Problem 1: [1, Problem 125] Let $(X, \mathcal{F})$ be a measure space and $f : X \mapsto \mathbb{R}$. Show that the following is equivalent

1) $\{f(x) > c\} \in \mathcal{F}, \forall c \in \mathbb{R};$
2) $\{f(x) \geq c\} \in \mathcal{F}, \forall c \in \mathbb{R};$
3) $\{f(x) < c\} \in \mathcal{F}, \forall c \in \mathbb{R};$
4) $\{f(x) \leq c\}, \forall c \in \mathbb{R};$
5) $f^{-1}(B) \in \mathcal{F}, \forall B \subseteq \mathbb{R}, B$ is Borel.

Problem 2: [1, Problem 128] Let $f(t)$ be a continuous function and $g(x)$ be measurable. Show that $f(g(x))$ is also measurable.

Problem 3: If $f_1$ and $f_2$ are measurable, then $f_1 + f_2, f_1 \cdot f_2$ are also measurable. If $f_n$ is a sequence of measurable functions, then $f = \lim_{n \to \infty} f_n$ is also measurable.

Problem 4: Let $f$ be a measurable function. Show that

1) If $f$ does not vanish, then $1/f$ is measurable.
2) $f_-$, $f_+$ and $|f|$ are measurable.
3) If $f$ on $[0, 1]$ and differentiable, then $f'$ is measurable.

Problem 5: Let $(f_n)$ be a sequence of measurable functions. Show that

1) $\sup_n f_n$ and $\inf_n f_n$ are measurable.
2) The set of points where $\lim_{n \to \infty} f_n$ exists is measurable.

Problem 6: [1, Problem 130] Let $f$ be differentiable on $[0, 1]$ (not necessarily continuously differentiable). Show that $f'$ is Lebesgue measurable on $[0, 1]$.

Problem 7: [1, Problem 132] A function on $\mathbb{R}$ is called Borel if and only if every subset $\{f < c\}, c \in \mathbb{R}$ is Borel. Show that every Lebesgue measurable function on $\mathbb{R}$ can be identified with a Borel function after alteration on a set of measure zero.

Problem 8: [1, Problem 133] Let $([0, 1], \mathcal{F}, \lambda)$ be the standard measure space on $[0, 1]$ with length $\lambda$. Show that for every Lebesgue measurable function $f$ on $[0, 1]$ and every $\epsilon > 0$ there is a subset $A \subseteq \mathcal{F}$ and a continuous function $g \in C[0, 1]$ such that

$$f(x) = g(x), x \in A \quad \text{and} \quad 1 - \lambda(A) < \epsilon.$$  

Problem 9: [1, Problem 135] Let $f \in C[a, b]$ show that the function $n(c)$ which equals to the number of solutions of the equation $f(x) = c$ is Lebesgue measurable.

Problem 10: [1, Problem 131] Let $f : [0, 1) \mapsto [0, 1] \times [0, 1)$ be the Cantor mapping

$$f(\sum_{n=1}^{\infty} a_n 2^{-n}) = \left( \sum_{k=1}^{\infty} a_{2k-1} 2^{-k}, \sum_{k=1}^{\infty} a_{2k} 2^{-k} \right),$$

where $a_n$ are the binary digits associated with $x \in [0, 1)$ such that

$$x = \sum_{n=1}^{\infty} a_n 2^{-n},$$

and such that for a rational $x \in [0, 1) \cap \mathbb{Q}$ we choose `$0$'-periodic representation. Show that the mapping $f$ sends measurable subsets of $[0, 1)$ into measurable subsets of $[0, 1] \times [0, 1)$. Moreover, show that $f$ preserves the measure.

References


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