Stability of Dynamical Systems

A very important application of complex numbers in finance is in analysing dynamical systems. We will look at systems having discrete or continuous time.

**Example (discrete): interest on investment**

An initial investment of $100 earns interest at an annual rate of 12% compounded monthly. If $x_n$ represents its value in dollars after $n$ months, then

$$x_{n+1} = 1.01x_n, \quad x_0 = 100.$$ 

The equation for $x_{n+1}$ is called a recurrence relation (or difference equation) and $x_0 = 100$ is the initial condition. This can easily be 'solved' to give

$$x_n = 100(1.01)^n.$$
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Example (continuous): interest on investment

A good approximation for a very short compounding interval is instantaneous compounding.

Let $x(t)$ be the value after $t$ years of a $100 investment earning an annual interest rate of $r$, compounded every $\Delta t$ years. Then

$$x(t + \Delta t) = x(t) + r \Delta t x(t), \quad x(0) = 100.$$ 

So,

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = rx(t)$$

As $\Delta t \to 0$, this becomes

$$\frac{dx}{dt} = rx, \quad x(0) = 100.$$ 

This is a linear differential equation with constant coefficients and an initial condition $x(0) = 100$. The solution is

$$x(t) = 100e^{rt}$$

Difference equations

Consider the sequence $x_0, x_1, x_2, \ldots$ of numbers which 'evolve' according to the difference equation

$$x_n + a_1 x_{n-1} + \cdots + a_r x_{n-r} = 0, \quad a_1, \ldots, a_r \in \mathbb{C}.$$ 

Note that this defines $x_n$ in terms of the preceding $r$ terms

$$x_n = -a_1 x_{n-1} - \cdots - a_r x_{n-r}.$$ 

This is a linear and homogeneous Discrete Time System. More complicated DTSs are also possible.

Example: Consider the difference equation

$$x_n - 2x_{n-1} + x_{n-2} = 0$$

and show that $x_n = n$ and $x_n = c$ (a constant) are two possible solutions.

Specifying $x_0$ and $x_1$ determines all subsequent $x_i$. Eg, if $x_0 = 3$ and $x_1 = 5$ then

$$x_2 = 2x_1 - x_0 = 2 \times 5 - 3 = 7, \quad x_3 = 2x_2 - x_1 = 2 \times 7 - 5 = 9,$$

and the solution is

$$x_n = 2n + 3.$$
Difference equations

The characteristic equation of the DTS

\[ x_n + a_1 x_{n-1} + \cdots + a_r x_{n-r} = 0 \quad \text{is} \quad \lambda^r + a_1 \lambda^{r-1} + \cdots + a_r = 0. \]

If \( \alpha \in \mathbb{C} \) is a root of the characteristic equation with multiplicity \( m \), then for \( k = 0, 1, \ldots, m-1 \), and \( A \in \mathbb{C} \),

\[ x_n = A n^k \alpha^n \]

is a solution of the DTS. (The general solution is a sum of such solutions.)

Check for \( r = 2 \) and \( k = 0 \):

\[ x_n + a_1 x_{n-1} + a_2 x_{n-2} = A \alpha^n + a_1 A \alpha^{n-1} + a_2 A \alpha^{n-2} = A \alpha^{n-2} (\alpha^2 + a_1 \alpha + a_2) = 0. \]

Check for \( r = 2 \) and \( k = 1 \): First note that the characteristic equation is

\[(\lambda - \alpha)^2 = 0 \Rightarrow \lambda^2 - 2\alpha \lambda + \alpha^2 = 0 \Rightarrow a_1 - 2\alpha, \ a_0 = \alpha^2.\]

\[ x_n - 2\alpha x_{n-1} + \alpha^2 x_{n-2} = A n \alpha^n - 2\alpha A (n-1) \alpha^{n-1} + \alpha^2 A \alpha^{n-2} \]

\[ = A \alpha^{n-2} (n-2(n-1) + n + 2) = 0. \]

Stability of Discrete Time Systems

We say a solution to the DTS is

- **stable** if \( |x_n| \to 0 \) as \( n \to \infty \),
- **unstable** if \( |x_n| \to \infty \) as \( n \to \infty \).

We don’t define stability for other asymptotic behaviour.

If

\[ x_n = n^k \alpha^n \]

for some \( k \geq 0, \alpha \in \mathbb{C} \) then

\[ |x_n| = n^k |\alpha|^n \]

and so the solution \( x_n = n^k \alpha^n \) is

- **stable** if \( |\alpha| < 1 \),
- **unstable** if \( |\alpha| > 1 \).

The general solution to the DTS is a sum of such solutions and every solution is stable if all roots of the characteristic equation have modulus less than 1. If any root has modulus greater than 1 then there are unstable solutions.
Stability of Discrete Time Systems

**Example:** Are the solutions to the DTS

\[ 2x_n - x_{n-1} + x_{n-2} = 0 \]

stable? The characteristic equation is

\[ 2\lambda^2 - \lambda + 1 = 0 \]

which has roots

\[ \alpha_{\pm} = \frac{1 \pm \sqrt{1 - 8}}{4} = \frac{1}{4} \pm \frac{i\sqrt{7}}{4}. \]

Since

\[ |\alpha_{\pm}| = \left|\frac{1}{4} \sqrt{1 + 7}\right| = \frac{\sqrt{8}}{4} < 1 \]

all solutions are stable.

**Example:** The DTS

\[ x_n + x_{n-1} + x_{n-3} + 2x_{n-4} = 0 \]

has unstable solutions because the product of the roots of the characteristic equation is 2 and hence there must be a root \( \alpha \) with \( |\alpha| > 1 \).

What about roots with modulus 1? Consider a DTS with roots \( re^{i\theta} \) and \( re^{-i\theta} \), where \( r, \theta \in \mathbb{R} \).

\[
x_n = A\left(re^{i\theta}\right)^n + B\left(re^{-i\theta}\right)^n \quad A, B \in \mathbb{C}
\]

\[
= Ar^n e^{in\theta} + Br^n e^{-in\theta}
\]

\[
= r^n(A + B)\cos(n\theta) + ir^n(A - B)\sin(n\theta)
\]

\[
= r^nC \cos(n\theta) + r^nD \sin(n\theta)
\]

If the modulus of the roots is 1, that is \( r = 1 \), then we have

\[ x_n = C \cos(n\theta) + D \sin(n\theta) \]

where \( C = A + B \), \( D = i(A - B) \).

**Example:** \( x_n - \sqrt{2} x^{n-1} + x_{n-2} = 0 \).
Consider \( x(t) \) that evolves with time according to the \( r \)\(^{th}\) order linear homogeneous constant coefficient differential equation

\[
\frac{d^r x}{dt^r} + a_1 \frac{d^{r-1} x}{dt^{r-1}} + \cdots + a_r x = 0.
\]

In the calculus half of this course you see that this has solutions of the form

\[ x(t) = A t^k e^{\alpha t} \]

where \( \alpha \) is a root of the characteristic equation

\[
\lambda^r + a_1 \lambda^{r-1} + \cdots + a_r = 0
\]

with multiplicity \( m > k \) and \( A \in \mathbb{C} \) is a constant. The general solution of the differential equation is a sum of solutions of this form.

We say a solution to the CTS is

- **stable** if \( |x(t)| \to 0 \) as \( t \to \infty \),
- **unstable** if \( |x(t)| \to \infty \) as \( t \to \infty \).

We don’t define stability for other asymptotic behaviour.

If

\[ x(t) = t^k e^{\alpha t} \]

for some \( k \geq 0 \), \( \alpha = a + bi \) for \( a, b \in \mathbb{C} \) then

\[ |x(t)| = |t^k e^{\alpha t}| = |t|^k |e^{at+bi}| = |t|^k |e^{at}| \]

and so the solution \( x(t) = t^k e^{\alpha t} \) is

- **stable** if \( \text{Re}(\alpha) < 0 \),
- **unstable** if \( \text{Re}(\alpha) > 0 \).

The general solution to the CTS is a sum of such solutions and every solution is stable if all roots of the characteristic equation have real part less than 0. If any root has real part greater than 0 then there are unstable solutions.
Example: Show that

\[ \frac{d^4 x}{dt^4} - 2 \frac{d^3 x}{dt^3} + x = 0 \]

has unstable solutions.

The characteristic equation is

\[ \lambda^4 - 2\lambda^3 + 1 = 0 \]

The sum of the roots is 2, so there must be a root with positive real part.

One root of this equation is 1 which has positive real part.

The other roots are approximately 1.839286755, -0.4196433777 \pm 0.6062907300i.