**Number systems**

The aim of this lecture is to introduce complex numbers. First we consider other number systems.

<table>
<thead>
<tr>
<th>Number system</th>
<th>Closed under</th>
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</thead>
<tbody>
<tr>
<td>Natural numbers ( \mathbb{N} ) = {0, 1, 2, 3, \ldots}</td>
<td>(+\times)</td>
</tr>
<tr>
<td>Integers ( \mathbb{Z} ) = {\ldots, -2, -1, 0, 1, 2, \ldots}</td>
<td>(+\ -\ \times)</td>
</tr>
<tr>
<td>Rational numbers ( \mathbb{Q} ) = {\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0}</td>
<td>(+\ -\ \times\ /)</td>
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<tr>
<td>Real numbers ( \mathbb{R} ) = {limits of convergence sequences in ( \mathbb{Q} }}</td>
<td>(+\ -\ \times\ /)</td>
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Which of the following equations can be solved in the number systems above?

- \(x + 5 = 7\)
- \(x + 7 = 5\)
- \(5x = 10\)
- \(5x + 20 = 10\)
- \(x^2 = 2\)
- \(x^2 + 5x + 6 = 0\)
- \(x^2 + 5x + 3 = 0\)
- \(x^2 + 1 = 0\)
- \(x^2 + 2x + 3 = 0\)
- \(x^3 + 7x^2 + 17x + 15 = 0\)

For the last 3 equations we need to extend \( \mathbb{R} \).
Complex numbers

$x^2 + 1 = 0$ has no real solutions, so let's 'make one up'. Call it the *imaginary unit* and denote it $i$. That is,

$$i^2 + 1 = 0 \iff i^2 = -1.$$

We will see later that this is all we need to solve all polynomial equations, but first we'll concentrate on arithmetic using this new number.

We want a number system that contains both $\mathbb{R}$ and $i$ with many of the properties of real numbers (so we can still solve all of the equations on the previous slide).

For us, complex numbers are the set

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, \ i^2 = -1\}$$

along with rules for $+$ and $\times$.

The expression $a + bi$ is called the *Cartesian form* of a complex number.

Complex arithmetic

Consider two complex numbers in Cartesian form,

$$z = a + bi, \quad w = c + di,$$

where $a, b, c, d \in \mathbb{R}$. To add and multiply $z$ and $w$, treat them like real polynomials in the variable $i$ but always replace $i^2$ with $-1$. So,

$$z + w = a + bi + c + di = (a + c) + (b + d)i,$$

$$zw = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

It's easy to see that with these rules, $\mathbb{C}$ is closed under addition and multiplication. That is, the sum and the product of two complex numbers are also complex numbers.

*Eg* \[
(1 + 2i) + (3 + 5i) = 4 + 7i \\
(1 + 2i)(3 + 5i) = 1 \times 3 + 1 \times 5i + 2i \times 3 + 2i \times 5i \\
= 3 + 5i + 6i + 10i^2 \\
= 3 + 5i + 6i + 10 \times (-1) = -7 + 11i
\]
Laws of arithmetic

It’s also easy to check that just like real numbers, $x, y, z \in \mathbb{C}$ obey the usual associative, commutative and distributive laws.

**Associative Laws:**

$$(x + y) + z = x + (y + z), \quad (xy)z = x(yz)$$

**Commutative Laws:**

$$x + y = y + x, \quad xy = yx$$

**Distributive Law:**

$$x(y + z) = xy + xz$$

(Beyond this course: The quaternions are not commutative, and the octonions are not commutative nor associative. See Wikipedia if you’re interested.)

Subtraction in $\mathbb{C}$

$\mathbb{C}$ has a zero (additive identity):

$$(0 + 0i) + z = (0 + 0i) + a + bi = (0 + a) + (0 + b)i = a + bi = z$$

We usually just write 0 for $0 + 0i$.

For each $z$ there is an additive inverse $-z = (-a) + (-b)i$ with the property that

$$z + (-z) = 0.$$ 

This allows us to define subtraction by

$$z - w = z + (-w)$$

which leads to exactly what you’d expect.

$$(2 + 3i) - (5 + 2i) = (2 + 3i) + ((-5) + (-2)i) = (2 - 5) + (3 - 2)i = -3 + i.$$
Division in $\mathbb{C}$

$\mathbb{C}$ has a one (multiplicative identity):

$$(1 + 0i)z = (1 + 0i)(a + bi) = 1(a + bi) + 0i(a + bi) = a + bi = z$$

We usually just write 1 for $1 + 0i$. For each $z \neq 0$ there is a multiplicative inverse

$$\frac{1}{z} = z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

with the property that

$$z z^{-1} = 1.$$ 

This allows us to define division by

$$\frac{z}{w} = zw^{-1},$$

that is,

$$\frac{a + bi}{c + di} = (a + bi) \left( \frac{c}{c^2 + d^2} - \frac{d}{c^2 + d^2}i \right) = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

The properties of $\mathbb{C}$ we've discussed make $\mathbb{C}$ a field. See definition 1 on page 3 of the printed notes.

Real and imaginary parts

For

$$z = a + bi \in \mathbb{C}, \quad a, b \in \mathbb{R},$$

The real part of $z$ is

$$\text{Re}(z) = a$$

and the imaginary part of $z$ is

$$\text{Im}(z) = b.$$ 

A complex number of the form $bi$ is called an imaginary number. If $\text{Re}(z) = 0$ we say $z$ is purely imaginary and if $\text{Im}(z) = 0$ we say $z$ is real.

**Eg** \quad $\text{Re}(3 - 4i) = 3, \quad \text{Im}(3 - 4i) = -4.$

Two complex numbers are equal if and only if their real and imaginary parts are equal.

Note that the imaginary part of a complex number is a real number.
Complex conjugation

For a complex number

\[ z = a + bi, \]

where \( a, b \in \mathbb{R} \), the complex conjugate of \( z \) is

\[ \bar{z} = a - bi. \]

**Exercise:** Verify the following properties by writing \( z = a + bi \) and \( w = c + di \).

\[
\begin{align*}
z\bar{z} &= a^2 + b^2 \\
\text{Re}(z) &= \frac{1}{2}(z + \bar{z}) \\
\text{Im}(z) &= \frac{1}{2i}(z - \bar{z}) \\
\bar{\bar{z}} &= z
\end{align*}
\]

Note that a complex number is real if \( z = \bar{z} \). For example, show that \( u = \bar{z}w + z\bar{w} \) is real.

\[
\bar{u} = \overline{\bar{z}w + z\bar{w}} = \bar{z}w + z\bar{w} = \bar{w}\bar{z} + \bar{z}w = z\bar{w} + \bar{z}w = \bar{z}w + z\bar{w} = u
\]

Division using complex conjugation

Complex conjugation provides an easy way to divide complex numbers without remembering the complicated formula for \( 1/z \).

The trick is to multiply the numerator and denominator of a quotient by the complex conjugate of the denominator.

\[
\frac{1 + 2i}{3 + 4i} = \frac{1 + 2i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i} = \frac{(1 + 2i)(3 - 4i)}{(3 + 4i)(3 - 4i)} = \frac{3 - 4i + 6i - 8i^2}{9 - 16i^2} = \frac{11 + 2i}{25}
\]

\[
= \frac{11}{25} + \frac{2}{25}i
\]

A useful fact to remember...

\[
\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-i^2} = \frac{-i}{1} = -i.
\]
\( \mathbb{C} \) is not ordered

So far we have seen that many properties of \( \mathbb{R} \) also apply to \( \mathbb{C} \).

An important exception is that \( \mathbb{R} \) is ordered whereas \( \mathbb{C} \) can not be.

For any pair of positive real numbers \( x \) and \( y \) one of the following is true.

\[
x < y, \quad y < x \quad \text{or} \quad x = y.
\]

Furthermore, the order of two positive real numbers is preserved when adding or multiplying by another positive real number.

It is not possible to extended or redefine \( < \) so that something like this is also true for \( \mathbb{C} \).

**Inequalities don’t make sense in \( \mathbb{C} \).**

Of course you can still compare real quantities made from a complex numbers, eg it makes sense to say things like \( \text{Re}(z) > \text{Re}(w) \).

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**The Argand plane**

We can represent complex numbers as points in \( \mathbb{R}^2 \) by plotting the real part on the horizontal axis and the imaginary part on the vertical axes. It is common to write \( z = x + iy \) where \( x, y \in \mathbb{R} \).
Modulus and argument

We can also locate points in the plane using polar coordinates.

The distance \( r \) from 0 to \( z \) is called the modulus of \( z \).

\[
\|z\| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.
\]

The angle from the positive real axis to \( z \) measured anti-clockwise is called the argument of \( z \) and written \( \text{arg}(z) \). This has infinitely many possible values for each \( z \). By convention, the principal argument of \( z \) is in \( (-\pi, \pi] \).

\[
\text{Arg}(z) \in (-\pi, \pi].
\]

Note that
- \( \text{Arg}(z) = -\text{Arg}(\overline{z}) \) unless \( \text{Arg}(z) = \pi \)
- \( |\overline{z}| = |z| \)

Polar form

If \( r = |z| \) and \( \theta = \text{Arg}(z) \), then

\[
z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)
\]

This is called the polar form of \( z \).

Eg Write \( z = 1 + i\sqrt{3} \) in polar form.

\[
|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2.
\]

\( z \) is in the first quadrant, so

\[
\text{Arg}(z) = \tan^{-1} \left( \frac{\sqrt{3}}{1} \right) = \frac{\pi}{3}.
\]

So the polar form of \( z \) is

\[
z = 2 \left( \cos \left( \frac{\pi}{3} \right) + i \sin \left( \frac{\pi}{3} \right) \right).
\]
To enter a complex number in MATLAB, use $i$ or $j$ for $i$.

If $z$ is a complex number, then $\text{real}(z)$, $\text{imag}(z)$, $\text{abs}(z)$ and $\text{angle}(z)$ are the real part, the imaginary part, the modulus and the principal argument of $z$. 