Some irreducible unitary representations of $G(K)$

for a simple algebraic group $G$

over a number field $K$

by

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Abstract. Let $K$ be an algebraic number field, and let $G_K$ be the group of $K$-rational points of a simply connected, simple linear algebraic group $G$ defined over $K$. We construct a new family of irreducible unitary representations of $G_K$, as follows. It is well known that $G_K$ embeds diagonally as a lattice in $G_A$, where $A$ is the ring of adèles of $K$. Let $\pi$ be an irreducible unitary representation of $G_A$. We show that $\pi|_{G_K}$, the restriction of $\pi$ to $G_K$, is irreducible and that $\pi$ is determined by $\pi|_{G_K}$ up to unitary equivalence. Many of these restrictions are not in the support of the regular representation of $G_K$.

1. Introduction and statement of the results

Let $G$ be a simple linear algebraic group defined over $K$. Let $G_K$ be the group of $K$-points in $G$, equipped with the discrete topology (this is the only locally compact topology on $G_K$). Since $G_K$ is not type I (see [Tho]), there is no hope for a classification of its irreducible unitary representations. The aim of the paper is to show that some natural unitary representations of $G_K$ are irreducible. All the representations we consider are assumed to be strongly continuous and unitary; equivalence of representations, without further qualification, means unitary equivalence.

Denote by $V$ the set of all places of $K$, that is, the set of all equivalence classes of valuations of $K$; $V$ is the union of the set $V_f$ of the finite places (or non-archimedean
valuations), and of the finite set $V_\infty$ of the finite places (or archimedean valuations). A statement which holds for all but finitely many places is said to hold for “almost all places”.

For any valuation $v$, let $K_v$ denote the corresponding completion of $K$. Let $\mathcal{O}$ be the ring of integers of $K$. For any non-archimedean valuation $v$, let $\mathcal{O}_v = \{x \in K_v : v(x) \leq 1\}$, the closure of $\mathcal{O}$ in $K_v$. (For instance, if $K = \mathbb{Q}$ and $v$ is the valuation given by a prime $p$, then $K_v$ is the field of $p$-adic numbers and $\mathcal{O}_v$ is the ring of $p$-adic integers.) The group $G_{K_v}$ of the $K_v$-points, equipped with the topology induced by $K_v$, is a simple $p$-adic Lie group when $v$ is finite and is a simple real Lie group when $v$ is infinite. Let $\pi$ be an irreducible representation of $G_{K_v}$ for some place $v$. Since $G_K$ is dense in $G_{K_v}$ and $\pi$ is strongly continuous, it is obvious that the restriction $\pi|_{G_K}$ of $\pi$ to $G_K$ is irreducible. It is equally clear that $\pi$ is determined by this restriction, that is, if $\pi$ and $\pi'$ are representations of $G_{K_v}$ such that $\pi|_{G_K}$ and $\pi'|_{G_K}$ are equivalent, then $\pi$ and $\pi'$ are equivalent. This may be generalised as follows.

Take finitely many distinct places $v_1, \ldots, v_n$ of $K$, and let $\pi_1, \ldots, \pi_n$ be irreducible representations of $G(K_{v_1}), \ldots, G(K_{v_n})$. Then the tensor product $\pi_1 \otimes \cdots \otimes \pi_n$ is an irreducible representation of $G(K_{v_1}) \times \cdots \times G(K_{v_n})$. The group $G_K$ embeds diagonally as a dense subgroup of $G(K_{v_1}) \times \cdots \times G(K_{v_n})$, by the Weak Approximation Theorem (see [PlR], Proposition 7.11). Hence, the (inner) tensor product

$$(\pi_1 \otimes \cdots \otimes \pi_n)|_{G_K}$$

is an irreducible representation of $G_K$ and determines $\pi_1 \otimes \cdots \otimes \pi_n$.

A setting for unification and generalisation of these facts is given by the embedding of $G_K$ in $G_A$, where $A$ is the ring of adèles of $K$. Recall that the adèle group $G_A$ is defined to be

$$\{(g_v)_{v \in V} \in \prod_{v \in V} G_{K_v} : g_v \in G_{\mathcal{O}_v} \text{ for almost all finite places } v\}.$$ 

The subgroup $G_{\mathcal{O}_v}$ of $G_{K_v}$ is compact and open, and $G_A$ is a locally compact group when endowed with the restricted product topology, for which the subgroup

$$\prod_{v \in V_\infty} G(K_v) \times \prod_{v \in V_f} G_{\mathcal{O}_v},$$
with its product topology, is open. By diagonal embedding, we identify $G_K$ with a subgroup of $G_A$, which is no longer dense; indeed, it is discrete. Moreover, under our assumptions on $G$, it has finite covolume (see [Bor], 5.6 Theorem); that is, $G_K$ is a lattice in $G_A$. For almost all valuations $v$, $G_K_v$ is not compact (see [Spr], 4.9 Lemma). These facts will play a crucial role in this paper.

For any place $v$, we identify $G_{K_v}$ with the subgroup

$$\{e\} \times \cdots \times \{e\} \times G_{K_v} \times \{e\} \times \cdots$$

of $G_A$. Since all $G_{K_v}$ are of type I (see [Ber]), the irreducible representations $\pi$ of $G_A$ are the restricted infinite tensor products of the form $\otimes_{v \in V} \pi_v$, where $\pi_v$ is an irreducible representation of $G_{K_v}$ which, for almost all finite valuations $v$, is spherical, i.e., the Hilbert space of $\pi_v$ contains a non-zero $G_{O_v}$-invariant vector. Observe that for any irreducible representation of $G_{K_v}$, there exists at most one such vector (up to scalar multiples), since $(G_{K_v}, G_{O_v})$ is a Gel’fand pair. In the rest of this paper, whenever we have an irreducible representation $\pi$ of $G_A$ we always write $\pi_v$ for its local component at $v$. Two irreducible representations $\pi$ and $\pi'$ are equivalent if and only if $\pi_v$ and $\pi'_v$ are equivalent for every valuation $v$. For all of this, see [GGPS], Chap. 3, §3, No 3; see also [Gui] and [Tad].

**Theorem A.** Let $G$ be a simply connected connected, simple linear algebraic group defined over $K$. Let $\pi$ and $\pi'$ be inequivalent irreducible representations of $G_A$. Then

(i) the restriction $\pi|_{G_K}$ of $\pi$ to $G_K$ is irreducible

(ii) the restrictions $\pi|_{G_K}$ and $\pi'|_{G_K}$ are inequivalent.

This theorem and its proof are in the spirit of [CoS] where similar facts are shown to be true for restrictions of representations of real semisimple Lie groups to lattices. Denote by $\hat{G}$ the unitary dual of a group $G$, that is, the set of (unitary) equivalence classes of its irreducible (unitary) representations. Then our result says that $\hat{G}_K$ contains a copy of the much better known space $\hat{G}_A$.

Our theorem applies, for example, to $SL_n$ and $Sp_n$ and yields irreducible representations of $SL_n(\mathbb{Q})$ and $Sp_n(\mathbb{Q})$ for $n \geq 2$.

Part (ii) in the above theorem may fail if $G$ is not simply connected. Indeed, for $G = PGL_2$ and $K = \mathbb{Q}$, it is easy to see that there exists a non-trivial unitary character
of $PGL_2(A)$ which is trivial on $PGL_2(Q)$. We do not know whether the irreducibility result (i) above remains true when $G$ is not simply connected.

It is perhaps surprising that the restrictions $\pi|_{G_K}$ are not contained in the support of the left regular representation $\lambda_{G_K}$ of $G_K$ for most irreducible representations $\pi$ of $G_A$. In particular, they are not equivalent to representations obtained by internal constructions, such as induction from amenable parabolic subgroups (see [HoR] for the case $GL_n(Q)$). More precisely, the following is true and easy to prove.

**Theorem B.** Let $G$ be a semisimple linear algebraic group defined over $K$, and let $\pi$ be an irreducible representation of $G_A$. The following are equivalent

(i) $\pi|_{G_K}$ is weakly contained in $\lambda_{G_K}$
(ii) $\pi_v$ is weakly contained in $\lambda_{G_{K_v}}$, for every place $v$
(iii) $\pi$ is weakly contained in $\lambda_{G_A}$.

In particular, if $\pi$ is any representation of $G_{K_v}$ for some place $v$, then $\pi|_{G_K}$ is not weakly contained in $\lambda_{G_K}$.

Recall that a unitary representation $\pi$ of a locally compact group $G$ is said to be weakly contained in another unitary representation $\rho$ if any diagonal matrix coefficient of $\pi$ is the limit, uniformly on compacta, of sums of diagonal matrix coefficients of $\rho$ (see [Dix], Chap. 18). The support of $\rho$ is the set of all irreducible representations which are weakly contained in $\rho$. The representations $\pi$ and $\rho$ are weakly equivalent if each is weakly contained in the other.

The question whether $\pi|_{G_K}$ and $\pi'|_{G_K}$ might be weakly equivalent for inequivalent irreducible representations $\pi$ and $\pi'$ is more difficult, and we do not have a complete answer. Our result uses Kazhdan’s Property (T); for an account of this, see [HaV].

**Theorem C.** Let $G$ be a simply connected, simple linear algebraic group, defined over $K$, and assume that $G_{K_v}$ has Kazhdan’s Property (T) for all $v \in V$. Let $\pi$ and $\pi'$ be inequivalent irreducible representations of $G_A$, not both weakly contained in the regular representation $\lambda_{G_A}$. Then $\pi|_{G_K}$ and $\pi'|_{G_K}$ are not weakly equivalent.

The assumption that $\pi$ and $\pi'$ are not both weakly contained in the regular representation is necessary for the following reason. Suppose, for simplicity, that $G$ has a trivial
centre. It was shown in [BCH] that the reduced $C^*$-algebra of $G_K$ is simple. Equivalently, any representation of $G_K$ which is weakly contained in $\lambda_{G_K}$ is weakly equivalent to $\lambda_{G_K}$. In particular, $\pi|_{G_K}$ and $\pi'|_{G_K}$ are always weakly equivalent if $\pi$ and $\pi'$ are weakly contained in $\lambda_{G_A}$.

Our results extend to $S$-arithmetic groups. Let $S$ be any set of valuations containing the set $V_\infty$. Let $O_S$ be the subring of $K$ consisting of all $x \in K$ for which $v(x) \leq 1$ for all $v \in V\setminus S$. For a semisimple algebraic group $G$, one may identify $G_{O_S}$ with a lattice in $G_S = \prod_{v \in S} G_{K_v}$ by diagonal embedding ([Bor]).

**Theorem D.** Let $G$ be a simply connected simple, linear algebraic group. Let $\pi$ and $\pi'$ be inequivalent irreducible representations of $G_S = \prod_{v \in S} G_{K_v}$. Assume that not all $\pi_v, v \in S$, are square integrable. Then

(i) the restriction $\pi|_{G_{O_S}}$ of $\pi$ to $G_{O_S}$ is irreducible

(ii) the restrictions $\pi|_{G_{O_S}}$ and $\pi'|_{G_{O_S}}$ are inequivalent.

If, in addition, $G_{K_v}$ has Property (T) for all $v \in S$, then $\pi|_{G_{O_S}}$ and $\pi'|_{G_{O_S}}$ are not even weakly equivalent.

Observe that if all $\pi_v$ are square integrable, then $\pi|_{G_{O_S}}$ is square integrable and hence cannot be irreducible. Indeed, it is well known that an infinite discrete group has no square integrable irreducible representations.

The paper is organised as follows. In Section 1, we give the proofs of Theorem A and Theorem B, and in Section 2 the proof of Theorem C. The proof of Theorem D is essentially just a combination of the proofs of Theorems A and C, with some minor changes, and we omit it.

**2. Proofs of Theorems A and B**

Let $G$ be a unimodular locally compact group, and let $\Gamma$ be a lattice in $G$. Denote by $\rho$ the quasi-regular representation of $G$ in $L^2(G/\Gamma)$, that is, the induced representation $\text{Ind}^G_1 \Gamma$. Let $\rho^0$ denote the restriction of $\rho$ to $L^2_0(G/\Gamma)$, the subspace orthogonal to the constant functions, i.e.,

$$\{f \in L^2(G/\Gamma) : \int_{G/\Gamma} f(x)dx = 0\}.$$
Our proof of Theorem A is based on the following elementary result, which makes up Corollaries 1.2 and 1.3 in [CoS]. For the convenience of the reader, we reproduce the proof.

**Lemma 1.** Let $\pi$ and $\pi'$ be irreducible representations of $G$. Then

(i) if $\pi$ is not a subrepresentation of $\pi \otimes \rho^0$, then $\pi|_\Gamma$ is irreducible

(ii) if $\pi$ is not a subrepresentation of $\pi' \otimes \rho$, then $\pi|_\Gamma$ and $\pi'|_\Gamma$ are inequivalent.

**Proof.** Let $H_\pi$ denote the Hilbert space of $\pi$, and realise the tensor product representation $\pi \otimes \rho$ in $L^2(G/\Gamma; H_\pi)$, with the action of $g$ in $G$ on a function $f$ given by

$$\pi \otimes \rho(g)f(g_1\Gamma) = \pi(g)f(g^{-1}g_1\Gamma) \quad \forall g_1 \Gamma \in G/\Gamma.$$ 

Define a map

$$\Phi : \text{Hom}_\Gamma(H_\pi, H_{\pi'}) \to \text{Hom}_G(H_\pi, L^2(G/\Gamma; H_{\pi'}))$$

by

$$\Phi(T)\xi(g\Gamma) = \pi'(g)T\pi(g^{-1})\xi \quad \forall g\Gamma \in G/\Gamma \quad \forall \xi \in H_\pi.$$ 

It is easy to verify that $\Phi$ is injective. Note that $\pi' \otimes \rho = \pi' \oplus \pi' \otimes \rho^0$. Hence, if $\pi$ is not a subrepresentation of $\pi \otimes \rho^0$, then $\dim(\text{Hom}_G(H_\pi, L^2(G/\Gamma; H_{\pi'}))) = 1$ and if $\pi$ is not a subrepresentation of $\pi' \otimes \rho$, then $\dim(\text{Hom}_G(H_\pi, L^2(G/\Gamma; H_{\pi'}))) = 0$. This proves the lemma. \qed

We need the following lemma.

**Lemma 2.** Let $G$ be a simply connected, semisimple linear algebraic group defined over $K$. Suppose that $G_{K_v}$ is not compact for some place $v$. Then the natural action of $G_{K_v}$ on $G_A/G_K$ is ergodic.

**Proof.** By Moore’s duality theorem (see [Zim], 2.2.3 Corollary), it suffices to show that $G_K$ acts ergodically on $G_A/G_{K_v}$. Now $G_A/G_{K_v}$ is isomorphic (as a $G_K$-space) to the subgroup $\prod_{w \neq v} G_{K_w}$ of $G_A$. Since $G_{K_v}$ is not compact and $G$ is simply connected, $G_K$ is a dense subgroup of $\prod_{w \neq v} G_{K_w}$, by the Strong Approximation Theorem (see [Zim], Theorem 7.12). This implies that $G_K$ acts ergodically on $\prod_{w \neq v} G_{K_w}$ (see [Zim], 2.2.13 Lemma). \qed
Proof of Theorem A. Let $\rho$ denote the natural induced representation of $G_A$ on $L^2(G_A/G_K)$ and $\rho^0$ denote the restriction of $\rho$ to the orthogonal complement of the constant functions.

(i) Let $\pi$ be an irreducible representation of $G_A$.

For almost all places $v$, $G_{K_v}$ is not compact (see [Spr], 4.9 Lemma). Further, for almost all finite places $v$, $\pi_v$ has a non-trivial $G_{O_v}$-invariant unit vector, $\xi_v$ say. Fix a finite place $v$ such that $\pi_v$ has a non-trivial $G_{O_v}$-invariant unit vector, and $G_{K_v}$ is not compact.

Suppose, by contradiction, that $\pi|_{G_K}$ is reducible. Then, by Lemma 1, $\pi$ is contained in $\pi \otimes \rho^0$. Hence, $\pi_v$ is contained in $\pi_v \otimes (\rho^0|_{G_{K_v}})$.

By Lemma 2, $1_{G_{K_v}}$, the trivial one-dimensional representation of $G_{K_v}$, is contained only once in the restriction of $\rho$ to $G_{K_v}$. That is, $1_{G_{K_v}}$ is not contained in $\rho^0|_{G_{K_v}}$. Hence, arguing as in [CoS], there exists some $\sigma$ in $\hat{G}_{K_v} \setminus \{1_{G_{K_v}}\}$, in the support of $\rho^0|_{G_{K_v}}$, such that $\pi_v$ is contained in $\pi_v \otimes \sigma$. But then $\pi_v$ is contained in $\pi_v \otimes \sigma \otimes \sigma$ and, by induction, we see that $\pi_v$ is contained in $\pi_v \otimes \sigma \otimes \sigma \otimes \cdots$ for any $n$ in $\mathbb{N}$.

Now the kernel of $\sigma$ is finite. Indeed, since $G$ is simply connected and simple and since $G_{K_v}$ is non-compact, every normal subgroup of $G_{K_v}$ is either contained in the centre of $G_{K_v}$ or coincides with $G_{K_v}$ (see [Mar], Chap. I, (2.3.2) Corollary).

It is known that then there is a real number $r$ in $[2, \infty)$ such that all the matrix coefficients of $\sigma$ lie in $L^r(G)$ (see [BoW], Chap. XI, 3.6 Proposition; see also [Cow] for the archimedean case). Hence, for some integer $N$, the $N$-fold tensor product $\sigma \otimes \cdots \otimes \sigma$ is contained in an infinite multiple of the regular representation $\lambda_{G_{K_v}}$. This implies that $\pi_v$ is contained in $\lambda_{G_{K_v}}$, so $\pi_v$ is square integrable.

However, the spherical function $g \mapsto \langle \pi_v(g)\xi_v, \xi_v \rangle$ is not square integrable. Indeed, the Plancherel measure for the positive definite spherical functions has no atoms (see [Mac], Theorem 5.2.10, for the $p$-adic case and [Hel], Chap. IV, Theorem 7.5, for the real case). This establishes the desired contradiction.

(ii) Let $\pi$ and $\pi'$ be inequivalent irreducible representations of $G_A$. Suppose, by contradiction, that $\pi$ is contained in $\pi' \otimes \rho$ and that $\pi'$ is contained in $\pi \otimes \rho$. Since $\rho = 1_{G_A} \oplus \rho^0$, this implies that $\pi$ is contained in $\pi' \otimes \rho^0$ and that $\pi'$ is contained in $\pi \otimes \rho^0$. 

7
Choose a place $v$ such that $\pi_v$ is not square integrable. By Lemma 2, there exist irreducible representations $\sigma$ and $\sigma'$ of $G_{K_v}$, different from $1_{G_{K_v}}$, such that $\pi_v$ is contained in $\pi'_v \otimes \sigma$ and $\pi'_v$ is contained in $\pi_v \otimes \sigma$. Therefore, $\pi_v$ is contained in $\pi_v \otimes \sigma' \otimes \sigma$ and hence in $\pi_v \otimes \sigma' \otimes N \otimes \sigma \otimes N$ for every $N$ in $N$. As in (i), this contradicts the fact that $\pi_v$ is not square integrable and, in view of Lemma 1, completes the proof.

We now proceed to the proof of Theorem B. We first observe that the equivalence of (i) and (iii) in this theorem is valid in a more general situation.

**Lemma 3.** Let $G$ be a locally compact group, and let $H$ be a closed subgroup of $G$. Assume that $1_G$ is weakly contained in the quasi-regular representation $\rho = \text{Ind}_H^G 1_H$. Let $\pi$ be a representation of $G$. Then the restriction $\pi|_H$ is weakly contained in the regular representation $\lambda_H$ of $H$ if and only if $\pi$ is weakly contained in $\lambda_G$.

**Proof.** It is well known that 

$$\text{Ind}_H^G \pi|_H = \pi \otimes \text{Ind}_H^G 1_H = \pi \otimes \rho \quad \text{and} \quad \text{Ind}_H^G \lambda_H = \lambda_G.$$ 

Suppose that $\pi|_H$ is weakly contained in $\lambda_H$. Then $\pi \otimes \rho$ is weakly contained in $\lambda_G$, by the continuity of induction (see [Fel], Theorem 4.1). Since $1_G$ is weakly contained in $\rho$, this implies that $\pi$ is weakly contained in $\lambda_G$.

Conversely, if $\pi$ is weakly contained in $\lambda_G$, then $\pi|_H$ is weakly contained in $\lambda_G|_H$, and hence in $\lambda_H$, as $\lambda_G|_H$ is a multiple of $\lambda_H$. \qed

**Remark.** The assumption that $1_G$ is weakly contained in the quasi-regular representation $\rho = \text{Ind}_H^G 1_H$ means that $G/H$ is amenable in the sense of Eymard [Eym].

**Proof of Theorem B.** Lemma 3 implies that (i) and (iii) of Theorem B are equivalent. It remains to show that (ii) and (iii) are equivalent.

Let $\pi$ be an irreducible representation of $G_A$. If $\pi$ is weakly contained in $\lambda_{G_A}$, then $\pi_v$ is weakly contained in $\lambda_{G_A}|_{G_{K_v}}$, and hence in $\lambda_{G_{K_v}}$.

Conversely, suppose that, for every place $v$, $\pi_v$ is weakly contained in $\lambda_{G_{K_v}}$. Let $\xi$ be a vector in the Hilbert space $\otimes_{v \in V} \mathcal{H}_{\pi_v}$ of the form $\otimes \xi_v$, where every $\xi_v$ is a unit vector, and $\xi_v$ is invariant under $G_{\mathcal{O}_v}$ for almost all finite places $v$. Define $\phi = (\pi(\cdot)\xi, \xi)$,
the positive definite function associated with ξ. It suffices to show that ϕ is the limit, uniformly on compacta in GA, of positive definite functions with compact support.

Let K be a compact subset of GA. By the definition of the topology on GA, there exist a finite set F of places containing V∞ and compact subsets Kv of GKv for v in F such that K is contained in KF \times \prod_{v \notin F} GKv, where KF = \prod_{v \in F} K_v. Now

ϕ = \bigotimes_{v \in F} \langle π_v(\cdot)ξ_v, ξ_v \rangle

on KF \times \prod_{v \notin F} GKv. Since πv is weakly contained in λGKv, there exists, for every v in F, a sequence ψn(v) of normalised positive definite functions on GKv with compact supports converging to \langle π_v(\cdot)ξ_v, ξ_v \rangle uniformly on Kv. Set

ψn = \bigotimes_{v \in F} ψ_n(v) \otimes \bigotimes_{v \notin F} χ_v,

where χ_v denotes the characteristic function of GOv. Then ψn is a positive definite function on GA with compact support and limn ψ_n = ϕ, uniformly on K. Hence, (ii) and (iii) are equivalent.

Given a representation π of GKv, we may lift it to a representation ˜π of GA, equal to π \otimes \bigotimes_{w \neq v} 1GKw. Clearly, π|GK = ˜π|GK. Now GKw is non-compact and hence non-amenable for almost all places w, and, for any such w, 1GKw is not weakly contained in the regular representation. Hence π|GK is not weakly contained in λGK. This proves the last assertion of Theorem B. □

2. Proof of Theorem C

Lemma 4. Let G be a semisimple linear algebraic group defined over K, and let π and π′ be irreducible representations of GA. If π is weakly contained in π′, then π and π′ are equivalent.

Proof. If π is weakly contained in π′, then πv is weakly contained in π′v for every place v.

Recall that a representation π of the group G is weakly contained in a representation ρ if and only if the C∗-kernel of ρ (this is the kernel of the extension of ρ to the C∗-algebra of G) is contained in the C∗-kernel of π.
It is known that every $G_{K_v}$ is a CCR-group, that is, the range of the $C^*$-algebra of $G_{K_v}$ under an irreducible representation consists of compact operators (see [Ber]; see also [Dix], 15.5.6 for the real case). This implies that $\pi_v$ is equivalent to $\pi'_v$ for all places $v$ (see [Dix], Theorem 4.3.7). Hence, $\pi$ and $\pi'$ are equivalent. □

Let $G$ be a semisimple algebraic group defined over $K$. As in the proof of Theorem A, let $\rho$ and $\rho^0$ denote the natural representation of $G_A$ on $L^2(G_A/G_K)$ and the restriction of $\rho$ to the orthogonal complement of the constant functions.

**Proof of Theorem C.** Let $\pi$ and $\pi'$ be inequivalent irreducible representations of $G_A$. Assume that $\pi$ is not weakly contained in the regular representation $\lambda_{G_A}$.

Suppose, by contradiction, that $\pi|_{G_K}$ and $\pi'|_{G_K}$ are weakly equivalent. Then

$$\text{Ind}_{G_K}^{G_A} \pi|_{G_K} = \pi \otimes \text{Ind}_{G_K}^{G_A} 1_{G_K} = \pi \oplus (\pi \otimes \rho^0)$$

and, by the continuity of induction, this is weakly equivalent to $\text{Ind}_{G_K}^{G_A} \pi'|_{G_K}$, which in turn is equal to $\pi' \oplus (\pi' \otimes \rho^0)$. Since $\pi$ is irreducible, this implies that $\pi$ is either weakly contained in $\pi'$ or in $\pi' \otimes \rho^0$ and that $\pi'$ is either weakly contained in $\pi$ or in $\pi \otimes \rho^0$. In view of Lemma 4, it follows that $\pi$ is weakly contained in $\pi' \otimes \rho^0$ and that $\pi'$ is weakly contained in $\pi \otimes \rho^0$.

By Theorem B, we may choose a place $v$ such that $G_{K_v}$ is non-compact and $\pi_v$ is not weakly contained in the regular representation $\lambda_{G_{K_v}}$. Since $\pi_v$ is weakly contained in $\pi'_v \otimes \rho^0|_{G_{K_v}}$ and $\pi'_v$ is weakly contained in $\pi_v \otimes \rho^0|_{G_{K_v}}$, there exist irreducible representations $\sigma$ and $\sigma'$ of $G_{K_v}$, weakly contained in $\rho^0|_{G_{K_v}}$, such that $\pi_v$ is weakly contained in $\pi'_v \otimes \sigma$ and $\pi'_v$ is weakly contained in $\pi_v \otimes \sigma'$. Therefore, $\pi$ is weakly contained in $\pi \otimes \sigma' \otimes \sigma$, and hence in $\pi \otimes (\sigma' \otimes \sigma)^{\otimes N}$ for any $N$ in $\mathbb{N}$.

Now $G_{K_v}$ has Kazhdan’s Property (T). This implies that $1_{G_{K_v}}$ is not weakly contained in the restriction $\rho^0|_{G_{K_v}}$ of $\rho^0$ to $G_{K_v}$. Indeed otherwise, $1_{G_{K_v}}$ would be contained in $\rho^0|_{G_{K_v}}$, and the action of $G_{K_v}$ on $G_A/G_K$ would not be ergodic, contradicting Lemma 2. So, $\sigma$ and $\sigma'$ are different from the trivial representation of $G_{K_v}$.

We now proceed as in the proof of Theorem A. The matrix coefficients of $\sigma' \otimes \sigma$ belong to $L^r(G_{K_v})$ for some finite $r$. Choose $N$ so large that $(\sigma' \otimes \sigma)^{\otimes N}$ are weakly contained in $\lambda_{G_{K_v}}$. Hence, $\pi_v$ is weakly contained in $\lambda_{G_{K_v}}$. This is a contradiction. □
Note. The acute reader will have noticed that we do not require the full force of Kazhdan’s Property (T), but rather that \( L^2(G_A/G_K)_0|_{G_{Kv}} \) does not weakly contain the trivial representation of \( G_{Kv} \) for all valuations \( v \). Thus in particular Theorem D also holds for the case where \( G = SL_2 \) and \( K = \mathbb{Q} \), by results of Selberg, which imply the isolation in \( \widehat{G_R} \), and estimates for modular forms which imply the isolation in \( \widehat{G_{\mathbb{Q}_p}} \). There has been much interest in this question (see, e.g., Burger, Li and Sarnak [BLS] and Burger and Sarnak [BS] for the real case), but, at the time of writing, a proof that this separation property (which is also of interest in other problems) always holds was not known to the authors.

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