Maxima, minima and saddle points

Definition

Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then

- $a \in \Omega$ is an absolute or global maximum of $f$ if $f(a) \geq f(x)$ for all $x \in \Omega$.
- $a \in \Omega$ is an absolute or global minimum of $f$ if $f(a) \leq f(x)$ for all $x \in \Omega$.
- $a \in \Omega$ is a local maximum of $f$ if there is an open set $A$ containing $a$ such that $f(a) \geq f(x)$ for all $x \in \Omega \cap A$.
- $a \in \Omega$ is a local minimum of $f$ if there is an open set $A$ containing $a$ such that $f(a) \leq f(x)$ for all $x \in \Omega \cap A$.
- $a \in \Omega$ is a stationary point of $f$ if $f$ is differentiable at $a$ and $\nabla f(a) = 0$.
- $a \in \Omega$ is a saddle point of $f$ if it is a stationary point of $f$ but is neither a local maximum nor minimum point of $f$. 
Maxima, minima and saddle points

**Theorem**

Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then local and maxima and minima can only occur at $a \in \Omega$ where $a$ satisfies one of the following:

1. $a$ is a stationary point,
2. $a$ lies on the boundary of $\Omega$ or
3. $f$ is not differentiable at $a$.

**Definition**

Points satisfying at least one of (1), (2) or (3) in the theorem above are called critical points.

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Maxima, minima and saddle points

Consider $\Omega$, the region of $\mathbb{R}^2$ bounded by $x = 0$, $y = 0$ and $y = x + 3$. Find the maximum and minimum values of $f : \Omega \to \mathbb{R}$, given by,

$$f(x, y) = x^3 - y^3 - 3xy.$$ 

$f$ is continuous and differentiable on $\Omega$ which is compact. Hence $f(\Omega)$ is compact and so maximum and minimum values exist and are attained on $\Omega$.

Since $f$ is differentiable everywhere, the extrema must occur at (1) stationary points $f$ or (2) boundary points of $\Omega$.

Stationary points of $f$ occur when

$$\nabla f = 0 \iff (3x^2 - 3y, -3y^2 - 3x) = (0, 0) \iff y = x^2 \text{ and } x = -y^2 \Rightarrow y = x^2 \Rightarrow x^4 + x = 0 \Rightarrow (x^3 + 1)x = 0.$$ 

Hence the only stationary points of $f$ are $(0, 0)$ and $(-1,1)$. Also note that

$$f(0,0) = 0 \quad \text{and} \quad f(-1,1) = 1.$$
Maxima, minima and saddle points

Divide the boundary into 3 pieces. First consider $B_1$.

$B_1 = \{(0, t) : 0 \leq t \leq 3\}$,
$B_2 = \{(t, 0) : -3 \leq t \leq 0\}$,
$B_3 = \{(t, t + 3) : -3 \leq t \leq 0\}$.

On $B_1$

$f(0, t) = 0^3 - t^3 - 0 = -t^3$

for $t \in [0, 3]$.

So the max on $B_1$ is at $(0, 0)$ where $f(0, 0) = 0$ and the min is at $(0, 3)$ where $f(0, 3) = -27$.

Next consider $B_2$.

$B_1 = \{(0, t) : 0 \leq t \leq 3\}$,
$B_2 = \{(t, 0) : -3 \leq t \leq 0\}$,
$B_3 = \{(t, t + 3) : -3 \leq t \leq 0\}$.

On $B_2$

$f(t, 0) = t^3 - 0^3 - 0 = t^3$

for $t \in [-3, 0]$.

So the max on $B_2$ is at $(0, 0)$ where $f(0, 0) = 0$ and the min is at $(-3, 0)$ where $f(-3, 0) = -27$. 
Maxima, minima and saddle points

Lastly consider $B_3$.

\[ B_1 = \{(0, t) : 0 \leq t \leq 3\}, \]
\[ B_2 = \{(t, 0) : -3 \leq t \leq 0\}, \]
\[ B_3 = \{(t, t+3) : -3 \leq t \leq 0\}. \]

On $B_3$

\[
 f(t, t+3) = t^3 - (t+3)^3 - 3t(t+3) = -3(4t^2 + 12t + 9)
\]

for $t \in [-3, 0]$. Now, $g(t) = f(t, t+3)$ has a stationary point when

\[ 8t + 12 = 0 \Rightarrow t = -\frac{3}{2}. \]

Extreme values can occur on $B_3$ at the end points (already considered) or the stationary point where

\[
\begin{pmatrix}
 3 \\
 3 \\
 -3 \\
 -3
\end{pmatrix}
\]

Hence the maximum of $f$ on $\Omega$ is 1 and the minimum value of $f$ on $\Omega$ is $-27$.

So we have a number of candidate points for the extreme values of $f$.

\[
 f(-1,1) = 1 \\
 f(0,0) = 0 \\
 f(0,3) = -27 \\
 f(-3,0) = -27 \\
 f(-1.5,1.5) = 0
\]
Classification of stationary points

The following functions $f : \mathbb{R}^2 \to \mathbb{R}$ have a stationary point at $(0, 0)$.

Is it a local maximum, minimum or saddle point?

(i) $f(x, y) = x^2 + y^2$
(ii) $f(x, y) = -x^2 - y^2$
(iii) $f(x, y) = x^2 - y^2$
(iv) $f(x, y) = xy$
(v) $f(x, y) = x^2 + y^4$
(vi) $f(x, y) = x^2 - y^4$
(vii) $f(x, y) = x^2 - 6xy + y^2$
(viii) $f(x, y) = 3x^2 - 2xy + 3y^2$

Local minimum at $(0, 0)$.

Local maximum at $(0, 0)$. 
(iii) \( f(x, y) = x^2 - y^2 \)

Along \( y = 0 \), \( f(x, 0) = x^2 \) and 

(0, 0) is a local minimum.

Along \( x = 0 \), \( f(0, y) = -y^2 \) and 

(0, 0) is a local maximum.

For all \( \epsilon > 0 \),

\[
\left( \frac{\epsilon}{2}, 0 \right) \in B((0, 0), \epsilon) \quad \text{with} \quad f \left( \frac{\epsilon}{2}, 0 \right) = \frac{\epsilon^2}{4}
\]

and

\[
\left( 0, \frac{\epsilon}{2} \right) \in B((0, 0), \epsilon) \quad \text{with} \quad f \left( 0, \frac{\epsilon}{2} \right) = -\frac{\epsilon^2}{4}.
\]

So,

\[
f \left( 0, \frac{\epsilon}{2} \right) < f(0, 0) < f \left( \frac{\epsilon}{2}, 0 \right).
\]

That is, (0, 0) is a stationary point that is neither a local max nor min and hence is a saddle point.

(iv) \( f(x, y) = xy \)

Along \( y = x \),

\[ f(x, x) = x^2 \]

which has a local minimum at (0, 0).

Along \( y = -x \),

\[ f(x, -x) = -x^2 \]

which has a local maximum at (0, 0).

So (0, 0) is neither a local maximum nor local minimum. Hence \( f \) has a saddle point at (0, 0).

Note that

\[
f(x, y) = \frac{1}{4} \left( (x + y)^2 - (x - y)^2 \right).
\]
Classification of stationary points

(v) \( f(x, y) = x^2 + y^4 \)

Local minimum at \((0, 0)\).

(iv) \( f(x, y) = x^2 - y^4 \)

Saddle point at \((0, 0)\).

(vii) \( f(x, y) = x^2 - 6xy + y^2 \)

Saddle point at \((0, 0)\).

(viii) \( f(x, y) = 3x^2 - 2xy + 3y^2 \)

Local minimum at \((0, 0)\).
Classification of stationary points

(vii)
\[ f(x, y) = x^2 - 6xy + y^2 \]
\[ = (x \ y) \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

Let
\[ H = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}. \]

\( H \) has eigenvalues and eigenvectors
\[ \lambda_1 = -2, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
\[ \lambda_2 = 4, \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

So we can orthogonally diagonalise \( H \).

Let
\[ P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]
and then
\[ P^{-1} = P^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \]

So
\[ P^T H P = D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}. \]

Now make a change of variables
\[ \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}. \]

Classification of stationary points

\[ \begin{pmatrix} X \\ Y \end{pmatrix} = P^T \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left( \frac{1}{\sqrt{2}} (x + y) \right) \]
\[ = \left( \frac{1}{\sqrt{2}} (y - x) \right). \]

So,
\[ f(x, y) = -2 \left( \frac{1}{\sqrt{2}} (x + y) \right)^2 + 4 \left( \frac{1}{\sqrt{2}} (y - x) \right)^2 = -(x + y)^2 + 2(x - y)^2. \]
Classification of stationary points

(viii)

\[ f(x, y) = (x \quad y) \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

The eigenvalues and eigenvectors of

\[ H = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \]

are

\[ \lambda_1 = 2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
\[ \lambda_2 = 4, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

Diagonalising and rotating the coordinates leads to

\[ f(x, y) = 2x^2 + 4y^2 = (x + y)^2 + 2(x - y)^2. \]

Classification of stationary points

The 'Taylor series' of \( f \) at a stationary point \((a, b)\) is

\[
\begin{align*}
    f(x, y) &= f(a, b) + \nabla f(a, b) \cdot ((x, y) - (a, b)) \\
    &+ \frac{1}{2!} (x - a \quad y - b) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a, b) & \frac{\partial^2 f}{\partial y \partial x}(a, b) \\ \frac{\partial^2 f}{\partial x \partial y}(a, b) & \frac{\partial^2 f}{\partial y^2}(a, b) \end{pmatrix} (x - a \quad y - b) \\
    &+ \cdots \text{(terms involving higher powers of } (x - a) \text{ and } (y - b) \text{)}
\end{align*}
\]

since \( \nabla f(a, b) = (0, 0) \).

For \((x, y)\) close to \((a, b)\) the nature of the stationary point will be determined by the eigenvalues of the matrix

\[
H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a, b) & \frac{\partial^2 f}{\partial y \partial x}(a, b) \\ \frac{\partial^2 f}{\partial x \partial y}(a, b) & \frac{\partial^2 f}{\partial y^2}(a, b) \end{pmatrix}.
\]
Classification of stationary points

Suppose \( f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^2 \) and has a stationary point at \( (a, b) \), that is, \( \nabla f(a, b) = 0 \). So Taylor’s theorem says that

\[
f(x, y) = f(a, b) + R_{1,(a,b)}(x, y)
\]

where the remainder term is given by

\[
R_{1,(a,b)}(x, y) = \frac{1}{2!} (x - a, y - b)^T H \begin{pmatrix} x - a \\ y - b \end{pmatrix}
\]

where

\[
H = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} (c, d) & \frac{\partial^2 f}{\partial y \partial x} (c, d) \\
\frac{\partial^2 f}{\partial x \partial y} (c, d) & \frac{\partial^2 f}{\partial y^2} (c, d)
\end{pmatrix}
\]

for some point \((c, d)\) between \((a, b)\) and \((x, y)\).

Can the eigenvalues of \( H \) be used to determine whether \( f \) has a local max, min or saddle point at \((a, b)\)? \( H \) is made of partial derivatives evaluated at an unknown point \((c, d)\). Can we determine the nature of the stationary point using partial derivatives calculated at \((a, b)\)? Yes, on a sufficiently small ball. Why?

Maxima, minima and saddle points

Definition

For \( f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R} \) the Hessian of \( f \) at \( a \) is the \( n \times n \) matrix

\[
H(f, a) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1^2} (a) & \frac{\partial^2 f}{\partial x_1 \partial x_2} (a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} (a) \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} (a) & \frac{\partial^2 f}{\partial x_2^2} (a) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} (a) \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} (a) & \frac{\partial^2 f}{\partial x_n \partial x_2} (a) & \cdots & \frac{\partial^2 f}{\partial x_n^2} (a)
\end{pmatrix}.
\]
Classification of stationary points

The signs of the eigenvalues of

\[ H(f,(a,b)) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix} \]

can be determined from the signs of the trace\(^1\) and determinant of \(H(f,(a,b))\).

\[ \text{Tr}(H(f,(a,b))) = \text{sum of eigenvalues} \]

and

\[ \text{det}(H(f,(a,b))) = \text{product of eigenvalues}. \]

These are continuous functions of the entries in the matrix which are continuous by the assumption that \(f\) is \(C^2\). Hence there must be an open ball around \((a,b)\) on which the trace and determinant (and hence the eigenvalues) of the Hessian have the same signs as those of the Hessian at \((a,b)\).

---

\(^1\)The trace of a matrix is the sum of its diagonal entries.
Maxima, minima and saddle points

Find the eigenvalues of the Hessian of \( f \) at \((0, 0)\) for each of the functions we considered last lecture.

\[
H(f, (0, 0)) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1^2}(0, 0) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(0, 0) \\
\frac{\partial^2 f}{\partial x_1 \partial x_2}(0, 0) & \frac{\partial^2 f}{\partial x_2^2}(0, 0)
\end{pmatrix}.
\]

(i) \( f(x, y) = x^2 + y^2 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\]
Eigenvalues are 2, 2.

(ii) \( f(x, y) = -x^2 - y^2 \)

\[
H(f, (0, 0)) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.
\]
Eigenvalues are -2, -2.

(iii) \( f(x, y) = x^2 - y^2 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.
\]
Eigenvalues are 2, -2.

(iv) \( f(x, y) = xy \)

\[
H(f, (0, 0)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Eigenvalues are 1, -1.

(v) \( f(x, y) = x^2 + y^4 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Eigenvalues are 2, 0.

(vi) \( f(x, y) = x^2 - y^4 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Eigenvalues are 2, 0.

(vii) \( f(x, y) = x^2 - 6xy + y^2 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 2 & -6 \\ -6 & 2 \end{pmatrix}.
\]
Eigenvalues are -4, 8.

(viii) \( f(x, y) = 3x^2 - 2xy + 3y^2 \)

\[
H(f, (0, 0)) = \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}.
\]
Eigenvalues are 4, 8.
Classification of stationary points

Definition

An $n \times n$ matrix $H$ is

- positive definite $\iff$ all eigenvalues are $> 0$
- positive semidefinite $\iff$ all eigenvalues are $\geq 0$
- negative definite $\iff$ all eigenvalues are $< 0$
- negative semidefinite $\iff$ all eigenvalues are $\leq 0$

Theorem (Alternative test — Sylvester’s criterion)

If $H_k$ is the upper left $k \times k$ submatrix of $H$ and $\triangle_k = \det H_k$ then $H$ is

- positive definite $\iff \triangle_k > 0$ for all $k$
- positive semidefinite $\Rightarrow \triangle_k \geq 0$ for all $k$
- negative definite $\iff \triangle_k < 0$ for all odd $k$ and $\triangle_k > 0$ for all even $k$
- negative semidefinite $\Rightarrow \triangle_k \leq 0$ for all odd $k$ and $\triangle_k \geq 0$ for all even $k$

Classification of stationary points

Theorem

Suppose $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is $C^2$ and $\nabla f(a) = 0$ at an interior point $a$ of $\Omega$. Then

- $H(f, a)$ is positive definite $\Rightarrow$ $f$ has a local minimum at $a$.
- $H(f, a)$ is negative definite $\Rightarrow$ $f$ has a local maximum at $a$.
- $f$ has a local minimum at $a$ $\Rightarrow$ $H(f, a)$ is positive semidefinite.
- $f$ has a local maximum at $a$ $\Rightarrow$ $H(f, a)$ is negative semidefinite.
Classication of stationary points

For \( f : \mathbb{R}^2 \to \mathbb{R} \) with a stationary point at \((a, b)\),

\[
\Delta_1 = \frac{\partial^2 f}{\partial x^2}(a, b) \quad \text{and} \quad \Delta_2 = \frac{\partial^2 f}{\partial x^2}(a, b) \frac{\partial^2 f}{\partial y^2}(a, b) - \left( \frac{\partial^2 f}{\partial x \partial y}(a, b) \right)^2.
\]

Then
- \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) (two positive eigenvalues) \( \Rightarrow \) \((a, b)\) is a local minimum.
- \( \Delta_1 < 0 \) and \( \Delta_2 > 0 \) (two negative eigenvalues) \( \Rightarrow \) \((a, b)\) is a local maximum.
- local minimum at \((a, b)\) \( \Rightarrow \) \( \Delta_1 \geq 0 \) and \( \Delta_2 \geq 0 \) (no negative eigenvalues).
- local maximum at \((a, b)\) \( \Rightarrow \) \( \Delta_1 \leq 0 \) and \( \Delta_2 \geq 0 \) (no positive eigenvalues).

Notes:
- \( \Delta_2 < 0 \) \( \Rightarrow \) \((a, b)\) is a saddle point (one positive and one negative eigenvalue).
- The semidefinite case can also be a saddle point.

---

Classication of stationary points

Find and classify the stationary points of

\[
f(x, y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x.
\]

Stationary points occur when \( \nabla f = 0 \), that is,

\[
(3x^2 + 12x - 12y + 9, 6y - 12x) = (0, 0)
\]

\( \Rightarrow \)

\[
\begin{align*}
3x^2 + 12x - 12y + 9 &= 0 \quad (1) \\
6y - 12x &= 0 \quad (2)
\end{align*}
\]

(2) \( \Rightarrow \) \( y = 2x \) which when substituted into (1) becomes

\[
3(x - 3)(x - 1) = 0.
\]

So \( x = 1 \) \( \Rightarrow \) \( y = 2 \) or \( x = 3 \) \( \Rightarrow \) \( y = 6 \).

So \( f \) has stationary points at \((1, 2)\) and \((3, 6)\).
Classification of stationary points

\[ f(x, y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x. \quad \Rightarrow \quad H(f, (x, y)) = \begin{pmatrix} 6x + 12 & -12 \\ -12 & 6 \end{pmatrix}. \]

At \((1, 2)\):

\[ H(f, (1, 2)) = \begin{pmatrix} 18 & -12 \\ -12 & 6 \end{pmatrix} \]

\[ \Delta_2 = 18 \times 6 - (-12) \times (-12) = -36 < 0. \]

So \((1, 2)\) is a saddle point of \(f\).

At \((3, 6)\):

\[ H(f, (3, 6)) = \begin{pmatrix} 30 & -12 \\ -12 & 6 \end{pmatrix} \]

\[ \Delta_1 = 30 > 0, \quad \Delta_2 = 30 \times 6 - (-12) \times (-12) = 36 > 0. \]

So \((3, 6)\) is a local minimum point of \(f\).

Classification of stationary points

Find and classify the stationary points of

\[ f(x, y, z) = yx^2 + zy^2 + z^2 - 2yx - 2zy + y - z. \]

\[ \nabla f = \mathbf{0} \Rightarrow \begin{cases} 
2xy - 2y = 0 \quad (1) \\
x^2 + 2zy - 2x - 2z + 1 = 0 \quad (2) \\
y^2 + 2z - 2y - 1 = 0 \quad (3)
\end{cases} \]

(1) is \(2y(x - 1) = 0\) so there are two cases

\[ y = 0:\]

\( (3) \Rightarrow z = \frac{1}{2}, \quad (2) \Rightarrow x = 0 \text{ or } x = 2. \]

So \((0, 0, \frac{1}{2})\) and \((2, 0, \frac{1}{2})\) are stationary points.

\[ x = 1:\]

\( (2) \Rightarrow z = 0 \text{ or } y = 1. \]

For \(z = 0\), \((3) \Rightarrow y = 1 \pm \sqrt{2}. \)

For \(y = 1\), \((3) \Rightarrow z = 1. \)

So, \((1, 1 \pm \sqrt{2}, 0)\) and \((1, 1, 1)\) are stationary points.

\( f \) has 5 stationary points: \((0, 0, \frac{1}{2}), \ (2, 0, \frac{1}{2}), \ (1, 1 + \sqrt{2}, 0), \ (1, 1 - \sqrt{2}, 0), \ (1, 1, 1). \)
Classification of stationary points

To classify we need \( H(f,(x,y)) = \begin{pmatrix} 2y & 2x - 2 & 0 \\ 2x - 2 & 2z & 2y - 2 \\ 0 & 2y - 2 & 2 \end{pmatrix} \).

\( H(f,(0,0,\frac{1}{2})) = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 2 \end{pmatrix} \)

\( \triangle_1 = 0, \quad \triangle_2 = \begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix} = -4 
\triangle_3 = \begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix} = -8 \)

(0,0,\frac{1}{2}) is a saddle point as the Hessian is neither positive semidefinite nor negative semidefinite.

\[ \text{[Eigenvalues are } 1, -2, 4.] \]

\( H(f,(1,1,1)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \)

\( \triangle_1 = 2 
\triangle_2 = 4 
\triangle_3 = 8 \)

(1,1,1) is a local minimum point as the Hessian is positive definite.

\[ \text{[Eigenvalues are } 2, 2, 2.] \]

Classification of stationary points

\( H(f,(2,0,\frac{1}{2})) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & -2 \\ 0 & -2 & 2 \end{pmatrix} \)

\( \triangle_1 = 0 
\triangle_2 = -4 
\triangle_3 = -8 \)

(2,0,\frac{1}{2}) is a saddle point as the Hessian is neither positive semidefinite nor negative semidefinite.

\[ \text{[E'values are } 1, -2, 4.] \]

\( H(f,(1,1 + \sqrt{2},0)) = \begin{pmatrix} 2 + 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2 \end{pmatrix} \)

\( \triangle_1 = 2 + 2\sqrt{2} 
\triangle_2 = 0 
\triangle_3 = -16 - 16\sqrt{2} \)

(1,1 + \sqrt{2},0) is a saddle point as the Hessian is neither positive semidefinite nor negative semidefinite.

\[ \text{[E'values are } -2, 4, \sqrt{2} + 2\sqrt{2}.] \]

\( H(f,(1,1 - \sqrt{2},0)) = \begin{pmatrix} 2 - 2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2 \end{pmatrix} \)

\( \triangle_1 = 2 - 2\sqrt{2} 
\triangle_2 = 0 
\triangle_3 = -16 + 16\sqrt{2} \)

(1,1 - \sqrt{2},0) is a saddle point as the Hessian is neither positive semidefinite nor negative semidefinite.

\[ \text{[E'values are } -2, 4, \sqrt{2} - 2\sqrt{2}.] \]