Lecture 8: Basis and Dimension

More Examples of Bases

From thm lecture 6 we know

**Thm 1**  Let $V = \text{vector space/ field } \mathbb{F}$. $B = \{v_1, \ldots, v_n\}$ is a basis iff every $v \in V$ is uniquely

**E.g. 1**  $V = M_{mn}(\mathbb{F})$. For $i \in$ define $E_{ij} =$

i.e. 0 everywhere but
\[ B = \{E_{ij}\} \text{ is a basis. e.g. for } M_{22} \text{ we can uniquely express} \]

\textbf{E.g. 2} \quad W = \text{line through origin in dirn} \ w \in \mathbb{R}^3.

\textbf{E.g. 3} \quad W = \text{plane through origin}
**Questions** 1. In the $\mathbb{R}^m$ examples, looks like the number of elements in a basis correspond to the dimension of a subspace. Can use this as a defn

2. Does any vector space have a basis and if so

**Aim lectures 8/9:** Answer 1 in affirmative and provide effective answer for 2.

More precise answer to 1 suggested by

**E.g. 3 revisited** $W =$ plane through origin
More generally have

**Thm 2**  Let $V =$ vector space/ field $F$

Let $S = \{w_1, \ldots, w_n\}$

Let $I = \{v_1, \ldots, v_m\}$

Then

Proof: Omitted. See §6.7.2 of notes. See e.g. 4 below for basic idea of proof.
**Thm 3 - Defn** Let $V = \text{vector space/ field } \mathbb{F}$.

Suppose $B_1 = \{v_1, \ldots, v_m\}$ and $B_2 = \{w_1, \ldots, w_n\}$ are bases for $V$. Then

In this case, say $V$ & dim $V :=$

When no such basis exists,

Proof: By thm 1

$B_1$ spans $V$ & $B_2$ lin indep $\implies$

so $m = n$.

**E.g. 4** We have a standard basis $\{\}$

so dim $\mathbb{R}^n =$
Thm 3 \implies any basis of \( \mathbb{R}^n \)

We’ll prove this without using thms 2 or 3.

Proof: Let \( B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \) be

& \( A = \) so \( A \) is \( n \times m \)

We wish to show \( m = \)

\( B \) lin indep \implies \( A \mathbf{x} = \mathbf{0} \) has

Gaussian elim \implies

(otherwise there are more

\( B \) spans so \( A \mathbf{x} = \mathbf{b} \) always

This suggests

Proof is a little delicate. Consider solns \( \mathbf{x}_i \)
to $A x_i = e_i$. Let $X$ be $m \times n$-matrix

$$AX = (A x_1$$

If $m < n$ then above argument $\implies$ there’s a non-zero soln $w$ to $X w =$

But $w = I w =$

Contradiction $\implies$

This proves thm 3 in this e.g.

**E.g. 5**

$$\dim M_{mn}(\mathbb{F})$$

$$\dim \mathbb{P}_k$$

**Corollary 1** For $V =$ vector space/ field $\mathbb{F}$ of $\dim d$
a) Any lin indep

b) Any spanning set

**E.g. 6** Can’t span $\mathbb{R}^3$ with < 3 vectors and any 4 vectors in $\mathbb{R}^3$ are lin dependent.

We need a lemma on extending lin indep sets

**Lemma** Let $V$ = vector space/ field $\mathbb{F}$

Let $I$ be a lin indep set. If $v \in V$ then $I \cup$

Geom clear as in

Proof lemma: Consider a lin reln

$\lambda v +$
If $\lambda \neq 0$ then solving

Hence,

$I \text{ lin indep} \implies$

Hence

**Corollary 2** Let $V = \text{vector space/ field } \mathbb{F}$. Let $B = \{v_1, \ldots, v_n\} \subset V$. The following conditions on $B$ are equivalent:

a) $B$ is a basis
b) $B$ is lin indep &
c)

Proof: a) $\implies$ b) and c) by

Check b) $\implies$ a): Suppose a) false so $B$ doesn’t span.
Pick \( \mathbf{v} \in \) Lemma \( \implies \{ \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) is
This contradicts Cor 1a) so a) must hold.
Check c) \( \implies \) a): Suppose a) false so
Say \( \mathbf{v}_1 \in \)

\textbf{E.g. 7} Let \( B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) be an o/n set of vectors in \( \mathbb{R}^n \).

\textbf{Remark} Concept of basis for infinite di-
imensional vector spaces is not as useful. In particular, the defn of span of an infinite set is usually modified.