Coordinates

So far we have seen vector spaces in a familiar form (e.g., $\mathbb{R}^n$) and in a more abstract form (i.e., sets satisfying the vector space axioms). We will bring these closer together by defining coordinates.

Every vector in $\mathbf{w} \in \mathbb{R}^2$ can be expressed as a linear combination of the non-zero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ provided they are not parallel. The scalars $x_1, x_2$ are called the coordinates of $\mathbf{w}$ with respect $\mathbf{v}_1, \mathbf{v}_2$. 
Coordinates

Suppose that \( \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3 \) are not parallel and \( W = \text{span}(\mathbf{v}_1, \mathbf{v}_2) \).

If we use the vectors \( \mathbf{v}_1, \mathbf{v}_2 \) determine coordinates on \( W \) then

- \( \mathbf{v}_1 \) has coordinates \((1,0)\) because \( \mathbf{v}_1 = 1 \mathbf{v}_1 + 0 \mathbf{v}_2 \).
- \( \mathbf{v}_2 \) has coordinates \((0,1)\) because \( \mathbf{v}_2 = 0 \mathbf{v}_1 + 1 \mathbf{v}_2 \).
- \( 2 \mathbf{v}_1 + \mathbf{v}_2 \) has coordinates \(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\).
- \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 \) has coordinates \(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\).

If \( \mathbf{v}_1, \mathbf{v}_2 \) were parallel, they could not be used to define coordinates on \( W \). Also, the do not define coordinates on \( \mathbb{R}^3 \).

What properties does a set of vectors need to in order to use them to make coordinates on a vector space?

Linear dependence

Let \( V \) be a vector space over \( \mathbb{F} \) and \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \subseteq V \).

\( S \) is linearly dependent if there are scalars \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F} \), not all zero, such that

\[
\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}.
\]

That is, there is a non-trivial linear combination of \( S \) that is zero.

Examples

\( S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \} \subseteq \mathbb{R}^3 \). Since

\[
1 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

zero can be written as a non-trivial linear combination of \( S \) and hence \( S \) is linearly dependent.

Prove that two vectors in \( \mathbb{R}^n \) are linearly dependent if and only if they are parallel.
Linear independence

Let \( V \) be a vector space over \( F \) and \( S = \{v_1, v_2, \ldots, v_n\} \subseteq V \).

\( S \) is linearly independent if \( S \) is not linearly dependent. That is, the only scalars \( \lambda_1, \lambda_2, \ldots, \lambda_n \in F \), satisfying
\[
\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0,
\]
are \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \).

Example

Show that
\[
S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
is linearly independent.

Consider the vectors
\[
a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, a_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \in F^m.
\]

Last lecture we saw that
\[
x_1 a_1 + \cdots + x_n a_n = Ax
\]
where
\[
A = (a_1 \ a_2 \ \ldots \ a_n) \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
\]

\( B = \{a_1, a_2, \ldots, a_n\} \) is linearly independent if and only if \( Ax = 0 \) has the unique solution \( x = 0 \). Note that this can only happen if \( A \) is square and \( \det(A) \neq 0 \).
Linear independence

Let

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}. \]

Show that

\[ B_1 = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \]

is linearly dependent and

\[ B_2 = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4 \} \]

is linearly independent.

Uniqueness of linear combinations

In order to use a set of vectors \( S \) to define coordinates on a vector space spanned by \( S \) we need to be sure that the coordinates are unique.

Let \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) be linearly independent and \( \mathbf{v} \in \text{span}(S) \). So there are scalars \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F} \), such that

\[ \mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n. \]

Suppose that there is another set of scalars \( \lambda'_1, \lambda'_2, \ldots, \lambda'_n \in \mathbb{F} \), such that

\[ \mathbf{v} = \lambda'_1 \mathbf{v}_1 + \lambda'_2 \mathbf{v}_2 + \cdots + \lambda'_n \mathbf{v}_n. \]

Subtracting these two linear combinations for \( \mathbf{v} \) gives

\[ 0 = (\lambda_1 - \lambda'_1) \mathbf{v}_1 + (\lambda_2 - \lambda'_2) \mathbf{v}_2 + \cdots + (\lambda_n - \lambda'_n) \mathbf{v}_n. \]

Since \( S \) is linearly independent,

\[ \lambda_1 - \lambda'_1 = \lambda_2 - \lambda'_2 = \cdots = \lambda_n - \lambda'_n = 0, \]

that is

\[ \lambda_1 = \lambda'_1, \quad \lambda_2 = \lambda'_2, \quad \cdots, \quad \lambda_n = \lambda'_n. \]

So there is only one way to write \( \mathbf{v} \) as a linear combination of \( S \).
Coordinates

If \( B = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is an ordered linearly independent set and \( \mathbf{v} \in \text{span}(B) \), then the coordinate vector of \( \mathbf{v} \) with respect to \( B \) is

\[
[\mathbf{v}]_B = \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{pmatrix}
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the unique scalars such that

\[
\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n.
\]

Examples

\( B = \{1, x, x^2\} \) is linearly independent because

\[
\lambda_1 + \lambda_2 x + \lambda_3 x^2 = 0 \implies \lambda_1 = \lambda_2 = \lambda_3 = 0.
\]

\[
[1 - 3x^2]_B = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad [a_0 + a_1 x + a_2 x^2]_B = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}
\]

Coordinates

Examples

For

\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.
\]

We already saw that

\( B_2 = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4 \} \)

is linearly independent. If \( \mathbf{v} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \), is \( \mathbf{v} \in \text{span}(B_2) \)? If so, what is \([\mathbf{v}]_{B_2}\)?

For

\( B_3 = \{1 + 2x, -1 + x + 2x^2, 3 + x^2\} \subseteq \mathbb{P}_2(\mathbb{R}) \).

Is \( B_3 \) linearly independent? Is \( 6 + 3x - x^2 \in \text{span}(B_3) \)? If so, what is the coordinate vector of \( 6 + 3x - x^2 \) with respect to \( B_3 \)?