Lecture 23: Intro to Eigenbases

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**Q** Why did we introduce abstract notion of vector spaces?

**A** 1. To handle infinite dim
2. Defn is

Let $V = \text{vector space/ field } \mathbb{F}$

Next few lectures, study lin maps of form

i.e. domain =

**Aim** $T : V \rightarrow V$ often picks out its own preferred coord system/basis for $V$. Wish to describe this
E.g. 1 \( V := 3\)-dim space of

\[
T : V \longrightarrow V \text{ is rotatn about axis } L = \text{Span } \mathbf{u} \text{ about angle } \theta.
\]

N.B. Can check geom that \( T \) is

Let \( P = \)

Preferred coord system has \( L \)

i.e. if \( \mathbf{w} \in P \) is

preferred basis \( \mathcal{B} = \{ \)
W.r.t $\mathcal{B}$

**E.g. 2** $V := \text{space of 2-dim}$

Let $L =$

Let $T : V \rightarrow V$ be reflection about $L$

Let $S : V \rightarrow V$ be orthog

Let $L' =$
Preferred Basis is
Matrix representing $T$ is

Matrix representing $S$ is

**Key Notion** The subspaces $L, L', P$ above are examples of invariant subspaces.

**Defn** Let $T : V \rightarrow V$ be lin. A subspace
$W$ of $V$

In this case, restricting domain to $W$ gives linear

**E.g.** $L, L', P$ in

**Remark** The matrices above obtained using “preferred” bases have “block diagonal” form with blocks representing $T|_W$ where $W$

**Eigenvectors**

Optimal scenario when invariant subspaces
are 1-dim so blocks have smallest size.

Q When’s this occur?
Ans: $W = \text{Span } \mathbf{v}$ is $T$-invariant iff

(*)

Defn (Eigenvector) If $\mathbf{v} \neq 0$ is as in (*) above we say that

E.g. 2 again

$\mathbf{u}, \mathbf{v}$ are
E-values are
\begin{align*}
\lambda & \quad T & \quad S \\
\mathbf{u} & \\
\mathbf{v} &
\end{align*}

E.g. 3
\[
A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, T = T_A:
\]

E.g. 4 \( V = C^\infty \)
\[ T : V \longrightarrow V \] is differentiation.

**Thm** Let \( T : V \longrightarrow V \) be

Suppose \( B = \{v_1, \ldots, v_n\} \) is

The matrix representing \( T \) wrt \( B \) is the diagonal matrix

Proof: Generalised matrix representation thm (lecture 11) shows
Change of Basis

Let $A$ be an $n \times n$-matrix so $T_A$:

Let $B = \{f_1, \ldots, f_n\}$

Q What’s matrix rep $T_A$ wrt

Let $M = (f_1 \ldots f_n)$

Thm 2 The matrix $C$ rep
Proof: Need prove first

**Lemma** $S_B(v) := [v]_B =$

Proof: $f_i = M e_i = M[f_i]_B$

$\therefore S_B(f_i) =$

The linear maps $S_B$

Resume proof thm: $C = ([A f_1]_B \ldots [A f_n]_B)$

**E.g. 5** Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has e-
vectors

\[ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]

with e-values 1,2. Find matrix \( A \) rep \( T \).

Ans: