Kernels and images

The zeros of a real valued function $f : \mathbb{R} \to \mathbb{R}$ are those values in the domain that solve the equation $f(x) = 0$.

$$\text{zeros}(f) = \{x \in \mathbb{R} : f(x) = 0\}.$$  

The range of a real valued function $f : \mathbb{R} \to \mathbb{R}$ is the set of values in the codomain that are the outputs of $f$. That is,

$$\text{range}(f) = \{y \in \mathbb{R} : y = f(x) \text{ for some } x \in \mathbb{R}\}.$$  

For linear transformations, the set of zeros is called the kernel and the range is called the image.

Each linear transformation has an associated matrix and we will use the terms kernel and image for the matrix as well.
Kernels and images

Kernels and images play an important role in understanding the existence and uniqueness of solutions to linear equations.

A linear equation (or system of linear equations) in \( x \) is an equation of the form

\[
T(x) = b,
\]

where \( T : V \to W \) is a linear transformation and \( b \) is fixed.

The image (or range) of \( T \) is the set of vectors \( b \) for which equation (1) has a solution.

The kernel of \( T \) is the set of solutions to the corresponding homogeneous equation, i.e.

\[
T(x) = 0.
\]

The kernel of a linear transformation or matrix

For a linear transformation \( T : V \to W \), the kernel or null space of \( T \) is the set

\[
\ker(T) = \{x \in V : T(x) = 0\}.
\]

Note that \( \ker(T) \) is a subspace of \( V \) because:

- \( T(0) = 0 \) and so \( 0 \in \ker(T) \).
- If \( x, y \in \ker(T) \) and \( \lambda \) is a scalar,

\[
T(x + y) = T(x) + T(y) = 0 + 0 = 0, \quad T(\lambda x) = \lambda T(x) = \lambda 0 = 0
\]

and so \( x + y \in \ker(T) \) and \( \lambda x \in \ker(T) \).

[The converse is also true — every subspace is the kernel of a linear map.]

Examples

- Find the kernel of \( T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}) \) defined by \( T(p(x)) = p''(x) \).
- For some \( v \in \mathbb{R}^3 \) let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the orthogonal projection onto \( \text{span}(v) \).
  Give a geometric description of \( \ker(T) \).
The kernel of a linear transformation or matrix

For an $m \times n$ matrix $A$ the kernel or null space of $A$ is the set

$$\ker(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}.$$

If $T(x) = Ax$ then $\ker(T) = \ker(A)$.

The dimension of the $\ker(T)$ or $\ker(A)$ is called the nullity of $T$ or $A$. That is

$$\text{nullity}(T) = \dim(\ker(T)), \quad \text{nullity}(A) = \dim(\ker(A)),$$

**Example**

Find a basis for $\ker(A)$ where

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 3 \end{pmatrix}$$

and hence find the nullity of $A$.

**Inhomogeneous linear equations**

Find the general solution to

$$Ax = b$$

where

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Notice that the solution has the form

$$x = x_h + x_p$$

where $x_h$ is the general solution to the corresponding homogeneous equation

$$Ax = 0,$$

that is, $x_h \in \ker(A)$, and $x_p$ is a particular solution. This is true in general.

**Exercise:** If $x_1$ and $x_2$ are two solutions to $Ax = b$ then $x_1 - x_2$ is a solution to the corresponding homogeneous equation.
The image of a linear transformation

For a linear transformation $T : V \to W$, the image or range of $T$ is the set

$$\text{im}(T) = T(V) = \{ w : w = T(v) \text{ for some } v \in V \}.$$ 

Note that $\text{im}(T)$ is a subspace of $W$ because:

- $0 \in V$ and $T(0) = 0$ so $0 \in \text{im}(T)$.
- Let $\lambda$ be a scalar and suppose $w_1, w_2 \in \text{im}(T)$ so there are $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Since $V$ is a vector space, $v_1 + v_2 \in V$ and $T$ is linear so

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2,$$

and so $w_1 + w_2 \in \text{im}(T)$ and $\lambda w_1 \in \text{im}(T)$.

Examples

- Find the image of $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined by $T(p(x)) = p''(x)$.
- For some $v \in \mathbb{R}^3$ let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto $\text{span}(v)$. Give a geometric description of $\text{im}(T)$.

The image of a linear transformation or matrix

For an $m \times n$ matrix $A$ the image or range of $A$ is the set

$$\text{im}(A) = \{ y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n \}.$$ 

$Ax$ is a linear combination of the columns of $A$ so $\text{im}(A) = \text{col}(A)$, the column space of $A$.

If $T(x) = Ax$ then $\text{im}(T) = \text{im}(A)$.

The dimension of the $\text{im}(T)$ or $\text{im}(A)$ is called the rank of $T$ or $A$. That is

$$\text{rank}(T) = \dim(\text{im}(T)), \quad \text{rank}(A) = \dim(\text{im}(A)).$$

Example

Find a basis for $\text{im}(A)$ where

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 3 \end{pmatrix}$$

and hence find the rank of $A$. 

JM Kress (UNSW Maths & Stats)  MATH1251 Complex Numbers  Semester 2 2010 7 / 10
**Rank-Nullity Theorem**

In practice, to find a basis for the kernel and image of a matrix $A$ we row reduce $A$ to row echelon form $U$.

- To find the kernel of $A$ we solve $Ax = 0$. Each non-leading column of $U$ gives a parameter in the general solution and hence a basis vector for $\text{ker}(A)$. So,

  \[ \text{nullity}(A) = \text{number of non-leading columns in } U. \]

- A basis for $\text{im}(A) = \text{col}(A)$ can be found by selecting columns of $A$ corresponding to leading columns of $U$. So,

  \[ \text{rank}(A) = \text{number of leading columns of } U. \]

For an $m \times n$ matrix $A$, the **Rank-Nullity Theorem** states that

\[ n = \text{rank}(A) + \text{nullity}(A). \]

For a linear transformation $T : V \rightarrow W$, the **Rank-Nullity Theorem** states that

\[ \dim(V) = \text{rank}(T) + \text{nullity}(T). \]

**Examples**

Verify the Rank-Nullity Theorem for the previous examples.

- $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T(p(x)) = p''(x)$.
- For some $\mathbf{v} \in \mathbb{R}^3$ let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto $\text{span}(\mathbf{v})$.
- $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 3 \end{pmatrix}$.