Let $T : V \rightarrow V$ have an e-basis $\{v_1, \ldots, v_n\}$ & let $L_i = \text{Span } v_i$.

**Aim lecture:** See how study of $T$ decomposes into study of

**E.g. 1** $y_1(t) =$ popn of hobbits in

$y_2(t) =$ popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

\[
\begin{align*}
y'_1(t) &= 3y_1(t) \\
y'_2(t) &= 2y_2(t)
\end{align*}
\] (*)
Soln: Easy, solve 2 eqns separately

Suppose now we put the two popns in

Typical DEs describing popn growth is

\[
y'_1(t) = 3y_1(t) - 2y_2(t) \tag{†}
\]

\[
y'_2(t) = -y_1(t) + 2y_2(t)
\]

These are “coupled” DEs i.e. \( y'_1, y'_2 \) each depend on both \( y_1 \& y_2 \). We’ll use diag to

\textbf{Notn} \( y(t) = \)
\[ y'(t) = \frac{dy}{dt} := \]

In e.g. 1, \( y'(t) = A y(t) \) where

\[ A = \]

**Note** In decoupled case (*) above, still have \( y'(t) = A y(t) \) but now

**Change of Var & DEs**

Consider more generally \( y(t) = (y_1(t), \ldots) \), & system of \( n \) linear DEs

\[ y'(t) = A y(t) \]

where \( A \in \mathbb{M}_{n,n}(\mathbb{R}) \).
Lemma For $C \in M_{n,n}(\mathbb{R})$

\[ \frac{d}{dt} \]

Proof: Clear from case $n = 2$. Say

$C = \frac{d}{dt}(C \ y) = $

Back to solving $y'(t) = A \ y(t)$, suppose we can diag $A = MDM^{-1}$ with
\[ D = \]

\[ MDM^{-1}y = \]
\[ \therefore DM^{-1} \]

Let’s change var to \( x(t) = \)

See \( \frac{dx}{dt} \)

This is decoupled. Can solve the \( n \) linear DEs

to get \( x_i(t) = \alpha_i \)

**Upshot** The soln to \( y'(t) = A y(t) \) is

\[ y(t) = M x(t) = \]
where \( \lambda_i \) =

& \( f_i \) = corresp e-vectors.

**E.g. 1 completed**

We diag \( A \)

\[
\det(\lambda I - A) = \begin{vmatrix}
\lambda - 3 & 2 \\
1 & \lambda - 2
\end{vmatrix}
\]

The e-values are

E-vectors?

\( \lambda = 4 \) : \( \ker(\lambda I - A) = \)
An e-vector is
\[ \lambda = 1 : \ker(\lambda I - A) = \]

An e-vector is
Hence (from upshot) soln given by
\[ y(t) = \]
i.e. \[ y_1(t) = \]

E.g. 2 Suppose in e.g. 1 that initial popn is \[ y(0)^T = (4000, 1000) \]. Solve the IVP.

Ans: We need only solve for \( \alpha_1, \alpha_2 \).
From Gaussian elim or guessing see

\[ \alpha_1 = \]

The soln is thus

\[ y(t) = \]

2nd Order ODEs

We can convert any 2nd order ODE into a pair of linear ODEs in 2 var as in following

E.g. 3 Solve IVP

\[ y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2, y'(0) = 3 \]

Ans: Let \( y_1 = y, y_2 = \)

\[ y_1' = \]
i.e. \( y' = \)

Diag \( A \): \( \det(\lambda I - A) = \)

Hence e-values are

E-vectors:
\( \lambda = 2 : \ker(\lambda I - A) = \)

An e-vector is
\( \lambda = 1 : \ker(\lambda I - A) = \)
An e-vector is
Hence, (from upshot) general soln is
\[ y(t) = \]

Need now find integration constants.