Eigenvectors and eigenvalues

Find the eigenvectors and eigenvalues of

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \]

\[ A = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix} \]

\[ A = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix} \]
Eigenbases and diagonalisation

For $n = 2$ and $n = 3$ we have seen examples of $n \times n$ matrices with enough linearly independent eigenvectors to form a basis for $\mathbb{R}^n$.

We have also seen an example where this does not work.

In MATH1251 we only study the case with $n$ linearly independent eigenvectors.\(^1\)

The eigenvalues of an $n \times n$ matrix are the roots of its characteristic polynomial. Since this is an $n^{th}$ order polynomial, if we allow the eigenvalues to be complex and take account of the multiplicities, then there will always be $n$ eigenvalues.

An eigenvalue corresponding to a repeated root may or may not have an eigenspace of dimension greater than 1.

One case where we can be sure to have a basis of eigenvectors for $\mathbb{R}^n$ is when $A$ has $n$ distinct eigenvalues. We will prove this by induction on the next slide.

\(^1\)‘Jordan decomposition’ is a more a more general theory that covers all possible cases.

Distinct eigenvalues

A set containing a single eigenvector of $A$ is linearly independent.

Assume that any set of $k$ eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $A$ corresponding to $k$ distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ is linearly independent.

Consider a set of $k + 1$ eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ of $A$ corresponding to $k + 1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}$. Suppose they are linearly dependent and so there are scalars $\alpha_1, \ldots, \alpha_k, \alpha_{k+1}$, not all zero, such that

$$0 = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k + \alpha_{k+1} \mathbf{v}_{k+1}.$$  

Multiply this equation on the left by $A - \lambda_{k+1} \mathbf{I}$.

$$0 = (A - \lambda_{k+1} \mathbf{I})(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k + \alpha_{k+1} \mathbf{v}_{k+1})$$

$$= \alpha_1 (\lambda_1 - \lambda_{k+1}) \mathbf{v}_1 + \cdots + \alpha_k (\lambda_k - \lambda_{k+1}) \mathbf{v}_k + \alpha_{k+1} (\lambda_{k+1} - \lambda_{k+1}) \mathbf{v}_{k+1}$$

$$= \alpha_1 (\lambda_1 - \lambda_{k+1}) \mathbf{v}_1 + \cdots + \alpha_k (\lambda_k - \lambda_{k+1}) \mathbf{v}_k.$$

This contradicts the assumption because it is a non-trivial linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that is zero. Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ must be linearly independent.

We have shown by induction that any set of eigenvectors corresponding to distinct eigenvalues must be linearly independent.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix $A$. Suppose $T$ (and $A$) has $n$ linearly independent eigenvectors $\{v_1, \ldots, v_n\}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ and so

$$B = \{v_1, \ldots, v_n\}$$

is a basis for $\mathbb{R}^n$.

The generalised matrix representation theorem says that the matrix of $T$ with respect to the basis $B$ is

$$D = \begin{pmatrix} [T(v_1)]_B & [T(v_2)]_B & \cdots & [T(v_n)]_B \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

With respect to a basis of eigenvectors, $T$ is represented by a diagonal matrix.

### Diagonalisation of $A$

Suppose $B = \{v_1, \ldots, v_n\}$ is a set of linearly independent eigenvectors of $T : \mathbb{R}^n \to \mathbb{R}^n$ which has matrix $A$ with respect to the standard basis. Define

$$M = (v_1 \ v_2 \ \cdots \ v_n) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Note that $M$ is the matrix of the transformation that maps a coordinate vector with respect to $B$ to a coordinate vector with respect to the standard basis. That is, for $v \in \mathbb{R}^n$,

$$v = M[v]_B \quad \text{and} \quad [v]_B = M^{-1}v.$$

Since $T$ has matrix $D$ with respect to $B$, we must have

$$[T(v)]_B = D[v]_B \Rightarrow M^{-1}T(v) = DM^{-1}v \Rightarrow T(v) = MDM^{-1}v$$

and since the matrix of $T$ in the standard basis is $A$, we have

$$A = MDM^{-1}.$$
Diagonalisation of $A$

We say the $n \times n$ matrix $A$ is **diagonalisable** if and only if there exists an invertible matrix $M$ and diagonal matrix $D$ such that

$$A = MDM^{-1}.$$ 

On the previous slide we have seen that we given an eigenvector basis $B = \{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, we can achieve this with

$$M = (v_1 \ v_2 \ \ldots \ v_n) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$ 

In fact this is a necessary condition. If such an $M$ and $D$ exist then the columns of $M$ are eigenvectors of $A$ and the diagonal entries of $D$ are their corresponding eigenvalues.

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Diagonalisation

Where possible, diagonalise the following matrices.

- $A = \begin{pmatrix} 19 & 6 \\ 18 & 16 \end{pmatrix}$

- $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

- $A = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix}$

- $A = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$
Real symmetric matrices

A square matrix $A$ is symmetric if $A = A^T$.

Let $A$ be a real $n \times n$ symmetric matrix. Then

- the eigenvalues of $A$ are all real.
- there is an orthonormal basis for $\mathbb{R}^n$ composed of eigenvectors of $A$.

Example

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
= 0
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
= 4
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\] and \[
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\]
are orthogonal and have length $\sqrt{2}$, so \[
\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}
\]
is an orthonormal basis for $\mathbb{R}^2$. Then \[
M = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\]
is an orthogonal matrix and \[
M^{-1} = M^T.
\]
We have the orthogonal diagonalisation:

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}
\end{pmatrix}
\]