^2\right)^{\frac{1}{2}} \geq \sum_{i \sim j} |f_{\pi, i}^2 - f_{\pi, j}^2|
\]

Thus,

\[R(f) \geq \frac{(6.14)}{4 \left(\sum_{i=1}^{n} (d(i)f_i^2 + d(i)f_{\pi, i}^2)\right)^2} \geq \frac{1}{4} \left(\sum_{i \sim j} |f_i^2 - f_j^2| + |f_{\pi, i}^2 - f_{\pi, j}^2|\right)^2,
\]

yielding the result. \(\square\)

**Proof of Theorem 4.1 (Dynamic Cheeger inequality).** Let \(g = Nf\) be the eigenvector of \(L_d\) corresponding to \(\lambda_2\) and \(1 \in \mathbb{R}^n\) be the vector with each element equal to 1. We order the vertices of \(G\) according to \(f\) by \(f_i \leq f_{i+1}\), and let \(r\) denote the largest integer such that \(D(S_r) \leq D(V)/2\). Define the positive and negative parts of \(f_i\) by

\[f_i^+ = \begin{cases} f_i - f_r & \text{if } f_i \geq f_r \\ 0 & \text{otherwise} \end{cases}
\]

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By Lemma 3.4 (3), \(\lambda_2 = R(f)\), and
\[
\sum_{i=1}^{n} d(i) f_i = \langle Nf, N1 \rangle = 0.
\]
Since
\[
\sum_{i=1}^{n} d(i) f_i^2 \leq \sum_{i=1}^{n} d(i)(f_i - f_r)^2 = \sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_i^-)^2)
\]
and
\[
(f_i - f_j)^2 = (f_i^+ - f_j^+) - (f_i^- - f_j^-))^2
\]
\[
= (f_i^+ - f_j^+)^2 + 2(f_i^+ f_j^- + f_j^+ f_i^-) + (f_i^- - f_j^+)^2
\]
\[
\geq (f_i^+ - f_j^+)^2 + (f_i^- - f_j^+)^2
\]
we have
\[
\lambda_2 = R(f) = \frac{\sum_{i\sim j} (f_i - f_j)^2 + (f_{\pi_v} - f_{\pi_v})^2}{2 \sum_{i=1}^{n} d(i)f_i^2}
\]
\[
\geq \frac{\sum_{i\sim j} (f_i - f_j)^2 + (f_{\pi_v} - f_{\pi_v})^2}{2 \sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_i^-)^2)}
\]
\[
\geq \frac{\sum_{i\sim j} (f_i^+ - f_j^+)^2 + (f_{\pi_v}^+- f_{\pi_v})^2 + (f_i^- - f_j^-)^2 + (f_{\pi_v}^- - f_{\pi_v})^2}{2 \sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_i^-)^2)}
\]
\[
\geq \min\{R(f^+), R(f^-)\}, \tag{6.15}
\]
where we have used the fact that for any positive real numbers \(a, b, c\) and \(d\)
\[
\frac{a + b}{c + d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\}.
\]
Without loss of generality, we assume \(R(f^+) \leq R(f^-)\) so that (6.15) becomes \(\lambda_2 \geq R(f^+)\). Next we apply Lemma 6.3 to \(R(f^+)^2\)
\[
2\sqrt{\lambda_2} \geq \frac{\sum_{i\sim j} |(f_i^+)^2 - (f_j^+)^2| + |(f_{\pi_v}^+)^2 - (f_{\pi_v})^2|}{\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_{\pi_v})^2)}, \tag{6.16}
\]
followed by the application of Corollary 6.2 and Lemma 6.1 to obtain
\[
\geq \frac{\sum_{i=1}^{n} |(f_i^+)^2 - (f_j^+)^2| |C(S_i)| + |C(\pi_v^{-1}(S_i))|}{\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_{\pi_v}^+)^2)}
\]
\[
\geq \frac{\sum_{i=1}^{n} (f_i^+)^2 (|C(S_{i-1})| + |C(\pi_v^{-1}(S_{i-1}))|) - |C(S_i)|}{\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(f_{\pi_v}^+)^2)} \tag{6.17}
\]

\[18\]
where we have used the fact that $|C(S_n)| = |C(S_0)| = |C(\pi_v^{-1}(S_n))| = |C(\pi_v^{-1}(S_0))| = 0$. On substituting the ordered cut value (6.1) into (6.17)

\[
(6.17) \geq \alpha \sum_{i=1}^{n} (f_i^+)^2 |\hat{D}(S_{i-1}) - \hat{D}(S_i) + \hat{D}(\pi_v^{-1}(S_{i-1})) - \hat{D}(\pi_v^{-1}(S_i))| \\
\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(\pi_v f_i^+)^2)
\]

Finally we split the summation on the numerator of (6.18) over the index $r < i$, $r = i$ and $r > i$. First for $i < r$, $\hat{D}(S_r) = D(S_i)$ because $D(S_r) \leq D(V)/2$, hence

\[
\sum_{i<r} (f_i^+)^2 |\hat{D}(S_{i-1}) - \hat{D}(S_i) + \hat{D}(\pi_v^{-1}(S_{i-1})) - \hat{D}(\pi_v^{-1}(S_i))| \\
= \sum_{i<r} (f_i^+)^2 \left| \sum_{j\leq i-1} d(j) - \sum_{j\leq i} d(j) + \sum_{j\leq i-1} d(\pi_v^{-1}j) - \sum_{j\leq i} d(\pi_v^{-1}j) \right| \\
= \sum_{i<r} (f_i^+)^2 \left( d(i) + d(\pi_v^{-1}i) \right) .
\]

Next, for $i = r$, the summation over this term is zero because $f^+(r) = 0$. Lastly for $i > r$, $\hat{D}(S_i) = D(S_i)$, thus

\[
\sum_{i>r} (f_i^+)^2 |\hat{D}(S_{i-1}) - \hat{D}(S_i) + \hat{D}(\pi_v^{-1}(S_{i-1})) - \hat{D}(\pi_v^{-1}(S_i))| \\
= \sum_{i>r} (f_i^+)^2 \left| \sum_{j>i+1} d(j) - \sum_{j>i} d(j) + \sum_{j>i+1} d(\pi_v^{-1}j) - \sum_{j>i} d(\pi_v^{-1}j) \right| \\
= \sum_{i>r} (f_i^+)^2 \left( d(i) + d(\pi_v^{-1}i) \right) .
\]

Combining the partial sums (6.19), (6.20) we have

\[
(6.18) = \alpha \sum_{i=1}^{n} d(i)(f_i^+)^2 + \sum_{i=1}^{n} d(\pi_v^{-1}i)(f_i^+)^2 \\
\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(\pi_v f_i^+)^2) \\
= \alpha \sum_{i=1}^{n} d(i)(f_i^+)^2 + \sum_{i=1}^{n} d(i)(\pi_v f_i^+)^2 \\
\sum_{i=1}^{n} (d(i)(f_i^+)^2 + d(i)(\pi_v f_i^+)^2) \\
= \alpha .
\]

By comparing the definition of the ordered cut value $\alpha$ to the dynamic Cheeger constant $h^d$, we immediately see that $\alpha \geq h^d$, so the required inequality follows from (6.16) to (6.21).

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References


