Lecture 4: More on Linear Combns and Span

**Matrix Interpretn of Lin Combn & Span in \( \mathbb{F}^m \)**

\[
\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \quad \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \in \mathbb{F}^m
\]

**Propn** The lin combn

\[
x_1 \mathbf{a}_1 + \ldots + x_n \mathbf{a}_n = A \mathbf{x}
\]

Proof:
Corollary \( \text{Span}(a_1, \ldots, a_n) \) consists of vectors \( \mathbf{b} \) such that \( A \mathbf{x} = \mathbf{b} \) has a soln.

Defn For an \( m \times n \)-matrix \( A \) as above, the
column space of $A$, denoted $\text{col}(A)$ is
\[
\text{Span}(a_1, \ldots, a_n)
\]

**Determining Span in $\mathbb{F}^m$**

**Stupid E.g. 1** Consider what are called the standard basis vectors

\[
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & e_2 &= \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, & \ldots, & e_n &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\
&\in \mathbb{F}^m
\end{align*}
\]

Corresponding matrix is $A = I$. You can always solve
E.g. 2  Is $\mathbf{b} = (0, 1, 2)^T$ in 
$\text{Span}((1, 2, 3)^T, (4, 5, 6)^T)$?
i.e. can you solve $A \mathbf{x} = \mathbf{b}$?

Hence we can solve $A \mathbf{x} = \mathbf{b}$ so

E.g. 3  Find all $\mathbf{b}$ in $\text{Span}((1, 0, 2)^T, (3, 1, 2)^T)$.
Try to solve
Hence, $A \mathbf{x} = \mathbf{b}$ is solvable iff Span here is

**E.g. 4** Do $\{(1, 0, 2)^T, (2, 1, 4)^T, (3, -2, 6)^T\}$ span $\mathbb{R}^3$?

Hence, can’t always solve $A \mathbf{x} = \mathbf{b}$:

**Determining Span in $M_{mn}(\mathbb{F})$ and $\mathbb{P}$**. Reduce questions to solving a system of linear eqns as follows.

**E.g. 5**

Is

$$
\begin{pmatrix}
0 & 1 \\
2 & 0
\end{pmatrix}
\in \text{Span}(\begin{pmatrix}1 & 2 \\
3 & 0\end{pmatrix}, \begin{pmatrix}4 & 5 \\
6 & 0\end{pmatrix})?
$$


Equivalently, can you solve

Comparing entries we see we need to solve for $\lambda, \mu \in \mathbb{F}$

Same

**E.g. 6** Which polynomials $b_1 + b_2x + b_3x^2$ lie in $\text{Span}(1 + 2x^2, 3 + x + 2x^2)$? i.e. when can you solve
Compare coefficients:

Hence $b_1 + b_2x + b_3x^2$ is in span iff

**Span in $\mathcal{R}[\mathbb{R}]$ and other vector spaces**

No systematic method here unfortunately.

**E.g. 7**

$$\cos 3x \cos 2x = \frac{1}{2} \cos x + \frac{1}{2} \cos 5x$$
E.g. 8

\[ e^{3x} \not\in \text{Span}(e^x, e^{2x}) \]

Why?

E.g. 9 Let \( \mathbb{R}^\infty \) be the vector space

with addition and scalar multiplication defined coordinate-wise. Let \( e_i = \)

Then \( \text{Span}(e_1, \)