MATH1251
Mathematics for Actuarial Studies and Finance
Chapter 9
Eigenvalues and Eigenvectors
Lecture 21

Dr. Jonathan Kress

School of Mathematics and Statistics
University of New South Wales

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Powers of matrices

Suppose that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear map with eigenbasis $\{\mathbf{v}_1, \mathbf{v}_2\}$ and corresponding eigenvalues $\lambda_1, \lambda_2$, that is

$$T(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1 \quad \text{and} \quad T(\mathbf{v}_2) = \lambda_2 \mathbf{v}_2$$
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It’s easy to apply $T$ repeatedly to its eigenvectors, ie

$$T^2(v_1) = T(T(v_1)) =$$
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A similar calculation for any $k \in \mathbb{Z}^+$ shows that

$$T^k(v_1) = \lambda_1^k v_1 \quad \text{and} \quad T^k(v_2) = \lambda_2^k v_2.$$
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If $T$ is represented by the matrix $A$ in the standard basis, then we also have have that

$$T^k(\mathbf{v}) = T^{k-1}(T(\mathbf{v})) = \underbrace{A \cdots A}_{k \text{ times}} \mathbf{v}.$$
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$$T^k(v) = T^{k-1}(T(v)) = T^{k-1}(Av) = \ldots$$
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$$T^k(v) = T^{k-1}(T(v)) = T^{k-1}(Av) = T^{k-2}(AAv) = \cdots = A^2 AA \cdots Av = A^k v$$

and

$$A^k v_1 = \lambda_1^k v_1 \quad \text{and} \quad A^k v_2 = \lambda_2^k v_2.$$
Powers of matrices

Note that multiplying diagonal matrices is easy:

\[
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\begin{pmatrix}
\mu_1 & 0 & \cdots & 0 \\
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\vdots & \vdots & \ddots & \vdots \\
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\end{pmatrix}
= 
\begin{pmatrix}
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The matrix representing \( T \) in its eigenbasis \( B = \{v_1, v_2\} \) is

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([T^k(v_1)]_B \ [T^k(v_2)]_B) =
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Powers of matrices

Suppose \( A \) is an \( n \times n \) matrix with eigenvectors \( \{v_1, v_2, \ldots, v_n\} \) that form a basis for \( \mathbb{R}^n \), then if

\[
M = (v_1 \ldots v_n) \quad \text{and} \quad D = \begin{pmatrix}
\lambda_1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \lambda_n
\end{pmatrix},
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the corresponding eigenvectors, \( M \) diagonalises \( A \), that is,

\[
M^{-1}AM = D \quad \text{or} \quad MDM^{-1} = A.
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Powers of matrices

Suppose $A$ is an $n \times n$ matrix with eigenvectors $\{v_1, v_2, \ldots, v_n\}$ that form a basis for $\mathbb{R}^n$, then if

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Then, for $k \in \mathbb{Z}^+$,

$$A^k = AAA \cdots AA$$
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This gives an easy way to find powers of diagonalisable matrices.
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Powers of matrices

Verify the diagonalisation

\[ A = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0.6 & 0 \\ 0 & 1.2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \]

and find \( A^k \).
Discrete time systems

Let

\[ x_1(k) = \text{defence expenditure of hobbits in year } k \]
\[ x_2(k) = \text{defence expenditure of orcs in year } k \]
Discrete time systems

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Suppose the populations are separate and year to year changes in expenditure are governed by the recursion relation

\[ x_1(k + 1) = 0.8x_1(k) \]
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These can be easily solved because they are uncoupled.

\[ x_1(k) = A(0.8)^k \quad \text{and} \quad x_2(k) = A(0.6)^k. \]
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These can be easily solved because they are uncoupled.

\[ x_1(k) = A(0.8)^k \quad \text{and} \quad x_2(k) = A(0.6)^k. \]

We can write this in matrix form

\[
\begin{pmatrix}
  x_1(k + 1) \\
  x_2(k + 1)
\end{pmatrix} = \begin{pmatrix}
  0.8 & 0 \\
  0 & 0.6
\end{pmatrix} \begin{pmatrix}
  x_1(k) \\
  x_2(k)
\end{pmatrix}
\]

What if the matrix is not diagonal, ie the recursion relations for \( x_1(k) \) and \( x_2(k) \) are coupled?
Discrete time systems

Let

\[
\mathbf{x}(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{pmatrix}
\]

and \( A \) be an \( n \times n \) matrix. Consider the recurrence relation

\[
\mathbf{x}(k + 1) = A\mathbf{x}(k).
\]

Note that

\[
\begin{align*}
\mathbf{x}(1) &= A\mathbf{x}(0) \\
\mathbf{x}(2) &= A\mathbf{x}(1) = A^2\mathbf{x}(0) \\
\mathbf{x}(3) &= A\mathbf{x}(2) = A^3\mathbf{x}(0) \\
\vdots \\
\mathbf{x}(k) &= A^k\mathbf{x}(0)
\end{align*}
\]

So, to write down the general solution we need an expression for \( A^k \).
Suppose $A$ is an $n \times n$ matrix with eigenvectors $\{v_1, v_2, \ldots, v_n\}$ that form a basis for $\mathbb{R}^n$, then with $\lambda_1, \lambda_2 \ldots, \lambda_n$ the corresponding eigenvectors. Then the solution to

$$x(k + 1) = Ax(k)$$

is

$$x(k) = \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \cdots + \alpha_n \lambda_n^k v_n$$

for arbitrary scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. 
Suppose the hobbit and orc defence expenditures are coupled and governed by the recursion relation

\[
x_1(k + 1) = 0.8x_1(k) + 0.4x_2(k)
\]
\[
x_2(k + 1) = 0.2x_1(k) + x_2(k)
\]
Suppose the hobbit and orc defence expenditures are coupled and governed by the recursion relation

\[
\begin{align*}
x_1(k+1) &= 0.8x_1(k) + 0.4x_2(k) \\
x_2(k+1) &= 0.2x_1(k) + x_2(k)
\end{align*}
\]

We can write this in matrix form

\[
\begin{pmatrix}
x_1(k+1) \\
x_2(k+1)
\end{pmatrix} =
\begin{pmatrix}
0.8 & 0.4 \\
0.2 & 1
\end{pmatrix}
\begin{pmatrix}
x_1(k) \\
x_2(k)
\end{pmatrix}
\]

Find the general solution to this recurrence relation and also the expenditure in the \(k\)th year if initially the hobbits spend 1 piece of gold and the orcs spend 0.