Properties of vectors in $\mathbb{R}^n$

An important reason for studying vectors in $\mathbb{R}^n$ in MATH1151 was to develop intuition and examples that are useful in the more abstract study of vector spaces. What are the essential algebraic and geometric properties of vectors?

Algebraic properties

- There are things called vectors and things called scalars.
- Can add any two vectors and the result is another vector.
- Can multiply any vector by a scalar and the result is another vector.
- Vector addition is commutative and associative.
- There is a zero vector, denoted $\mathbf{0}$.
- For each vector $\mathbf{v}$ there is $-\mathbf{v}$ (i.e., can define subtraction).
- Multiplying by the scalar 1 leaves any vector unchanged.
- There are distributive laws for scalar multiplication.

Geometric properties

- Length of a vector.
- Angle between two vectors.
Other examples

The study of vector spaces concentrates on algebraic properties. There are lots of examples of sets of things that obey the algebraic properties of vectors.

Examples

- \( \mathbb{R}^n \) or \( \mathbb{C}^n \).
- Real or complex \( m \times n \) matrices \( M_{mn}(\mathbb{R}) \) or \( M_{mn}(\mathbb{C}) \).
- Real valued functions on \( \mathbb{R} \) (or complex functions on \( \mathbb{C} \)).
- Differentiable functions on an interval \( I \subseteq \mathbb{R} \).
- Integrable functions on an interval \( I \subseteq \mathbb{R} \).
- Integrable functions on \( \mathbb{R} \) such that \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx \) converges.
- \( \mathbb{P}(\mathbb{R}) = \) polynomial functions on \( \mathbb{R} \).
- \( \mathbb{P}_n(\mathbb{R}) = \) polynomial functions on \( \mathbb{R} \) of degree \( \leq n \).
- Trigonometric polynomials of degree up to \( n \), eg functions of the form,
  \[
  T(\theta) = a_0 + \sum_{k=1}^{n} \left( a_k \cos(k\theta) + b_k \sin(k\theta) \right).
  \]
- Solutions to a linear homogeneous ODE, eg solutions to,
  \[
  y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0.
  \]

Consider the following:

- a non-empty set \( V \) (the vectors),
- a field \( F \) (the scalars, eg \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \)),
- a map \( + : V \times V \to V \) (vector addition) and
- a map \( * : F \times V \to V \) (scalar multiplication).

If \( u, v \in V \) and \( \lambda \in F \), addition is usually written as \( u + v \) and scalar multiplication as \( \lambda \cdot v \) or simply \( \lambda v \).

\((V, F, +, \star)\)

\((V, F, +, \star)\) is called a vector space if it obeys the vector space axioms, which are essentially the algebraic properties of vectors mentioned on the previous slide.

Note that it must be clear how the operations \( + \) and \( \star \) act.

Note: vectors are usually typeset using a bold font. Hand written vectors should be written with a tilde below or an arrow above.
1. **Closure under Addition.** If \( u, v \in V \), then \( u + v \in V \).

2. **Associative Law of Addition.** If \( u, v, w \in V \), then \( (u + v) + w = u + (v + w) \).

3. **Commutative Law of Addition.** If \( u, v \in V \), then \( u + v = v + u \).

4. **Existence of Zero.** There exists an element \( 0 \in V \) such that, for all \( v \in V \), \( v + 0 = v \).

5. **Existence of Negative.** For each \( v \in V \) there exists an element \( w \in V \) (usually written as \( -v \)), such that \( v + w = 0 \).

6. **Closure under Multiplication by a Scalar.** If \( v \in V \) and \( \lambda \in F \), then \( \lambda v \in V \).

7. **Associative Law of Multiplication by a Scalar.** If \( \lambda, \mu \in F \) and \( v \in V \), then \( \lambda(\mu v) = (\lambda\mu)v \).

8. If \( v \in V \) and \( 1 \in F \) is the scalar one, then \( 1v = v \).

9. **Scalar Distributive Law.** If \( \lambda, \mu \in F \) and \( v \in V \), then \( (\lambda + \mu)v = \lambda v + \mu v \).

10. **Vector Distributive Law.** If \( \lambda \in F \) and \( u, v \in V \), then \( \lambda(u + v) = \lambda u + \lambda v \).

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**Vector arithmetic**

Consider a vector space \( V \).

**Subtraction:** For \( u, v \in V \), there is a unique \( -v \) and so we can define \( u - v = u + (-v) \).

**Some properties:**

- **Uniqueness of 0.**
  Suppose 0 and \( 0' \) both have the zero property, then
  \[
  0 = 0 + 0' = 0' + 0 = 0'.
  \]

- **Cancellation property.**
  If \( u, v, w \in V \) and \( u + v = u + w \) then \( v = w \).
  This is true because \( -u \) exists and can be added to both sides. (The associative law is also needed here.)

- **Uniqueness of negatives.**
  This follows from the cancellation property because if \( u \) and \( v \) are both negatives of \( w \) then \( w + u = w + v = 0 \).
Vector arithmetic

Some more properties:

- **Multiplication by the zero scalar.** \( 0 \mathbf{v} = \mathbf{0}, \)

  \[ \mathbf{v} + \mathbf{0} = \mathbf{v} = 1\mathbf{v} = (1 + 0)\mathbf{v} = \mathbf{1v} + \mathbf{0v} = \mathbf{v} + \mathbf{0v}. \]

- **Multiplication of the zero vector.** \( \lambda \mathbf{0} = \mathbf{0}. \)
- **Multiplication by \(-1.\)** \((-1)\mathbf{v} = -\mathbf{v}\) (the additive inverse of \(\mathbf{v}\)).

  \[ \mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = 0\mathbf{v} = \mathbf{0}. \]

- **Zero products.** If \(\lambda \mathbf{v} = \mathbf{0},\) then either \(\lambda = 0\) or \(\mathbf{v} = \mathbf{0}.\)
- **Cancellation Property.** If \(\lambda \mathbf{v} = \mu \mathbf{v}\) and \(\mathbf{v} \neq \mathbf{0}\) then \(\lambda = \mu.\)

The vector space axioms lead to the usual vector arithmetic. If \(\mathbf{u}, \mathbf{v} \in V\) then

\[
2(3\mathbf{u} + 4\mathbf{v}) - 3\mathbf{v} = (2(3\mathbf{u}) + 2(4\mathbf{v})) - 3\mathbf{v} \\
= ((2 \times 3)\mathbf{u} + (2 \times 4)\mathbf{v}) - 3\mathbf{v} \\
= (6\mathbf{u} + 8\mathbf{v}) - 3\mathbf{v} \\
= 6\mathbf{u} + (8\mathbf{v} - 3\mathbf{v}) \\
= 6\mathbf{u} + (8\mathbf{v} + (-3)\mathbf{v}) \\
= 6\mathbf{u} + (8 + (-3))\mathbf{v} \\
= 6\mathbf{u} + 5\mathbf{v}
\]
Consider $V = \mathbb{C}^n$ with the usual coordinate wise addition rule and scalars $\mathbb{C}$. But define scalar multiplication by

$$\lambda(z_1, z_2, z_3, \ldots, z_n) = (\bar{\lambda}z_1, \bar{\lambda}z_2, \bar{\lambda}z_3, \ldots, \bar{\lambda}z_n).$$

Is this a vector space?