Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations with matrices $A$ and $B$ and $\lambda$ is a scalar. Define $T + S$ and $\lambda T$ by

$$(T + S)(x) = T(x) + S(x) \quad \text{and} \quad (\lambda T)(x) = \lambda T(x).$$

**Exercise**

- Show that $T + S$ is a linear transformation.
- Show that $\lambda T$ is a linear transformation.

So linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are a vector space!

**Exercise**

- Show that the matrix of $T + S$ is $A + B$.
- Show that the matrix of $\lambda T$ is $\lambda A$. 
Composition and invertible linear transformations

If \( T : \mathbb{R}^n \to \mathbb{R}^m \) and \( S : \mathbb{R}^p \to \mathbb{R}^n \) are linear with matrices \( A \) and \( B \), then \( T \) composed with \( S \) is defined by

\[
(T \circ S)(x) = T(S(x)).
\]

Exercise

- Show that \( T \circ S \) is a linear transformation with matrix \( AB \).

So the algebra of linear transformations is the algebra of matrices.

Exercise:

Suppose that \( T : \mathbb{R}^n \to \mathbb{R}^n \) is an invertible linear transformation with matrix \( A \).

- Show that \( T^{-1} \) is a linear transformation with matrix \( T^{-1} \) is \( A^{-1} \).

Example

Let \( T_{\theta} \) be the linear transformation that rotates \( \mathbb{R}^2 \) through an angle \( \theta \) about \( O \).

\[
R_{\theta} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{pmatrix}.
\]

Find the matrix of the inverse of \( T_{\theta} \), that is, the matrix of \( T_{\theta}^{-1} \).

Algebra of linear transformations

The previous slides discussed linear maps between \( \mathbb{R}^n \) and \( \mathbb{R}^m \). The same applies for linear maps between more general vector spaces.

Check that the following maps on \( C^\infty(\mathbb{R}) \) are linear:

- multiplication by a fixed \( C^\infty(\mathbb{R}) \) function,
- differentiation.

Explain why the following maps on \( C^\infty(\mathbb{R}) \) are linear:

- \( T(f) = f'' \),
- \( T(f) = -f \),
- \( T(f) = g \), where \( g(x) = e^x f''(x) - (x - 1)f'(x) + 2f(x) \).

These maps can be constructed by composition and linear combination from known linear maps, i.e., differentiation and multipication by a function.

We say

\[
e^x f''(x) - (x - 1)f'(x) + 2f(x) = x
\]

is a linear equation because it is of the form \( T(v) = b \) for a linear map \( T \). If the right hand side were 0, we would call it a linear homogeneous equation.

\(^1\)Real functions that infinitely differentiable.
Generalised matrix representation theorem

To be able to relate the algebra of linear maps between vector spaces in general to the algebra of matrices, we need the **generalised matrix representation theorem**.

Let
- \( T : V \rightarrow W \) be a linear map,
- \( B_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \) be an ordered basis for \( V \),
- \( B_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\} \) be an ordered basis for \( W \),
- \( A = (a_1 \ a_2 \ldots a_n) \) where \( a_i = [T(\mathbf{v}_i)]_{B_W} \).

Then for any \( \mathbf{v} \in V \),

\[
    [T(\mathbf{v})]_{B_W} = A[\mathbf{v}]_{B_V}.
\]

We say \( A \) represents \( T \) with respect to \( B_V \) and \( B_W \).

Now all the previous statements can be extended to linear transformations between general vector spaces.

Generalised matrix representation theorem

Consider a linear map \( T : \mathbb{P}_3(\mathbb{R}) \rightarrow \mathbb{P}_2(\mathbb{R}) \) defined by

\[
    T(p(x)) = (x + 1)p''(x) - p'(x).
\]

Find \( \ker(T) \).

Find the matrix \( A \) of \( T \) with respect to the ordered bases

\[
    B_3 = \{1, x, x^2, x^3\} \quad \text{and} \quad B_2 = \{1, x, x^2\}.
\]

Consider the following linear maps from \( \mathbb{P}(\mathbb{R}) \) to \( \mathbb{P}(\mathbb{R}) \)
- \( S_1(p(x)) = p'(x) \), ie differentiation,
- \( S_2(p(x)) = (x + 1)p(x) \), ie multiplication by \( x + 1 \).

Find the matrices \( A_1 \) of \( S_1 \) and \( A_2 \) of \( S_2 \) with respect to the basis \( \{1, x, x^2, x^3, x^4, \ldots\} \). Note that \( A_1 \) and \( A_2 \) are infinite in size!

If \( S : \mathbb{P}(\mathbb{R}) \rightarrow \mathbb{P}(\mathbb{R}) \) has the same definition as \( T \) but applied to polynomials of arbitrary degree, use \( A_1 \) and \( A_2 \) to construct the matrix of \( S \) and show how this is related to \( A \).
Find $\ker(A)$.

How does this relate to $\ker(T)$?

Solve the homogeneous linear equation

$$(x + 1)p''(x) - p'(x) = 0$$

and the inhomogeneous linear equation

$$(x + 1)p''(x) - p'(x) = x.$$